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ON THE APPROXIMATION OF FUNCTIONS BY JACOBI – DUNKL EXPANSION IN THE WEIGHTED SPACE $\mathbb{L}_2^{(\alpha,\beta)}$

ПРО НАБЛИЖЕННЯ ФУНКЦІЙ ЗА ДОПОМОГОЮ РОЗКЛАДІВ ЯКОБІ – ДАНКЛА У ВАГОВОМУ ПРОСТОРІ $\mathbb{L}_2^{(\alpha,\beta)}$

We prove some new estimates useful in applications for the approximation of certain classes of functions characterized by the generalized continuity modulus from the space $\mathbb{L}_2^{(\alpha,\beta)}$ by partial sums of the Jacobi – Dunkl series. For this purpose, we use the generalized Jacobi – Dunkl translation operator obtained by Vinogradov in the monograph [Theory of approximation of functions of real variable, Fizmatgiz, Moscow (1960) (in Russian)].

Доведено деякі нові оцінки, корисні в застосуваннях, для наближень певних класів функцій, що характеризуються узагальненим модулем неперервності з простору $\mathbb{L}_2^{(\alpha,\beta)}$, частковими сумами рядів Якобі – Данкла. З цією метою використано узагальнений оператор трансляції Якобі – Данкла, що був отриманий Виноградовим у монографії [Theory of approximation of functions of real variable. Fizmatgiz, Moscow (1960) (in Russian)].

1. Introduction. It is well-known that many problems for partial differential equations are reduced to a power series expansion of the desired solution in terms of special functions or orthogonal polynomials (such as Laguerre, Hermite, Jacobi, etc. polynomials). In particular, this is associated with the separation of variables as applied to problems in mathematical physics (see, e.g., [10, 11]).

In [2], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator. In this paper, we also discuss this subject. More specially, we prove some estimates (similar to those proved in [2]) in certain classes of functions characterized by a generalized continuity modulus and connected with the discrete Jacobi – Dunkl transform associated with the Jacobi – Dunkl operator defined on $\mathbb{T} = [-\pi/2, \pi/2]$ by

$$\Lambda_{\alpha,\beta} f(\theta) := \frac{d}{d\theta} f(\theta) + \frac{\mathcal{A}'_{\alpha,\beta}(\theta)}{\mathcal{A}_{\alpha,\beta}(\theta)} \frac{f(\theta) - f(-\theta)}{2}, \quad f \in \mathcal{C}^1\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right),$$

where

$$\alpha \geq \beta \geq -\frac{1}{2}, \quad \alpha \neq -\frac{1}{2}.$$

This paper is organized as follows. In Section 2, we state some basic notions and results from the discrete harmonic analysis associated with the Jacobi – Dunkl transform that will be needed throughout this paper. Some estimates are proved in Section 3.

2. Preliminaries. In this section, we will recall some properties of Jacobi and Jacobi – Dunkl polynomials, we develop some results from the discrete harmonic analysis related to the differential-difference operator $\Lambda_{\alpha,\beta}$. Further details can be found in [3 – 5, 7, 8, 12]. In the following, we fix parameters α and β subject to the constraints

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$$\alpha \geq \beta \geq -\frac{1}{2}, \quad \alpha \neq -\frac{1}{2},$$

and set

$$\rho = \alpha + \beta + 1.$$

The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1,$$

where $a, b, c, z \in \mathbb{C}$ with $c \notin \mathbb{Z}_-$ and $(a)_n$ is the Pochhammer symbol given by

$$(a)_n := \begin{cases} a(a+1)\ldots(a+n-1) & \text{if} \quad n \in \mathbb{N}^*, \\ 1 & \text{if} \quad n = 0, \end{cases}$$

where $\mathbb{N}^* = \{1, 2, \ldots\}$.

The Jacobi polynomials $\varphi_n^{(\alpha,\beta)}(\theta)$, $n \in \mathbb{N}$, $\theta \in \mathbb{T}$, are defined by

$$\varphi_n^{(\alpha,\beta)}(\theta) := \mathcal{R}_n^{(\alpha,\beta)}(\cos(2\theta)) = {}_2F_1\left(-n, n+\rho; \alpha+1; \sin^2\theta\right)$$

with $\mathcal{R}_n^{(\alpha,\beta)}(x)$, $n \in \mathbb{N}$, is the normalized Jacobi polynomial of degree n such that $\mathcal{R}_n^{(\alpha,\beta)}(1) = 1$. Note that, for all $n \in \mathbb{N}$, we have

$$\left|\varphi_n^{(\alpha,\beta)}(\theta)\right| \leq 1 \quad \forall \theta \in \mathbb{T} \tag{1}$$

and

$$\varphi_n^{(\alpha,\beta)}(-\theta) = \varphi_n^{(\alpha,\beta)}(\theta) \quad \forall \theta \in \mathbb{T}. \tag{2}$$

The Jacobi operator $\Delta_{\alpha,\beta}$ defined on $\mathcal{C}^2\left(\left]0, \frac{\pi}{2}\right[\right)$ is given by

$$\Delta_{\alpha,\beta} := \frac{d^2}{d\theta^2} + \frac{\mathcal{A}'_{\alpha,\beta}}{\mathcal{A}_{\alpha,\beta}} \frac{d}{d\theta}$$

with

$$\mathcal{A}_{\alpha,\beta}(\theta) := \begin{cases} 2^{2\rho}(\sin|\theta|)^{2\alpha+1}(\cos\theta)^{2\beta+1} & \text{if} \quad \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}, \\ 0 & \text{if} \quad \theta = 0. \end{cases}$$

For all $n \in \mathbb{N}$, $\varphi_n^{(\alpha,\beta)}$ is the unique even $\mathcal{C}^\infty$-solution on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ of the differential equation

$$\Delta_{\alpha,\beta} f = -\lambda_n^2 f,$$

$$f(0) = 1,$$

$$f'(0) = 0,$$

where

$$\lambda_n = \lambda_n^{(\alpha,\beta)} := 2\mathrm{sgn}(n)\sqrt{|n|(|n|+\rho)}, \quad n \in \mathbb{Z}.$$

The Jacobi function $\varphi_n^{(\alpha,\beta)}$, $n \in \mathbb{N}$, satisfies the following inequalities.

Lemma 1. *The following inequalities are valid for Jacobi functions* $\varphi_n^{(\alpha,\beta)}$:
a) *for* $\theta \in (0, \pi/4]$, *we have*

$$1 - \varphi_n^{(\alpha,\beta)}(\theta) \le c_1 \lambda_n^2 \theta^2, \tag{3}$$

b) *for every* $\gamma > 0$, *there is a number* $c_2 = c_2(\gamma, \alpha, \beta) > 0$ *such that, for all* n *and* θ *with* $\gamma < n\theta < \dfrac{\pi n}{4}$, *we obtain*

$$\left|\varphi_n^{(\alpha,\beta)}(\theta)\right| \le c_2 (n\theta)^{-\alpha-1/2}. \tag{4}$$

Proof. See [9] (Proposition 3.5 and Lemma 3.1).
The Jacobi – Dunkl operator $\Lambda_{\alpha,\beta}$ is defined by

$$\Lambda_{\alpha,\beta} f(\theta) := \frac{d}{d\theta} f(\theta) + \frac{\mathcal{A}'_{\alpha,\beta}(\theta)}{\mathcal{A}_{\alpha,\beta}(\theta)} \frac{f(\theta) - f(-\theta)}{2}, \quad f \in \mathcal{C}^1\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right),$$

with

$$\frac{\mathcal{A}'_{\alpha,\beta}(\theta)}{\mathcal{A}_{\alpha,\beta}(\theta)} = (2\alpha + 1)\cot\theta + (2\beta + 1)\tan\theta, \quad \theta \in \left]\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}.$$

From [7], for all $n \in \mathbb{Z}$, the differential-difference equation

$$\Lambda_{\alpha,\beta} f(\theta) = i\lambda_n f(\theta), \quad n \in \mathbb{Z},$$

$$f(0) = 1,$$

admits a unique $\mathcal{C}^\infty$-solution $\psi_n^{(\alpha,\beta)}(\theta)$ on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. It is related to the Jacobi polynomial and to its derivative by

$$\psi_n^{(\alpha,\beta)}(\theta) := \begin{cases} \varphi_{|n|}^{(\alpha,\beta)}(\theta) - \dfrac{i}{\lambda_n} \dfrac{d}{d\theta} \varphi_{|n|}^{(\alpha,\beta)}(\theta) & \text{if} \quad n \in \mathbb{Z}^*, \\ 1 & \text{if} \quad n = 0. \end{cases}$$

We note that, for all $n \in \mathbb{Z}$ and $\theta \in \mathbb{T}$, we have

$$\psi_{-n}^{(\alpha,\beta)}(\theta) = \psi_n^{(\alpha,\beta)}(-\theta) = \overline{\psi_n^{(\alpha,\beta)}(\theta)}, \tag{5}$$

$$\left|\psi_n^{(\alpha,\beta)}(\theta)\right| \le 1,$$

and, for all f and g such that $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Lambda_{\alpha,\beta} f(\theta) g(\theta) \mathcal{A}_{\alpha,\beta}(\theta) d\theta$ exists, we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Lambda_{\alpha,\beta} f(\theta) g(\theta) \mathcal{A}_{\alpha,\beta}(\theta) d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \Lambda_{\alpha,\beta} g(\theta) \mathcal{A}_{\alpha,\beta}(\theta) d\theta. \tag{6}$$

For all $n, p \in \mathbb{Z}$, we have the orthogonality formula given by (see [7])

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_n^{(\alpha,\beta)}(\theta)\overline{\psi_p^{(\alpha,\beta)}(\theta)}\mathcal{A}_{\alpha,\beta}(\theta)d\theta = \left(w_n^{(\alpha,\beta)}\right)^{-1}\delta_{n,p}, \tag{7}$$

where

$$w_n^{(\alpha,\beta)} = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\psi_n^{(\alpha,\beta)}(\theta)\right|^2 \mathcal{A}_{\alpha,\beta}(\theta)d\theta\right)^{-1}, \quad w_0^{(\alpha,\beta)} = \frac{\Gamma(\rho+1)}{2^{2\rho}\Gamma(\alpha+1)\Gamma(\beta+1)}$$

and

$$w_n^{(\alpha,\beta)} = \frac{(2|n|+\rho)\Gamma(\alpha+|n|+1)\Gamma(\rho+|n|)}{2^{2\rho+1}(\Gamma(\alpha+1))^2\Gamma(|n|+1)\Gamma(\beta+|n|+1)} \quad \forall n \in \mathbb{Z}^*.$$

We obtain the following asymptotic equality as $n \to +\infty$:

$$w_n^{(\alpha,\beta)} \asymp \frac{|n|^{2\alpha+1}}{2^{2\rho}\left(\Gamma(\alpha+1)\right)^2}.$$

By using the relation (see [7])

$$\frac{d}{d\theta}\varphi_{|n|}^{(\alpha,\beta)}(\theta) = -\frac{\lambda_n^2}{4(\alpha+1)}\sin(2\theta)\,\varphi_{|n|-1}^{(\alpha+1,\beta+1)}(\theta),$$

the function $\psi_n^{(\alpha,\beta)}$ can be written in the form

$$\psi_n^{(\alpha,\beta)}(\theta) = \varphi_{|n|}^{(\alpha,\beta)}(\theta) + i\,\frac{\lambda_n}{4(\alpha+1)}\sin(2\theta)\varphi_{|n|-1}^{(\alpha+1,\beta+1)}(\theta). \tag{8}$$

Let $\mathbb{L}_2^{(\alpha,\beta)}$ denote the space of square integrable functions $f(\theta)$ on the closed interval $\mathbb{T}$ with the weight function $\mathcal{A}_{\alpha,\beta}(\theta)$ and the norm

$$\|f\| = \sqrt{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(\theta)|^2\mathcal{A}_{\alpha,\beta}(\theta)d\theta}.$$

We define the weighted spaces $l^2(\mathbb{Z}) := l^2\left(\mathbb{Z}, w_n^{(\alpha,\beta)}\right)$ by

$$l^2(\mathbb{Z}) = \left\{(f_n)_{n\in\mathbb{Z}} : \mathbb{Z} \longrightarrow \mathbb{C} : \sum_{n=-\infty}^{+\infty} |f_n|^2 w_n^{(\alpha,\beta)} < +\infty\right\}.$$

The Jacobi – Dunkl expansion of a function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ is defined by (see [6, 7])

$$f(\theta) = \sum_{n=-\infty}^{+\infty} \mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)w_n^{(\alpha,\beta)} \quad \forall\theta \in \mathbb{T}, \tag{9}$$

where

$$\mathcal{F}f(n) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta)\overline{\psi_n^{(\alpha,\beta)}(\theta)}\mathcal{A}_{\alpha,\beta}(\theta)d\theta \quad \forall n \in \mathbb{Z}.$$

The sequence $\{\mathcal{F}f(n), n \in \mathbb{Z}\}$ is called the discrete Jacobi – Dunkl transform of f. For $n \in \mathbb{N}$, we denote the partial sum of (9) by

$$\mathcal{S}_n^f(\theta) := \sum_{k=-n}^{n} \mathcal{F}f(k)\psi_k^{(\alpha,\beta)}(\theta)w_k^{(\alpha,\beta)} \quad \forall \theta \in \mathbb{T}.$$

From (7), we get, for all $f \in \mathbb{L}_2^{(\alpha,\beta)}$ (see [6]),

$$\lim_{n\to+\infty} \left\| \mathcal{S}_n^f - f \right\| = 0.$$

We state some properties of the discrete Jacobi – Dunkl transform $\mathcal{F}$ (see [7]).

Theorem 1 (Plancherel formula). *If $f \in \mathbb{L}_2^{(\alpha,\beta)}$, then $\mathcal{F}f$ belongs to $l^2(\mathbb{Z})$ and we have*

$$\|f\| = \sqrt{\sum_{n=-\infty}^{+\infty} |\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)}}. \tag{10}$$

Proof. See [7] (Theorem 3.4).

The generalized Jacobi – Dunkl translation operator is defined for $f \in \mathbb{L}_2^{(\alpha,\beta)}$ and $\theta, h \in \mathbb{T}$ by

$$\mathcal{T}^h f(\theta) := \begin{cases} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\varphi)W(h,\theta,\varphi)\mathcal{A}_{\alpha,\beta}(\varphi)d\varphi & \text{if} \quad h,\theta \in G_{\alpha,\beta}, \\ f(\theta+h) & \text{if} \quad h \notin G_{\alpha,\beta} \quad \text{or} \quad \theta \notin G_{\alpha,\beta}, \end{cases}$$

where

$$G_{\alpha,\beta} := \begin{cases} \mathbb{R} \setminus \{n\pi\}_{n\in\mathbb{Z}} & \text{if} \quad \alpha > \beta \geq -\dfrac{1}{2}, \\ \mathbb{R} \setminus \left\{\dfrac{n\pi}{2}\right\}_{n\in\mathbb{Z}} & \text{if} \quad \alpha = \beta > -\dfrac{1}{2}, \\ \emptyset & \text{if} \quad \alpha = \beta = -\dfrac{1}{2}, \end{cases}$$

and $W(h,\theta,\varphi)$ is a certain function satisfies the following properties (see [12]):

$$W(h,\theta,\varphi) = W(\theta,h,\varphi),$$

$$W(h,\theta,-\varphi) = W(-h,-\theta,\varphi),$$

$$W(h,\theta,\varphi) = W(h,-\varphi,-\theta).$$

In particular, the product formula

$$\mathcal{T}^h\psi_n^{(\alpha,\beta)}(\theta) = \psi_n^{(\alpha,\beta)}(h)\psi_n^{(\alpha,\beta)}(\theta). \tag{11}$$

holds. Some properties of generalized Jacobi – Dunkl translation operator are fulfilled.

Theorem 2. *If* $f \in \mathbb{L}_2^{(\alpha,\beta)}$, *then* $\mathcal{T}^h f \in \mathbb{L}_2^{(\alpha,\beta)}$ *and we have*

$$\|\mathcal{T}^h f\| \leq \|f\| \quad \forall h \in \mathbb{T}.$$

Proof. See [12] (Theorem 3).

Proposition 1. *Let* $f \in \mathbb{L}_2^{(\alpha,\beta)}$ *and* $n \in \mathbb{Z}$. *Then*

$$\mathcal{F}(\mathcal{T}^h f)(n) = \psi_n^{(\alpha,\beta)}(h)\mathcal{F}f(n) \quad \forall h \in \mathbb{T}.$$

Proof. See [12] (Remark 4).

For every $f \in \mathbb{L}_2^{(\alpha,\beta)}$, we define the differences $\Delta_h^m f$ of order m, $m = 1, 2, \ldots,$ with step $h > 0$, $0 < h < \pi/2$, by

$$\Delta_h^1 f(\theta) = \Delta_h f(\theta) := \mathcal{T}^h f(\theta) + \mathcal{T}^{-h} f(\theta) - 2f(\theta),$$

$$\Delta_h^m f(\theta) = \Delta_h\big(\Delta_h^{m-1} f(\theta)\big) \quad \text{for} \quad m \geq 2.$$

Also, we can write that

$$\Delta_h^m f(\theta) = \Big(\mathcal{T}^h + \mathcal{T}^{-h} - 2I_{\mathbb{L}_2}\Big)^m f(\theta),$$

where $I_{\mathbb{L}_2}$ is the identity operator in $\mathbb{L}_2^{(\alpha,\beta)}$.

The generalized modulus of continuity of a function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ is defined by

$$\omega_m(f,\delta) = \sup_{0<h\leq\delta} \|\Delta_h^m f\|, \quad \delta > 0.$$

Let $W_{2,\psi}^{r,m}(\Lambda_{\alpha,\beta})$, $r = 0, 1, \ldots,$ denote the class of functions $f \in \mathbb{L}_2^{(\alpha,\beta)}$ that have generalized derivatives satisfying the estimate

$$\omega_m\big(\Lambda_{\alpha,\beta}^r f, \delta\big) = O(\psi(\delta^m)), \quad \delta \to 0,$$

where $\psi(.)$ is any nonnegative function given on $[0, +\infty)$, $\psi(0) = 0$ and

$$\Lambda_{\alpha,\beta}^0 f = f,$$

$$\Lambda_{\alpha,\beta}^r f = \Lambda_{\alpha,\beta},$$

$$\Lambda_{\alpha,\beta}^{r-1} f, \quad r = 1, 2, \ldots,$$

i.e.,

$$W_{2,\psi}^{r,m}(\Lambda_{\alpha,\beta}) = \Big\{f \in \mathbb{L}_2^{(\alpha,\beta)} : \Lambda_{\alpha,\beta}^r f \in \mathbb{L}_2^{(\alpha,\beta)} \quad \text{and} \quad \omega_m\big(\Lambda_{\alpha,\beta}^r f, \delta\big) = O(\psi(\delta^m)), \quad \delta \to 0\Big\}.$$

3. Main results. Taking into account what was said in the previous section, for some classes of functions characterized by the generalized modulus of continuity, we can prove two estimates for the serie

$$E_N(f) = \sqrt{\sum_{|n|\geq N} |\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)}},$$

which are useful in applications. To prove the main results, we shall need some preliminary results.

Lemma 2. *For* $f \in \mathbb{L}_2^{(\alpha,\beta)}$, *we get*

$$\mathcal{F}(\Lambda_{\alpha,\beta} f)(n) = i\lambda_n \mathcal{F}f(n)$$

for all $n \in \mathbb{Z}$.

Proof. Since

$$\lambda_{-n} = -\lambda_n \quad \forall n \in \mathbb{Z},$$

it follows from this and (5) that

$$\Lambda_{\alpha,\beta}\overline{\psi_n^{(\alpha,\beta)}(\theta)} = \Lambda_{\alpha,\beta}\psi_{-n}^{(\alpha,\beta)}(\theta) = i\lambda_{-n}\psi_{-n}^{(\alpha,\beta)}(\theta) = -i\lambda_n\overline{\psi_n^{(\alpha,\beta)}(\theta)}.$$

Therefore, thanks to the formula (6), we conclude that

$$\mathcal{F}(\Lambda_{\alpha,\beta} f)(n) = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Lambda_{\alpha,\beta} f(\theta)\overline{\psi_n^{(\alpha,\beta)}(\theta)}\mathcal{A}_{\alpha,\beta}(\theta)d\theta =$$

$$= -\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta)\Lambda_{\alpha,\beta}(\overline{\psi_n^{(\alpha,\beta)}(\theta)})\mathcal{A}_{\alpha,\beta}(\theta)d\theta =$$

$$= \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i\lambda_n f(\theta)\overline{\psi_n^{(\alpha,\beta)}(\theta)}\mathcal{A}_{\alpha,\beta}(\theta)d\theta = i\lambda_n\mathcal{F}f(n).$$

Lemma 2 is proved.

***Remark* 1.** From Lemma 2, we can see that, for all $f \in W_{2,\psi}^{r,m}(\Lambda_{\alpha,\beta})$,

$$\mathcal{F}\big(\Lambda_{\alpha,\beta}^r f\big)(n) = (i\lambda_n)^r\mathcal{F}f(n) \quad \forall n \in \mathbb{Z}$$

for all $r = 0, 1, 2, \ldots, m$.

Lemma 3. *Let* $\theta \in \mathbb{T}$. *If* $f \in \mathbb{L}_2^{(\alpha,\beta)}$ *with*

$$f(\theta) = \sum_{n=-\infty}^{+\infty} \mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)w_n^{(\alpha,\beta)},$$

then

$$\mathcal{T}^h f(\theta) = \sum_{n=-\infty}^{+\infty} \mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(h)\psi_n^{(\alpha,\beta)}(\theta)w_n^{(\alpha,\beta)}.$$

Proof. By product formula (11) of $\mathcal{T}^h$, we have

$$\mathcal{T}^h\psi_n^{(\alpha,\beta)}(\theta) = \psi_n^{(\alpha,\beta)}(h)\psi_n^{(\alpha,\beta)}(\theta).$$

Thus, for any polynomial

$$\mathcal{Q}_N(\theta) = \sum_{n=-N}^{N} \mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)w_n^{(\alpha,\beta)},$$

since $\mathcal{T}^h$ is linear, we obtain

$$\mathcal{T}^h \mathcal{Q}_N(\theta) = \sum_{n=-N}^{N} \mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(h)\psi_n^{(\alpha,\beta)}(\theta)w_n^{(\alpha,\beta)}. \tag{12}$$

By using the fact that $\mathcal{T}^h$ is a linear bounded operator in $\mathbb{L}_2^{(\alpha,\beta)}$ and the set of all polynomials $\mathcal{Q}_N(\theta)$ is everywhere dense in $\mathbb{L}_2^{(\alpha,\beta)}$, passage to the limit in (12) gives the desired equality.

Lemma 3 is proved.

Lemma 4. *Let $f \in W_{2,\psi}^{r,m}(\Lambda_{\alpha,\beta})$ and $0 < h < \pi/2$. Then, for all $n \in \mathbb{Z}$, we have*

$$\left\|\Delta_h^m\left(\Lambda_{\alpha,\beta}^r f\right)\right\|^2 = 2^{2m} \sum_{n=-\infty}^{+\infty} \lambda_n^{2r}\left|1 - \varphi_{|n|}^{(\alpha,\beta)}(h)\right|^{2m} |\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)},$$

where $m = 0, 1, 2, \ldots$ and $r = 0, 1, 2, \ldots, m$.

Proof. Take into account the result of Lemma 3, we get

$$\Delta_h f(\theta) = \mathcal{T}^h f(\theta) + \mathcal{T}^{-h} f(\theta) - 2f(\theta) =$$

$$= \sum_{n=-\infty}^{+\infty} \left(\psi_n^{(\alpha,\beta)}(h) + \psi_n^{(\alpha,\beta)}(-h) - 2\right)\mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)w_n^{(\alpha,\beta)}.$$

Since (see (8))

$$\psi_n^{(\alpha,\beta)}(h) = \varphi_{|n|}^{(\alpha,\beta)}(h) + i\frac{\lambda_n}{4(\alpha+1)} \sin(2h)\varphi_{|n|-1}^{(\alpha+1,\beta+1)}(h),$$

$$\psi_n^{(\alpha,\beta)}(-h) = \varphi_{|n|}^{(\alpha,\beta)}(-h) - i\frac{\lambda_n}{4(\alpha+1)} \sin(2h)\varphi_{|n|-1}^{(\alpha+1,\beta+1)}(-h),$$

by formula (2), we have

$$\Delta_h f(\theta) = 2 \sum_{n=-\infty}^{+\infty} \left(\varphi_{|n|}^{(\alpha,\beta)}(h) - 1\right)\mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)w_n^{(\alpha,\beta)}.$$

Using the proof of recurrence for m, we obtain

$$\Delta_h^m f(\theta) = 2^m \sum_{n=-\infty}^{+\infty} \left(\varphi_{|n|}^{(\alpha,\beta)}(h) - 1\right)^m \mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)w_n^{(\alpha,\beta)}.$$

Remark 1 gives

$$\Delta_h^m(\Lambda_{\alpha,\beta}^r f)(\theta) = i^r 2^m \sum_{n=-\infty}^{+\infty} \lambda_n^r\left(\varphi_{|n|}^{(\alpha,\beta)}(h) - 1\right)^m \mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)w_n^{(\alpha,\beta)}.$$

By appealing the Plancherel formula (10), we get

$$\left\|\Delta_h^m\left(\Lambda_{\alpha,\beta}^r f\right)\right\|^2 = 2^{2m} \sum_{n=-\infty}^{+\infty} \lambda_n^{2r}\left|1 - \varphi_{|n|}^{(\alpha,\beta)}(h)\right|^{2m} |\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)}.$$

Lemma 4 is proved.

Theorem 3. *For functions* $f \in \mathbb{L}_2^{(\alpha,\beta)}$ *in the class* $W_{2,\psi}^{r,m}(\Lambda_{\alpha,\beta})$, *there exists a fixed constant* $c > 0$ *such that, for all* $N > 0$, *we have*

$$E_N(f) = O(\lambda_N^{-r}\psi[(c/N)^m])$$

as $N \to \infty$, *where* $r = 0, 1, 2, \ldots,$ $m = 1, 2, \ldots,$ *and* $\psi(.)$ *is any nonnegative function given on* $[0, +\infty)$.

Proof. Let $f \in W_{2,\psi}^{r,m}(\Lambda_{\alpha,\beta})$, by the Hölder inequality for sums, we obtain

$$E_N^2(f) - \sum_{|n|\ge N} \varphi_{|n|}^{(\alpha,\beta)}(h)|\mathcal{F}f(n)|^2 w_n = \sum_{|n|\ge N} \Big(1 - \varphi_{|n|}^{(\alpha,\beta)}(h)\Big)|\mathcal{F}f(n)|^2 w_n =$$

$$= \sum_{|n|\ge N} \left(|\mathcal{F}f(n)|^{2-\frac{1}{m}} w_n^{1-\frac{1}{2m}}\right)\left(\Big(1 - \varphi_{|n|}^{(\alpha,\beta)}(h)\Big)|\mathcal{F}f(n)|^{\frac{1}{m}} w_n^{\frac{1}{2m}}\right) \le$$

$$\le \left(\sum_{|n|\ge N} |\mathcal{F}f(n)|^2 w_n\right)^{\frac{2m-1}{2m}} \left(\sum_{|n|\ge N} \Big(1 - \varphi_{|n|}^{(\alpha,\beta)}(h)\Big)^{2m} |\mathcal{F}f(n)|^2 w_n\right)^{\frac{1}{2m}} =$$

$$= (E_N(f))^{\frac{2m-1}{m}} \left(\sum_{|n|\ge N} \Big(1 - \varphi_{|n|}^{(\alpha,\beta)}(h)\Big)^{2m} |\mathcal{F}f(n)|^2 w_n\right)^{\frac{1}{2m}}.$$

Since

$$\lambda_n^2 \ge \lambda_N^2 \quad \text{for all} \quad |n| \ge N,$$

we conclude that

$$E_N^2(f) - \sum_{|n|\ge N} \varphi_{|n|}^{(\alpha,\beta)}(h)|\mathcal{F}f(n)|^2 w_n \le$$

$$\le (E_N(f))^{\frac{2m-1}{m}} \left(\lambda_N^{-2r} \sum_{|n|\ge N} \lambda_n^{2r} \Big(1 - \varphi_{|n|}^{(\alpha,\beta)}(h)\Big)^{2m} |\mathcal{F}f(n)|^2 w_n\right)^{\frac{1}{2m}} \le$$

$$\le (E_N(f))^{\frac{2m-1}{m}} \left(\lambda_N^{-2r} 2^{2m} \sum_{|n|\ge N} \lambda_n^{2r} (1 - \varphi_{|n|}^{(\alpha,\beta)}(h))^{2m} |\mathcal{F}f(n)|^2 w_n\right)^{\frac{1}{2m}}.$$

From Lemma 4, we have

$$2^{2m} \sum_{|n|\ge N} \lambda_n^{2r} \Big(1 - \varphi_{|n|}^{(\alpha,\beta)}(h)\Big)^{2m} |\mathcal{F}f(n)|^2 w_n \le \big\|\Delta_h^m\big(\Lambda_{\alpha,\beta}^r f\big)\big\|^2.$$

Thus,

$$E_N^2(f) \le \sum_{|n|\ge N} \varphi_{|n|}^{(\alpha,\beta)}(h)|\mathcal{F}f(n)|^2 w_n + (E_N(f))^{\frac{2m-1}{m}} \lambda_N^{-\frac{r}{m}} \big\|\Delta_h^m\big(\Lambda_{\alpha,\beta}^r f\big)\big\|^{\frac{1}{m}}. \tag{13}$$

From (4), we get

$$\sum_{|n|\geq N}\varphi_{|n|}^{(\alpha,\beta)}(h)|\mathcal{F}f(n)|^2 w_n \leq c_2(Nh)^{-\alpha-1/2}E_N^2(f).$$

For $f\in W_{2,\psi}^{r,m}(\Lambda_{\alpha,\beta})$, there exists a constant $C>0$ such that

$$\left\|\Delta_h^m\big(\Lambda_{\alpha,\beta}^r f\big)\right\|\leq C\psi(h^m).$$

Choose a constant c_3 such that the number $c_4=1-c_2c_3^{-\alpha-1/2}$ is positive. Setting $h=c_3/N$ in the inequality (13), we have

$$c_4E_N^2(f)\leq (E_N(f))^{\frac{2m-1}{m}}\lambda_N^{-\frac{r}{m}}C^{\frac{1}{m}}(\psi[(c_3/N)^m])^{\frac{1}{m}}.$$

By raising both sides to the power m and simplifying by $(E_N(f))^{2m-1}$, we finally obtain

$$c_4^m E_N(f)\leq C\lambda_N^{-r}\psi[(c_3/N)^m]$$

for all $N>0$.

Hence, the theorem is proved with $c=c_3$.

Theorem 4. *Let $\phi(t)=t^\nu$. Then*

$$f\in W_{2,\psi}^{r,m}(\Lambda_{\alpha,\beta})$$

is equivalent to

$$E_N(f)=O\big(N^{-r-m\nu}\big),$$

where $r=0,1,2,\ldots,$ $m=1,2,\ldots,$ and $0<\nu<2$.

Proof. Assume that $f\in W_{2,\psi}^{r,m}(\Lambda_{\alpha,\beta})$, by using the fact that

$$\lambda_N=2\sqrt{N(N+\rho)}\geq 2N.$$

Then, from this and according to the Theorem 3, we conclude that

$$E_N(f)=O\big(N^{-r-m\nu}\big).$$

This shows us this implication.

We prove necessity. Let

$$E_N(f)=O\big(N^{-r-m\nu}\big),$$

i.e.,

$$\sum_{|n|\geq N}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)}=O\big(N^{-2r-2m\nu}\big). \tag{14}$$

It is easy to show that there exists a function $f\in\mathbb{L}_2^{(\alpha,\beta)}$ such that $\Lambda_{\alpha,\beta}^r f\in\mathbb{L}_2^{(\alpha,\beta)}$ and

$$\Lambda_{\alpha,\beta}^r f(\theta)=i^r\sum_{n=-\infty}^{+\infty}\lambda_n^r\mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)w_n^{(\alpha,\beta)}.$$

From the formula above and Plancherel identity (10), we have

$$\left\|\Delta_h^m\left(\Lambda_{\alpha,\beta}^r f\right)\right\|^2 = 2^{2m}\sum_{n=-\infty}^{+\infty}\lambda_n^{2r}\left|1-\varphi_{|n|}^{(\alpha,\beta)}(h)\right|^{2m}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)}.$$

This sum is divided into two

$$\|\Delta_h^m(\Lambda_{\alpha,\beta}^r f)\|^2 = \mathcal{I}_1 + \mathcal{I}_2,$$

where

$$\mathcal{I}_1 = \sum_{|n|<N} 2^{2m}\lambda_n^{2r}|1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2m}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)}$$

and

$$\mathcal{I}_2 = \sum_{|n|\geq N} 2^{2m}\lambda_n^{2r}|1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2m}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)}$$

with $N = \left[h^{-1}\right]$ is the integer part of h^{-1}.

Let us now estimate each of them, we estimate $\mathcal{I}_2$, it follows from (1) that

$$\mathcal{I}_2 \leq 2^{4m}\sum_{|n|\geq N}\lambda_n^{2r}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)}.$$

Note that

$$\lambda_n^2 = 4n^2\left(1+\frac{\rho}{|n|}\right) \leq 4n^2(1+\rho) \quad \text{for all} \quad |n|\geq 1, \quad n\in\mathbb{Z}. \tag{15}$$

It follows from this that

$$\mathcal{I}_2 \leq 2^{4m}(4\rho+4)^r\sum_{|n|\geq N} n^{2r}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)} =$$

$$= c_5\sum_{j=0}^{+\infty}\sum_{N+j\leq|n|\leq N+j+1} n^{2r}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)} \leq$$

$$\leq c_5\sum_{j=0}^{+\infty}(N+j+1)^{2r}\sum_{N+j\leq|n|\leq N+j+1}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)} =$$

$$= c_5\sum_{j=0}^{+\infty}a_j(\mathcal{V}_j-\mathcal{V}_{j+1}),$$

where $a_j = (N+j+1)^{2r}$ and $\mathcal{V}_j = \sum_{|n|\geq N+j}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)}$. Furthermore, for all integers $M\geq 1$, the summation by parts gives

$$\sum_{j=0}^{M}a_j(\mathcal{V}_j-\mathcal{V}_{j+1}) = a_0\mathcal{V}_0 - a_M\mathcal{V}_{M+1} + \sum_{j=1}^{M}\mathcal{V}_j(a_j-a_{j-1}) \leq$$

$$\leq a_0\mathcal{V}_0 + \sum_{j=1}^{M}\mathcal{V}_j(a_j-a_{j-1}).$$

Moreover, by the finite increments theorem, we have

$$a_j - a_{j-1} \leq 2r(N+j+1)^{2r-1}.$$

On the other hand, by (14), there exists $c_6 > 0$ such that, for all $N > 0$,

$$E_N^2(f) \leq c_6 N^{-2r-2m\nu}.$$

For $N \geq 1$, we obtain

$$\sum_{j=0}^{M} a_j(\mathcal{V}_j - \mathcal{V}_{j+1}) \leq a_0\mathcal{V}_0 + \sum_{j=1}^{M}\mathcal{V}_j(a_j - a_{j-1}) \leq$$

$$\leq c_6\left(1+\frac{1}{N}\right)^{2r} N^{-2m\nu} + 2rc_6\sum_{j=1}^{M}\left(1+\frac{1}{N+j}\right)^{2r-1}(N+j)^{-1-2m\nu} \leq$$

$$\leq c_6 2^{2r} N^{-2m\nu} + 2^{2r} r c_6 \sum_{j=1}^{M}(N+j)^{-1-2m\nu}.$$

Finally, by the integral comparison test, we get

$$\sum_{j=1}^{M}(N+j)^{-1-2m\nu} \leq \sum_{\mu=N+1}^{+\infty}\mu^{-1-2m\nu} \leq \int_{N}^{+\infty} t^{-1-2m\nu}dt = \frac{1}{2m\nu}N^{-2m\nu}.$$

Letting $M \to +\infty$, we see that, for $r \geq 0$ and $m, \nu > 0$, there exists a constant c_7 such that, for all $N \geq 1$,

$$\mathcal{I}_2 \leq c_7 N^{-2m\nu}.$$

Consequently, for all $h > 0$, we have

$$\mathcal{I}_2 \leq c_7 h^{2m\nu}. \tag{16}$$

Now, we estimate $\mathcal{I}_1$. From formulae (3) and (15), we obtain

$$\mathcal{I}_1 \leq 2^{2m} c_1^{2m} h^{4m} \sum_{|n|<N} \lambda_n^{2r+4m} |\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)} \leq$$

$$\leq c_8 h^{4m} \sum_{|n|<N} n^{2r+4m}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)} \leq$$

$$\leq c_8 h^{4m} \sum_{j=0}^{N-1} \sum_{j\leq|n|\leq j+1} n^{2r+4m}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)} \leq$$

$$\leq c_8 h^{4m} \sum_{j=0}^{N-1}(j+1)^{2r+4m} \sum_{j\leq|n|\leq j+1}|\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)} =$$

$$= c_8 h^{4m} \sum_{j=0}^{N-1} a_j (\mathcal{V}_j - \mathcal{V}_{j+1}),$$

where $a_j = (j+1)^{2r+4m}$ and $\mathcal{V}_j = \sum_{|n|\geq j} |\mathcal{F}f(n)|^2 w_n^{(\alpha,\beta)}$.

Using a summation by parts and proceeding as with $\mathcal{I}_2$ and the fact that $\mathcal{V}_j \leq c_6 j^{-2r-2m\nu}$ by hypothesis, we get

$$\mathcal{I}_1 \leq c_8 h^{4m} \sum_{j=0}^{N-1} a_j (\mathcal{V}_j - \mathcal{V}_{j+1}) \leq c_8 h^{4m} \left(a_0 \mathcal{V}_0 + \sum_{j=1}^{N-1} \mathcal{V}_j (a_j - a_{j-1}) \right) \leq$$

$$\leq c_8 h^{4m} \left(\mathcal{V}_0 + c_6 (2r+4m) \sum_{j=1}^{N-1} (j+1)^{2r+4m-1} j^{-2r-2m\nu} \right).$$

From the inequality $j+1 \leq 2j$, we conclude that

$$\mathcal{I}_1 \leq c_8 h^{4m} \left(\mathcal{V}_0 + c_9 \sum_{j=1}^{N-1} j^{4m-2m\nu-1} \right).$$

As a consequence of a series comparison, we have the inequality

$$\mu \sum_{j=1}^{N-1} j^{\mu-1} \leq N^{\mu} \quad \text{for} \quad \mu > 0 \quad \text{and} \quad N \geq 2.$$

If $\mu = 4m - 2m\nu > 0$ for $\nu < 2$, then we obtain

$$\mathcal{I}_1 \leq c_8 h^{4m} \big(\mathcal{V}_0 + c_{10} N^{4m-2m\nu} \big) \leq c_8 h^{4m} \big(\mathcal{V}_0 + c_{10} h^{2m\nu-4m} \big),$$

since $N \leq 1/h$.

If h is sufficiently small, then $\mathcal{V}_0 \leq c_{10} h^{2m\nu-4m}$. Then we have

$$\mathcal{I}_1 \leq c_{11} h^{2m\nu}. \tag{17}$$

Combining the estimates (16) and (17) for $\mathcal{I}_1$ and $\mathcal{I}_2$ gives

$$\big\| \Delta_h^m \big(\Lambda_{\alpha,\beta}^r f \big) \big\| = O(h^{m\nu}).$$

Consequently,

$$\omega_m \big(\Lambda_{\alpha,\beta}^r f, \delta \big) = O(\delta^{m\nu}) = O(\psi(\delta^m)).$$

Therefore, the necessity is proved and the proof of the theorem is completed.

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