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N. V. Krylov (Univ. Minnesota, Minneapolis, MN, USA)

ON TIME INHOMOGENEOUS STOCHASTIC ITÔ EQUATIONS WITH DRIFT IN L_{d+1}

ПРО НЕОДНОРІДНІ ЗА ЧАСОМ СТОХАСТИЧНІ РІВНЯННЯ ІТО З ПЕРЕНОСОМ В L_{d+1}

We prove the solvability of Itô stochastic equations with uniformly nondegenerate bounded measurable diffusion and drift in $L_{d+1}(\mathbb{R}^{d+1})$. Actually, the powers of summability of the drift in x and t could be different. Our results seem to be new even if the diffusion is constant. The method of proving the solvability belongs to A. V. Skorokhod. Weak uniqueness of solutions is an open problem even if the diffusion is constant.

Доведено розв'язність стохастичних рівнянь Іто з рівномірно невиродженою та обмеженою матрицею дифузії і з переносом в $L_{d+1}(\mathbb{R}^{d+1})$. Справді, показники інтегровності по x і t можуть відрізнятися. Цей результат є новим навіть коли дифузія стала. Метод, який ми використовуємо, належить А. В. Скороходу. Питання про слабку єдиність є відкритим навіть коли дифузія стала.

1. Introduction. Let $\mathbb{R}^d$ be a Euclidean space of points $x = (x^1, \ldots, x^d)$, $d \geq 2$. We fix some $p, q \in [1, \infty]$ such that

$$\frac{d}{p} + \frac{1}{q} \leq 1 \tag{1.1}$$

with further restrictions on them to be specified later. The goal of this article is to study the solvability of Itô's stochastic equations of the form

$$x_t = x^{(0)} + \int_0^t \sigma\left(t^{(0)} + s, x_s\right) dw_s + \int_0^t b\left(t^{(0)} + s, x_s\right) ds, \tag{1.2}$$

where w_t is a d-dimensional Wiener process, σ is a uniformly nondegenerate, bounded, Borel function with values in the set of symmetric $(d \times d)$-matrices, b is a Borel measurable $\mathbb{R}^d$- valued function given on $(-\infty, \infty) \times \mathbb{R}^d$ such that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |b(t,x)|^p \, dx \right)^{q/p} dt < \infty \tag{1.3}$$

if $p \geq q$ or

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} |b(t,x)|^q \, dt \right)^{p/q} dx < \infty$$

if $p \leq q$. If $p = \infty$ or $q = \infty$ we interpret this conditions in a natural way. Observe that the case $p = q = d + 1$ is not excluded and in this case the condition becomes $b \in L_{d+1}(\mathbb{R}^{d+1})$. Under this condition the solvability of (1.2) was proved in [17].

We are talking, of course, about weak solutions and prove their existence in Theorem 3.1. In Theorem 6.1 we prove the existence of strong Markov processes corresponding to diffusion σ and drift b with the above properties. If b is bounded, as we know from [16], there exist strong Markov and strong Feller processes with diffusion σ and drift b for which the Harnack inequality holds and the caloric functions are Hölder continuous. We are far from proving such fine properties.

The main technical tools are collected in Section 4 where we prove new mixed norms estimates for the distributions of semimartingales. The treatment there, actually, follows very closely the work by A. I. Nazarov [11] written in terms of PDEs.

There is a vast literature about stochastic equations with irregular drift. Probably one of the first authors starting this area was N. I. Portenko, see his book [12], where he constructed diffusion processes with sufficiently regular σ and $b \in L_p(\mathbb{R}^{d+1})$, $p > d+2$. This condition on b was later refined in many articles with various ambitious goals in them to the requirement that b be such that (1.3) holds not under condition (1.1) but rather

$$\frac{d}{p} + \frac{2}{q} \leq 1. \tag{1.4}$$

We refer the reader to the recent articles [2, 10, 15] and the references therein for the discussion of many powerful results obtained under condition (1.4), when the case of equality is treated as "critical". It could be critical in some respects but not for obtaining our results, that seem to be the first ones about the existence of solutions and Markov processes with our condition on the drift. Still it is worth emphasizing that our condition is different if $p \geq q$ (and, hence, $p \geq d+1$) or $p < q$, whereas there is no such distinction attached to (1.4).

We assume that $d \geq 2$ and denote

$$B_R = \{x \in \mathbb{R}^d : |x| < R\}, \qquad D_i = \frac{\partial}{\partial x^i}, \qquad D_{ij} = D_i D_j, \qquad \partial_t = \frac{\partial}{\partial t}.$$

For $p, q \in [1, \infty]$, we introduce the space $L_{p,q}$ as the space of Borel functions on $\mathbb{R}^{d+1}$ such that

$$\|f\|_{p,q}^q := \int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |f(t,x)|^p \, dx \right)^{q/p} dt < \infty$$

if $p \geq q$ or

$$\|f\|_{p,q}^p := \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} |f(t,x)|^q \, dt \right)^{p/q} dx < \infty$$

if $p \leq q$ with natural interpretation of these definitions if $p = \infty$ or $q = \infty$. To better memorize these formulas observe that p is associated with integration with respect to x, q with that with respect to t and the interior integral is always elevated to the power ≤ 1. In case $p = q = d+1$ we abbreviate $L_{d+1,d+1} = L_{d+1}$, $\|\cdot\|_{d+1,d+1} = \|\cdot\|_{d+1}$.

2. An example of nonexistence.

***Example* 2.1.** Suppose that numbers α and β satisfy

$$0 < \alpha \leq \beta < 1, \quad \alpha + \beta = 1, \tag{2.1}$$

and set

$$b(t,x) = -\frac{1}{t^\alpha |x|^\beta}\frac{x}{|x|} I_{0<|x|\le 1, t\le 1}.$$

Observe that if $d/p + 1/q = 1+\varepsilon,\ \varepsilon > 0$, one can take $\beta = d/(p+p\varepsilon),\ \alpha = 1/(q+q\varepsilon)$ and then

$$\int_0^1 \left(\int_{|x|\le 1} |b(t,x)|^p\,dx \right)^{q/p} dt < \infty, \qquad \int_{|x|\le 1} \left(\int_0^1 |b(t,x)|^q\,dt \right)^{p/q} dx < \infty.$$

Also note that if $p \le qd$ (say $p = q$), condition (2.1) is satisfied.

However, it turns out that no matter which $\alpha,\ \beta$ we take satisfying (2.1) there is no solutions of the equation $dx_t = dw_t + b(t,x_t)\,dt$ starting at zero, where w_t is a d-dimensional Wiener process.

To prove this assume the contrary. Namely, assume the there is a stopping time τ such that $P(\tau > 0) > 0$ and for $t \le \tau$ there is x_t such that

$$x_t = w_t + \int_0^t b(s,x_s)\,ds.$$

We may assume that $\tau \le 1$ and before τ the process is in B_1. Then, for $t \le \tau$,

$$\begin{aligned} dx_t &= -\frac{1}{t^\alpha |x_t|^\beta}\frac{x_t}{|x_t|} I_{x_t\ne 0}\,dt + dw_t, \\ d|x_t|^2 &= -2\frac{|x_t|}{t^\alpha |x_t|^\beta}\,dt + d\,dt + 2x_t\,dw_t. \end{aligned} \tag{2.2}$$

We will be interested in $|x_t|^{1+\beta} = \xi_t^{(1+\beta)/2}$, where $\xi_t = |x_t|^2$. By Itô's formula for any $\varepsilon > 0$ we have

$$\begin{aligned} d(\xi_t+\varepsilon)^{(1+\beta)/2} &= \frac{1+\beta}{2}(\xi_t+\varepsilon)^{(\beta-1)/2}\,d\xi_t + \frac{\beta^2-1}{8}(\xi_t+\varepsilon)^{(\beta-3)/2}4|x_t|^2\,dt = \\ &= I_t(\varepsilon)\,dt + J_t(\varepsilon)\,dt + (1+\beta)(\xi_t+\varepsilon)^{(\beta-1)/2}x_t\,dw_t, \end{aligned} \tag{2.3}$$

where

$$I_t(\varepsilon) = -(1+\beta)(\xi_t+\varepsilon)^{(\beta-1)/2}\frac{|x_t|^\alpha}{t^\alpha},$$

$$J_t(\varepsilon) = \frac{1+\beta}{2}\left[d + (\beta-1)(\xi_t+\varepsilon)^{-1}|x_t|^2\right](\xi_t+\varepsilon)^{(\beta-1)/2}.$$

Since $(\xi_t+\varepsilon)^{-\alpha/2}|x_t|^\alpha \uparrow I_{x_t\ne 0}$ as $\varepsilon \downarrow 0$, by the dominated convergence theorem

$$\int_0^t I_s(\varepsilon)\,ds \to -(1+\beta)\int_0^t I_{x_s\ne 0}\frac{1}{s^\alpha}\,ds,$$

which is finite.

Furthermore, since $|x_s|^{\beta-1}x_s$ is bounded on each trajectory, by the dominated convergence theorem

$$\int\limits_0^t \Big|(\xi_s+\varepsilon)^{(\beta-1)/2}x_s - |x_s|^{\beta-1}x_s\Big|^2\,ds \to 0,$$

and we conclude from (2.3) that for $t \le \tau$

$$|x_t|^{1+\beta} = -(1+\beta)\int\limits_0^t I_{x_s\neq 0}\frac{1}{s^\alpha}\,ds + \lim_{\varepsilon\downarrow 0}\int\limits_0^t J_s(\varepsilon)\,ds + (1+\beta)\int\limits_0^t |x_s|^{\beta-1}x_s I_{x_s\neq 0}\,dw_s \tag{2.4}$$

and the above limit exists and is finite. Since $2J_s(\varepsilon) \ge (\xi_s+\varepsilon)^{(\beta-1)/2}$, it follows that

$$\int\limits_0^t |x_s|^{\beta-1}\,ds = \lim_{\varepsilon\downarrow 0}\int\limits_0^t (\xi_s+\varepsilon)^{(\beta-1)/2}\,ds$$

and the left-hand side is finite. In particular,

$$\int\limits_0^\tau I_{x_s=0}\,ds = 0. \tag{2.5}$$

Now by the dominated convergence theorem (2.4) implies that

$$|x_t|^{1+\beta} = -(1+\beta)\int\limits_0^t \frac{1}{s^\alpha}\,ds +$$

$$+\frac{1}{2}(1+\beta)\int\limits_0^t (d+\beta-1)|x_s|^{\beta-1}\,ds + (1+\beta)\int\limits_0^t |x_s|^{\beta-1}x_s\,dw_s.$$

Next, use $\alpha \le \beta$ and Hölder's inequality to conclude that

$$\int\limits_0^t |x_s|^{-\alpha}\,ds = \int\limits_0^t \left(\frac{1}{s^\alpha|x_s|^\beta}\right)^{\alpha/\beta} s^{\alpha^2/\beta}\,ds \le \left(\int\limits_0^t \frac{1}{s^\alpha|x_s|^\beta}\,ds\right)^{\alpha/\beta}\left(\int\limits_0^t s^{\alpha^2/(\beta-\alpha)}\,ds\right)^{(\beta-\alpha)/\beta}.$$

Since $\alpha^2/(\beta-\alpha)+1 = (\alpha^2+1-2\alpha)/(\beta-\alpha) = \beta^2/(\beta-\alpha)$,

$$\int\limits_0^t |x_s|^{-\alpha}\,ds \le N\left(\int\limits_0^t \frac{1}{s^\alpha|x_s|^\beta}\,ds\right)^{\alpha/\beta} t^\beta,$$

where $N = N(\alpha,\beta)$ (which is trivial if $\alpha=\beta$). Thus,

$$|x_t|^{1+\beta} + ct^\beta \le N_1\left(\int\limits_0^t \frac{1}{s^\alpha|x_s|^\beta}\,ds\right)^{\alpha/\beta} t^\beta + (1+\beta)\int\limits_0^t |x_s|^{\beta-1}x_s\,dw_s,$$

where $c>0$ is a constant. For equation (2.2) to make sense we should have

$$\int_0^\tau \frac{1}{s^\alpha |x_s|^\beta}\,ds < \infty \quad \text{(a. s.)}.$$

Therefore,

$$\gamma := \tau \wedge \inf\left\{t \geq 0 \colon N_1 \left(\int_0^t \frac{1}{s^\alpha |x_s|^\beta}\,ds\right)^{\alpha/\beta} \geq c/2\right\},$$

is a stopping time such that $P(\gamma > 0) = P(\tau > 0)$. It follows that for any $t > 0$

$$\int_0^t I_{s<\gamma}|x_s|^{\beta-1}x_s\,dw_s \geq 0,$$

which is only possible if $I_{s<\gamma}|x_s|^{\beta-1}x_s = 0$ for almost all (ω, s). Then $x_s = 0$ for $s < \gamma$ and (2.5) is only possible if $P(\tau = 0) = 1$.

3. An existence theorem. In this section, we state a result saying that in a wide class of cases there exists a probability space and a Wiener process on this space such that a stochastic equation having measurable coefficients as well as this Wiener process is solvable. In other words, according to conventional terminology, we are talking here about "weak" solutions of a stochastic equation. The main difference between "weak" solutions and usual ("strong") solutions consists in the fact that the latter can be constructed on any a priori given probability space on the basis of any given Wiener process.

Let $\sigma(t,x)$ be Borel $d \times d$ symmetric matrix valued, $b(t,x)$ be Borel $\mathbb{R}^d$-valued functions given on $\mathbb{R}^{d+1} := (-\infty,\infty) \times \mathbb{R}^d$. We assume that the eigenvalues of $\sigma(t,x)$ are between δ and δ^{-1}, where $\delta \in (0,1)$ is a fixed number. The set of such matrices we denote by $\mathbb{S}_\delta$.

Next, fix numbers $p, q \in (1,\infty)$, $\|b\| \in (0,\infty)$ and let $b^n(t,x)$, $n = 1,2,\ldots$, be $\mathbb{R}^d$-valued Borel functions on $\mathbb{R}^{d+1}_+$ and suppose that

$$\|b\|_{p,q}, \|b^n\|_{p,q} \leq \|b\|, \quad n = 1,2,\ldots, \qquad \frac{d}{p} + \frac{1}{q} = 1,$$

and $b^n \to b$ as $n \to \infty$ in $L_{p,q}$. Let $\sigma^n(t,x)$, $n = 1,2,\ldots$, be Borel functions on $\mathbb{R}^d$ with values in $\mathbb{S}_\delta$ such that $\sigma^n \to \sigma$ as $n \to \infty$ ($\mathbb{R}^{d+1}$-a.e.).

Theorem 3.1. *Take* $(t^0, x^0) \in \mathbb{R}^{d+1}$. (i) *There exists a probability space* $(\Omega, \mathcal{F}, P)$, *a filtration of* σ*-fields* $\mathcal{F}_t \subset \mathcal{F}$, $t \geq 0$, *a process* w_t, $t \geq 0$, *which is a* d*-dimensional Wiener process relative to* $\{\mathcal{F}_t\}$, *and an* $\mathcal{F}_t$*-adapted process* x_t *such that* (*a.s.*) *for all* $t \geq 0$ *equation* (1.2) *holds.*

(ii) *Furthermore, let* $(t^n, x^n) \in \mathbb{R}^{d+1}$, $n = 1,2,\ldots$, *and let* $(t^n, x^n) \to (t^0, x^0)$ *as* $n \to \infty$. *Assume that for each* $n = 1,2,\ldots$ *there exists a probability space* $(\Omega^n, \mathcal{F}^n, P^n)$, *a filtration of* σ*-fields* $\mathcal{F}^n_t \subset \mathcal{F}^n$, $t \geq 0$, *a process* w^n_t, $t \geq 0$, *which is a* d*-dimensional Wiener process relative to* $\{\mathcal{F}^n_t\}$, *and an* $\mathcal{F}^n_t$*-adapted process* x^n_t *such that (a.s.) for all* $t \geq 0$

$$x^n_t = x^n + \int_0^t \sigma^n(t^n + s, x^n_s)\,dw^n_s + \int_0^t b^n(t^n + s, x^n_s)\,ds.$$

Then the finite dimensional distributions of a subsequence of $x^n_\cdot$ *converge weakly to the corresponding distributions of one of the solutions of* (1.2) *described in* (i). *Moreover, if* $p \geq q$, *the set of distributions of* $x^n_\cdot$ *on* $C([0,\infty), \mathbb{R}^d)$ *is tight.*

The proof of this theorem, following a similar proof by A. V. Skorokhod, is given in Section 5, after we make a crucial step in the next section where we prove, in particular, that for solutions of (1.2), any Borel $f \geq 0$, and $T \in (0, \infty)$

$$E\int_0^T f(t, x_t)\,dt \leq N\|f\|_{p,q},$$

where N is independent of f and (t^0, x^0).

It is worth saying that deciding whether the solutions of (1.2) are weakly unique or not under our conditions is a very challenging open problem even if $\sigma^{ij} \equiv \delta^{ij}$.

***Remark* 3.1.** Theorem 3.1 is also true if $d/p + 1/q < 1$. This can be seen from its proof which becomes somewhat more technical in that case because of the form of our main estimate (4.9). Also the main interest in Theorem 3.1 is, of course, the lowest local integrability of b, when the condition $d/p + 1/q = 1$ is weaker than $d/p + 1/q < 1$ due to Hölder's inequality.

4. Estimates of the distributions of semimartingales. Here we first prove a version of Lemma 5.1 of [6]. The proof given in [6] uses somewhat advanced knowledge of very powerful results from the theory of fully nonlinear parabolic equations. We give a proof based on a simpler fact which in turn was one of the cornerstones of that theory.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $\mathcal{F}_t, t \geq 0$, be an increasing family of complete σ-fields $\mathcal{F}_t \subset \mathcal{F}$, $t \geq 0$, let m_t be an $\mathbb{R}^d$-valued continuous local martingale relative to $\mathcal{F}_t$, let A_t be a continuous $\mathcal{F}_t$-adapted nondecreasing process, let B_t be a continuous $\mathbb{R}^d$-valued $\mathcal{F}_t$-adapted process which has finite variation (a.e.) on each finite time interval. Assume that

$$A_0 = 0, \qquad m_0 = B_0 = 0, \qquad d\langle m\rangle_t \ll dA_t$$

and that we are also given progressively measurable relative to $\mathcal{F}_t$ nonnegative processes r_t and c_t. Finally, take an $\mathcal{F}_0$ measurable $\mathbb{R}^d$-valued x_0 and introduce

$$x_t = x_0 + m_t + B_t, \qquad \tau_t = \int_0^t r_s\,dA_s, \qquad \phi_t = \int_0^t c_s\,dA_s, \qquad a_t^{ij} = \frac{1}{2}\frac{d\langle m^i, m^j\rangle_t}{dA_t}.$$

Lemma 4.1. *Let γ be an $\mathcal{F}_t$-stopping time and set*

$$A = E\int_0^\gamma e^{-\phi_t}\mathrm{tr}\,a_s\,dA_t, \qquad B = E\int_0^\gamma e^{-\phi_t}\,|dB_t|.$$

Then, for any Borel $f(t,x) \geq 0$, we have

$$E\int_0^\gamma e^{-\phi_t}(r_t \det a_t)^{1/(d+1)} f(\tau_t, x_t)\,dA_t \leq N(d)(B^2 + A)^{d/(2d+2)}\|f\|_{d+1}. \tag{4.1}$$

Proof. Without losing generality we may assume that $A < \infty$ and $B < \infty$. Furthermore, just stopping the processes A_t, m_t, and B_t at time γ, we reduce the general case to the one in which $\gamma = \infty$. In that case we also observe that, as usual, it suffices to prove (4.1) for $f \in C_0^\infty(\mathbb{R}^{d+1})$.

After these reductions we use Theorem 2.2.4 of [8] according to which, for any $\lambda > 0$ on $\mathbb{R}^{d+1}$, there exists a nonnegative function $v(t,x)$ such that

(i) all Sobolev derivatives $\partial_t v,\ D_i v,\ D_{ij} v$ exist and are bounded on $\mathbb{R}^{d+1}$ and $v \le Ne^{-|x|/N}$ for all $t,\ x$ and a constant N;

(ii) for any nonnegative symmetric $(d\times d)$-matrix α and $r \ge 0$,

$$r\partial_t v + \alpha^{ij} D_{ij} v - \lambda(r + \operatorname{tr}\alpha)v + \sqrt[d+1]{r \det\alpha}\, f \le 0,$$

$$\partial_t v - \lambda v \le 0, \quad (\lambda v\delta_{ij} - D_{ij} v) \ge 0, \quad |Dv| \le \sqrt{\lambda} v \quad \text{(a. e.)}; \tag{4.2}$$

(iii) for any $y \in \mathbb{R}^d,\ t \in (-\infty,\infty)$, we have

$$v(t,y)e^{-\lambda t} \le N(d)\frac{1}{\lambda^{d/(2d+2)}} I_t, \tag{4.3}$$

where

$$I_t^{d+1} := \int_0^\infty ds \int_{\mathbb{R}^d} e^{-\lambda(d+1)(t+s)} f^{d+1}(t+s,x)\,dx.$$

Take a nonnegative $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ with unit integral, for $\varepsilon > 0$ denote

$$\zeta_\varepsilon(t,x) = \varepsilon^{-(d+1)}\zeta(\varepsilon t, \varepsilon x)$$

and use the notation $u^{(\varepsilon)} = u * \zeta_\varepsilon$. Then $v^{(\varepsilon)}$ is infinitely differentiable and in light of (4.2), for any nonnegative symmetric $(d\times d)$-matrix α and $r \ge 0$,

$$\begin{gathered} r\partial_t v^{(\varepsilon)} + \alpha^{ij} D_{ij} v^{(\varepsilon)} - \lambda(r + \operatorname{tr}\alpha)v^{(\varepsilon)} + \sqrt[d+1]{r\det\alpha}\, f^{(\varepsilon)} \le 0, \\ \partial_t v^{(\varepsilon)} - \lambda v^{(\varepsilon)} \le 0, \qquad \big(\lambda v^{(\varepsilon)}\delta_{ij} - D_{ij} v^{(\varepsilon)}\big) \ge 0, \qquad \big|Dv^{(\varepsilon)}\big| \le \sqrt{\lambda} v^{(\varepsilon)}. \end{gathered} \tag{4.4}$$

Next, by Itô's formula the process

$$v^{(\varepsilon)}(\tau_t, x_t)e^{-\phi_t - \lambda\tau_t} - \int_0^t e^{-\phi_s - \lambda\tau_s} D_i v^{(\varepsilon)}(\tau_s, x_s)\,dB_s^i +$$

$$+ \int_0^t e^{-\phi_s - \lambda\tau_s}\Big((\lambda r_s + c_s)v^{(\varepsilon)} - r_s\partial_t v^{(\varepsilon)} - a_s^{ij} D_{ij} v^{(\varepsilon)}\Big)(\tau_s, x_s)\,dA_s$$

is a local martingale. Here owing to (4.4)

$$\Big((\lambda r_s + c_s)v^{(\varepsilon)} - r_s\partial_t v^{(\varepsilon)} - a_s^{ij} D_{ij} v^{(\varepsilon)}\Big)\,dA_s - D_i v^{(\varepsilon)}\,dB_s^i \ge$$

$$\ge (r_s \det a_s)^{1/(d+1)} f^{(\varepsilon)}\,dA_s - \lambda \operatorname{tr} a_s v^{(\varepsilon)}\,dA_s - \sqrt{\lambda} v^{(\varepsilon)}\,|dB_s|.$$

Therefore, for

$$M^\varepsilon = \sup_{t\ge 0, x\in\mathbb{R}^d} v^{(\varepsilon)}(t,x)e^{-\lambda t},$$

the process

$$\kappa_t^\varepsilon := v^{(\varepsilon)}(\tau_t, x_t)e^{-\phi_t-\lambda\tau_t} + \int\limits_0^t e^{-\phi_s-\lambda\tau_s}(r_s \det a_s)^{1/(d+1)} f^{(\varepsilon)}(\tau_s, x_s)\, dA_s -$$

$$-\int\limits_0^t e^{-\phi_s}\Big(\lambda \mathrm{tr}\, a_s\, dA_s + \sqrt{\lambda}\, |dB_s|\Big)\, M^\varepsilon$$

is a local supermartingale. In addition, it is bounded from below by a summable quantity $(A, B < < \infty)$. Hence, it is a supermartingale and by Fatou's lemma

$$Ev^{(\varepsilon)}(0, x_0) = \kappa_0^\varepsilon \geq E\int\limits_0^\infty e^{-\phi_t-\lambda\tau_s}(r_t \det a_t)^{1/(d+1)} f^{(\varepsilon)}(\tau_t, x_t)\, dA_t - M^\varepsilon(\lambda A + \sqrt{\lambda}B).$$

By sending $\varepsilon \downarrow 0$ and using (4.3) and Fatou's lemma once more we obtain

$$E\int\limits_0^\infty e^{-\phi_t-\lambda\tau_s}(r_t \det a_t)^{1/(d+1)} f(\tau_t, x_t)\, dA_t \leq N(d)\frac{1}{\lambda^{d/(2d+2)}}\Big(1 + \lambda A + \sqrt{\lambda}B\Big)\, I_0.$$

We replace here $e^{-\lambda t}f$ by f and arrive at

$$E\int\limits_0^\infty e^{-\phi_t}(r_t \det a_t)^{1/(d+1)} f(\tau_t, x_t)\, dA_t \leq N(d)\frac{1}{\lambda^{d/(2d+2)}}\Big(1 + \lambda A + \sqrt{\lambda}B\Big)\, \|f\|_{d+1}.$$

Now we use the arbitrariness of λ. If $A < B^2$, then for $\lambda = B^{-2}$ we have

$$\frac{1}{\lambda^{d/(2d+2)}}\Big(1 + \lambda A + \sqrt{\lambda}B\Big) \leq 3B^{d/(d+1)} \leq 3(B^2 + A)^{d/(2d+2)}.$$

If $A \geq B^2$ and $A > 0$, then for $\lambda = A^{-1}$ the above inequality between the extreme terms still holds. Finally, if $A = B = 0$, then the left-hand side of (4.1) is zero.

The lemma is proved.

Lemma 4.2. *In the notation of Lemma* 4.1 *for any Borel* $f(x) \geq 0$ *we have*

$$E\int\limits_0^\gamma e^{-\phi_t}(\det a_t)^{1/d} f(x_t)\, dA_t \leq N(d)(B^2 + A)^{1/2}\|f\|_{L_d(\mathbb{R}^d)}. \tag{4.5}$$

Proof. We follow a probabilistic version of an argument in [11]. We again may concentrate on the case of $A + B < \infty$, $\gamma = \infty$, and $f \in C_0^\infty(\mathbb{R}^d)$. In that case observe that by Theorem 2.2.3 of [8] there exists a nonnegative function $v(x)$ defined on $\mathbb{R}^d$ such that

(a) $v \leq Ne^{-|x|/N}$ for all x and a constant N; the generalized derivatives $D_i v$ and $D_{ij}v$, $i, j = 1, \ldots, d$, are bounded on $\mathbb{R}^d$;

(b) for any nonnegative symmetric $(d \times d)$-matrix α (a. e.)

$$-\lambda v\, \mathrm{tr}\ \alpha + \alpha^{ij}D_{ij}v + \sqrt[d]{\det \alpha} f \leq 0, \quad |Dv| \leq \sqrt{\lambda}v,$$
$$(\lambda v\delta_{ij} - D_{ij}v) \geq 0; \tag{4.6}$$

(c) for any $x \in \mathbb{R}^d$

$$v(x) \le N(d)\lambda^{-1/2}\|f\|_{L_d(\mathbb{R}^d)}. \tag{4.7}$$

Then we closely follow the proof of Lemma 4.1. Take a nonnegative $\zeta \in C_0^\infty(\mathbb{R}^d)$ with unit integral, for $\varepsilon > 0$ denote $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta(\varepsilon x)$ and use the notation $u^{(\varepsilon)} = u * \zeta_\varepsilon$. Then $v^{(\varepsilon)}$ is infinitely differentiable and in light of (4.6), for any nonnegative symmetric $(d \times d)$-matrix α,

$$\begin{gathered}\alpha^{ij} D_{ij} v^{(\varepsilon)} - \lambda \operatorname{tr} \alpha v^{(\varepsilon)} + \sqrt[d]{\det \alpha}\, f^{(\varepsilon)} \le 0,\\ \Big(\lambda v^{(\varepsilon)} \delta_{ij} - D_{ij} v^{(\varepsilon)}\Big) \ge 0, \qquad \big|D v^{(\varepsilon)}\big| \le \sqrt{\lambda} v^{(\varepsilon)}.\end{gathered} \tag{4.8}$$

Next, by Itô's formula the process

$$v^{(\varepsilon)}(x_t)e^{-\phi_t} - \int_0^t e^{-\phi_s} D_i v^{(\varepsilon)}(x_s)\, dB_s^i + \int_0^t e^{-\phi_s - \lambda\tau_s}\big(c_s v^{(\varepsilon)} - a_s^{ij} D_{ij} v^{(\varepsilon)}\big)(x_s)\, dA_s$$

is a local martingale. Here owing to (4.8)

$$\Big(c_s v^{(\varepsilon)} - a_s^{ij} D_{ij} v^{(\varepsilon)}\Big)\, dA_s - D_i v^{(\varepsilon)}\, dB_s^i \ge$$

$$\ge (\det a_s)^{1/d} f^{(\varepsilon)}\, dA_s - \lambda \operatorname{tr} a_s v^{(\varepsilon)}\, dA_s - \sqrt{\lambda} v^{(\varepsilon)}\, |dB_s|.$$

Therefore, for

$$M^\varepsilon = \sup_{x \in \mathbb{R}^d} v^{(\varepsilon)}(x)$$

the process

$$\kappa_t^\varepsilon := v^{(\varepsilon)}(x_t)e^{-\phi_t} + \int_0^t e^{-\phi_s} f^{(\varepsilon)}(x_s)\, dA_s - \int_0^t e^{-\phi_s}\big(\lambda \operatorname{tr} a_s\, dA_s - \sqrt{\lambda}\, |dB_s|\big) M^\varepsilon$$

is a local supermartingale. In addition, it is bounded from below by a summable quantity $(A, B < < \infty)$. Hence, it is a supermartingale and by Fatou's lemma

$$E v^{(\varepsilon)}(x_0) = \kappa_0^\varepsilon \ge E \int_0^\infty e^{-\phi_t} (\det a_t)^{1/d} f^{(\varepsilon)}(x_t)\, dA_t - M^\varepsilon(\lambda A + \sqrt{\lambda} B).$$

By sending $\varepsilon \downarrow 0$ and using (4.7) and Fatou's lemma once more, we obtain

$$E \int_0^\infty e^{-\phi_t} (\det a_t)^{1/d} f(x_t)\, dA_t \le N(d) \frac{1}{\lambda^{1/2}} \Big(1 + \lambda A + \sqrt{\lambda} B\Big) \|f\|_{L_d(\mathbb{R}^d)}.$$

Now we use the arbitrariness of λ. If $A < B^2$, then for $\lambda = B^{-2}$ we have

$$\frac{1}{\lambda^{1/2}} \Big(1 + \lambda A + \sqrt{\lambda} B\Big) \le 3B^{1/2} \le 3(B^2 + A)^{1/2}.$$

If $A \ge B^2$ and $A > 0$, then for $\lambda = A^{-1}$ the above inequality between the extreme terms still holds. Finally, if $A = B = 0$, then the left-hand side of (4.5) is zero.

The lemma is proved.

Theorem 4.1. *Assume the notation of Lemma* 4.1 *and let* $p, q \in [1,\infty]$ *be such that*

$$\theta := 1 - \frac{d}{p} - \frac{1}{q} \ge 0,$$

then, for any Borel $f(t,x) \ge 0$, *we have*

$$I(p,q,f) := E\int\limits_0^\gamma e^{-\phi_t}\kappa_t f(r_t,x_t)\,dA_t \le N(d)(A+B^2)^{d/(2p)}\|f\|_{p,q}, \tag{4.9}$$

where $\kappa_t = r_t^{1/q}(\det a_t)^{1/p}c_t^\theta$ *and for any* $\alpha \ge 0$ *we set* $\alpha^0 = 1$ (*say, if* $\theta = 0$).

Proof. By Hölder's inequality, if $\theta > 0$,

$$I(p,q,f) \le \Big(I\big(p(1-\theta), q(1-\theta), f^{1/(1-\theta)}\big)\Big)^{1-\theta}.$$

It follows that it suffices to concentrate on $\theta = 0$. Then we observe that if $q = \infty$, then $p = d$ and

$$\|f\|_{p,q}^p = \int\limits_{\mathbb{R}^d} \sup_{t\ge0} f^d(t,x)\,dx.$$

In that case (4.12) follows from Lemma 4.2. If $p = \infty$, then $q = 1$, and

$$I(p,q,f) = E\int\limits_0^\gamma r_t f(\tau_t,x_t)\,dA_t \le E\int\limits_0^\gamma \sup_x f(\tau_t,x)\,d\tau_t \le \int\limits_0^\infty \sup_x f(t,x)\,dt = \|f\|_{p,q}.$$

In the third simple situation when $q = p = d+1$ estimate (4.12) follows from Lemma 4.1. We prove the lemma in the remaining cases by interpolating between the above ones.

If $p > q$ (and hence $p > d+1$) we take a nonnegative function $h(t)$ such that $(hf)/h = f$ $(0/0 := 0)$ and use

$$r_t^{1/q}(\det a_t)^{1/p} f = \Big(r_t^{1/q-1/p}h^{-1}\Big)\Big((r\det a_t)^{1/p} fh\Big)$$

along with Hölder's inequality. By performing simple manipulations we find

$$I(p,q,f) \le IJ :=$$

$$:= \Big(I\big(\infty,1,h^{-p/(p-d-1)}\big)\Big)^{(p-d-1)/p}\Big(I\big(d+1,d+1,(hf)^{p/(d+1)}\big)\Big)^{(d+1)/p}. \tag{4.10}$$

Here

$$I \le \left(\int\limits_0^\infty h^{-p/(p-d-1)}(t)\,dt\right)^{(p-d-1)/p}.$$

Also

$$J \le N(d)(B^2+A)^{d/(2p)}\Big\|(hf)^{p/(d+1)}\Big\|_{d+1}^{(d+1)/p} =$$

$$= N(d)(B^2+A)^{d/(2p)} \left(\int_0^\infty \left(\int_{\mathbb{R}^d} f^p(t,x)\,dx \right) h^p(t)\,dt \right)^{1/p}.$$

We now choose h so that

$$h^{-p/(p-d-1)}(t) = \left(\int_{\mathbb{R}^d} f^p(t,x)\,dx \right) h^p(t).$$

Then both quantities become

$$\left(\int_{\mathbb{R}^d} f^p(t,x)\,dx \right)^{q/p}, \qquad J \le N(d)(B^2+A)^{d/(2p)} \|f\|_{p,q}^{q/p}, \qquad I \le \|f\|_{p,q}^{q(p-d-1)/p}$$

and coming back to (4.10) we get (4.9).

In the remaining case $q > p$ (and $q > d+1$) we use

$$r_t^{1/q} (\det a_t)^{1/p} f = \Big((\det a_t)^{1/p-1/q} h^{-1} \Big) \Big((r \det a_t)^{1/q} f h \Big).$$

This time for $h = h(x)$

$$I(p,q,f) \le IJ :=$$

$$:= \Big(I\big(d, \infty, h^{-q/(q-d-1)}\big) \Big)^{(q-d-1)/q} \Big(I\big(d+1, d+1, (hf)^{q/(d+1)}\big) \Big)^{(d+1)/q}. \qquad (4.11)$$

Here

$$I \le N(d)(B^2+A)^{(d/p-d/q)(1/2)} \left(\int_{\mathbb{R}^d} h^{-qd/(q-d-1)}(x)\,dx \right)^{(q-d-1)/(qd)},$$

$$J \le N(d)(B^2+A)^{d/(2q)} \left(\int_{\mathbb{R}^d} h^q(x) \left(\int_0^\infty f^q(t,x)\,dt \right) dx \right)^{1/q}.$$

We choose h so that

$$h^{-qd/(q-d-1)}(x) = h^q(x) \left(\int_0^\infty f^q(t,x)\,dt \right)$$

and then easily come to (4.12).

The theorem is proved.

***Corollary* 4.1.** *Introduce a measure* (*Green's measure*) *on Borel subsets* Γ *of* $\mathbb{R}^{d+1}$ *by the formula*

$$G(\Gamma) = E \int_0^\gamma e^{-\phi_t} \kappa_t I_\Gamma(\tau_t, x_t)\,dA_t.$$

Assume that $A, B < \infty$ *and set* $p' = p/(p-1), q' = q/(q-1)$. *Then* $G(\Gamma)$ *is absolutely continuous and its density* $G(t,x)$ *is such that, if* $p \geq q$,

$$\left(\int_0^\infty \left(\int_{\mathbb{R}^d} G^{p'}(t,x)\,dx \right)^{q'/p'} dt \right)^{1/q'}$$

and, if $p \leq q$,

$$\left(\int_{\mathbb{R}^d} \left(\int_0^\infty G^{q'}(t,x)\,dt \right)^{p'/q'} dx \right)^{1/q'}$$

is dominated by

$$N(d)(B^2 + A)^{(1-\theta)d/(2p)}.$$

Theorem 4.2. *Under the assumptions of Theorem* 4.1 *let* $p_0 \in [1,\infty]$ *and* $q_0 \in [1,\infty)$ *be such that*

$$\theta_0 := 1 - \frac{d}{p_0} - \frac{1}{q_0} \geq 0.$$

Also assume that $d|B_t| \ll dA_t$ *and there exists a Borel* $h(t,x)$ *such that (*$P(d\omega) \times dA_t$*-a.e.)*

$$|b_t| \leq \kappa_t^0 h(\tau_t, x_t),$$

where $b_t = dB_t/dA_t$ *and* $\kappa_t^0 = r_t^{1/q_0}(\det a_t)^{1/p_0} c_t^{\theta_0}$. *Then for any Borel* $f(t,x) \geq 0$ *we have*

$$I(p,q,f) := E\int_0^\gamma e^{-\phi_t}\kappa_t f(\tau_t, x_t)\,dA_t \leq N(d,p_0,q_0)C\|f\|_{p,q}, \tag{4.12}$$

where

$$\kappa_t = r_t^{1/q}(\det a_t)^{1/p} c_t^{\theta}, \quad C = \left(A + \|h\|_{p_0,q_0}^{2p_0/(p_0-d)} \right)^{d/(2p)}$$

and for any number $\alpha \geq 0$ *we set* $\alpha^0 = 1$ (*say, if* $\theta = 0$).

Proof. Observe that $p_0 > d$ since $q_0 < \infty$. Then, we may assume that $A < \infty$ and $\|h\|_{p_0,q_0} < \infty$. Using stopping times we easily reduce the general situation to the one in which $B < \infty$. After that, in light of Theorem 4.1, we need only prove that

$$B \leq N(d,p_0,q_0)\left(A^{1/2} + \|h\|_{p_0,q_0}^{p_0/(p_0-d)} \right). \tag{4.13}$$

By Theorem 4.1

$$B = E\int_0^\tau e^{-\phi_t}|dB_t| \leq I(p_0,q_0,h) \leq N(d)(A+B^2)^{d/(2p_0)}\|h\|_{p_0,q_0}.$$

Here if $B^2 \leq A$, estimate (4.13) holds. If $A \leq B^2$, then the above inequality yields

$$B \leq N(d)B^{d/p_0}\|h\|_{p_0,q_0}, \qquad B^{(p_0-d)/p_0} \leq N(d)\|h\|_{p_0,q_0}$$

and we obtain (4.13) again.

The theorem is proved.

***Remark* 4.1.** In the case of $q = \infty, p = d$ an estimate of B in terms of $\|h\|_{p,q}$ is given in Theorem 5.2 of [6] if γ is the first exit time of x_t from a ball and in Theorem 2.17 of [9] if $A_t = t$ and $c_t = \lambda \operatorname{tr} a_t$, where $\lambda > 0$ is a number (and $\gamma = \infty$).

***Remark* 4.2.** As in [11] we note that estimate (4.12) also, obviously, holds if

$$|b_t| \le \sum_{k=1}^{n} \kappa_t^k h_k(\tau_t, x_t),$$

where $\kappa_t^k = r_t^{1/q_k} (\det a_t)^{1/p_k} c_t^{\theta_k}$, $p_k \in [1, \infty], q_k \in [1, \infty)$, $\theta_k = 1 - d/p_k - 1/q_k \ge 0$, and h_k are nonnegative Borel functions. In that case the constant C depends only on d, p, q, p_k, q_k, $\|h_k\|_{p_k,q_k}$, $k = 1, \ldots, n$, in a somewhat complicated way.

***Remark* 4.3.** The main case of applications of Theorem 4.2 in this article is when $p = p_0 < < \infty, q = q_0 < \infty, \theta = \theta_0 = 0$, $\gamma = T$, where T is a fixed number, $r_t = 1$, $c_t = 0$, $A_t = t \wedge T$,

$$|b_t| \le (\det a_t)^{1/p} h(t, x_t) I_{t \le T}.$$

In that case $2p/(p - d) = 2q$ and estimate (4.12) becomes

$$E \int_0^T (\det a_t)^{1/p} f(t, x) dt \le N(d, p) \left(T + \|h I_{(0,T)}\|_{p,q}^{2q} \right)^{d/(2p)} \|f\|_{p,q}.$$

We finish the section with somewhat unrelated result which we use later in Section 6 and which would be a simple consequence of Theorem 4.5.1 of [14] if we assumed that b is bounded.

Lemma 4.3. *Let x_t, $t \ge 0$, be an $\mathbb{R}^d$-valued process on a probability space $(\Omega, \mathcal{F}, P)$. Define $\mathcal{F}_t$ as the completion of the σ-field generated by x_s, $s \le t$. Let σ_t be an $\mathbb{S}_\delta$-valued and b be an $\mathbb{R}^d$-valued processes which are progressively measurable with respect to $\{\mathcal{F}_t\}$. Suppose that for any $T \in (0, \infty)$*

$$\int_0^T |b_t| \, dt < \infty \quad \text{(a. s.)}$$

and for any $C_0^\infty(\mathbb{R}^{d+1})$-function $u(t, x)$ the process

$$u(t, x_t) - \int_0^t L_s u(s, x_s) \, ds \tag{4.14}$$

is a local martingale with respect to $\{\mathcal{F}_t\}$, where for $a = \sigma^2$

$$L_t u(t, x) = \partial_t u(t, x) + \frac{1}{2} a_t^{ij} D_{ij} u(t, x) + b_t^i D_i u(t, x).$$

Then there exists a d-dimensional Wiener process $(w_t, \mathcal{F}_t)$, $t \ge 0$, such that

$$x_t = x_0 + \int_0^t \sigma_s \, dw_s + \int_0^t b_s \, ds.$$

Proof. First observe that by using cut-off functions one easily shows that (4.14) is a local martingale for any twice continuously differentiable function u. Then, we claim that the following processes are local martingales:

$$X_t := x_t - \int_0^t b_s\,ds,$$

$$B_t := x_t x_t^* - \int_0^t \big(a_s + b_s x_s^* + x_s b_s^*\big)\,ds,$$

$$A_t := X_t X_t^* - \int_0^t a_s\,ds.$$

Indeed, the first two processes are obtained from (4.14) for $u = x, xx^*$. Concerning the last one introduce γ_R as the minimum of $\tau_R = \inf\big\{t \ge 0 \colon |x_t| \ge R\big\}$ and

$$\inf\left\{t \ge 0\colon \int_0^t |b_s|\,ds + |B_t| \ge R\right\}.$$

Also let

$$\Phi_t = \int_0^t b_s I_{s<\gamma_R}\,ds.$$

Observe that $X_{t\wedge\gamma_R}$ and Φ_t are bounded and simple manipulations yield

$$A_{t\wedge\gamma_R} = \int_0^t X_{s\wedge\gamma_R}\,d\Phi_s^* - X_{t\wedge\gamma_R}\Phi_t^* + \int_0^t \big(d\Phi_s\big)X_{s\wedge\gamma_R}^* - \Phi_t X_{t\wedge\gamma_R}^* + B_{t\wedge\gamma_R},$$

which by the Lemma from Appendix 2 of [5] shows that $A_{t\wedge\gamma_R}$ is a martingale.

By the above claim the quadratic variation process of the local martingale X_t is

$$\int_0^t a_s\,ds.$$

After that our assertion follows directly from Theorem III.10.8 of [7].

The lemma is proved.

5. Proof of Theorem 3.1. Introduce

$$B(t) = \|bI_{(-\infty,t)}\|_{p,q}^q.$$

Lemma 5.1. *Suppose that* $p \ge q$ *and let* x_t *be a solution of* (1.2). *Then, for* $0 \le s < t < s+1 < \infty$ *and* $n = 1, 2, \ldots,$ *we have*

$$E|x_t - x_s|^n \le N\big(t - s + B^2(t_0 + t) - B^2(t_0 + s)\big)^{nd/(2p)}, \tag{5.1}$$

where $N = N\big(n, d, \delta, p, \|b\|\big)$.

Proof. We may assume that $t_0 = 0$. Then observe that for any integer $n = 1, 2, \ldots$

$$I_{n+1} := E\left(\int\limits_s^t b(u, x_u)\, du\right)^{n+1} =$$

$$= (n+1)!E \int\limits_{s \le u_1 \le \ldots \le u_n} b(u_1, x_{u_1}) \ldots b(u_n, x_{u_n}) E\left(\int\limits_{u_n}^t b(u, x_u)\, du \mid \mathcal{F}_{u_n}\right) du_1 \ldots du_n,$$

where the conditional expectation we can estimate by using Remark 4.3.

Then we get

$$I_{n+1} \le N(n+1) I_n \Big(t - s + \|bI_{(s,t)}\|_{p,q}^{2q}\Big)^{d/(2p)} \|b\|_{p,q},$$

where N depends only on d, p, and δ. Here

$$\|bI_{(s,t)}\|_{p,q}^{2q} = \Big(B(t) - B(s)\Big)^2 \le B^2(t) - B^2(s).$$

Therefore,

$$I_{n+1} \le N(n+1) I_n \Big(t - s + B^2(t) - B^2(s)\Big)^{d/(2p)} \|b\|_{p,q}.$$

The induction on n yields

$$I_n \le N^n n! \left(t - s + B^2(t) - B^2(s)\right)^{nd/(2p)} \|b\|_{p,q}^n.$$

Also, as is well-known,

$$E\left|\int\limits_s^t \sigma(u, x_u)\, dw_u\right|^n \le N(n, \delta)(t - s)^{n/2}.$$

It follows that the left-hand side of (5.1) is less than a constant N times

$$(t - s)^{n/2} + \Big(t - s + B^2(t) - B^2(s)\Big)^{nd/(2p)},$$

which less than twice the factor of N in (5.1) because $p > d$ and $t - s \le 1$.

The lemma is proved.

Lemma 5.2. *Under the assumptions in Theorem* 3.1 (ii) *the set of distributions of* $x^n_\cdot$ *on* $C\big([0, \infty), \mathbb{R}^d\big)$ *is tight if* $p \ge q$.

Proof. Define

$$B_n(t) = \left\|b^n I_{(-\infty, t^n + t)}\right\|_{p,q}^q$$

and let $\phi^n(s)$ be the inverse function of $\psi^n(t) := t^n + t + B_n^2(t^n + t)$. By Lemma 5.1 and Kolmogorov's criteria the set of distributions of $y^n_\cdot := x^n_{\phi^n(\cdot)}$ on $C\big([0, \infty), \mathbb{R}^d\big)$ is tight.

Observe that, as $n \to \infty$, $\psi^n(t)$ converges to $t_0 + t + B^2(t_0 + t)$ which is continuous and monotone. By Polya's theorem the convergence is uniform on any finite time interval, and hence, the functions $\psi^n(t)$ are equicontinuous on any finite time interval. Now define

$$\Phi(s) = \inf_{n \ge 1} \phi^n(s)$$

and take $S \in (0,\infty)$. By tightness, for any $\varepsilon > 0$ there is a compact set K_ε in $C\big([0,S],\mathbb{R}^d\big)$ such that $P^n(\{y^n_s, s \le S\} \in K_\varepsilon) \ge 1-\varepsilon$ for all n. Due to the uniform continuity of ψ^n and of the elements of K_ε, the elements of

$$\hat{K}_\varepsilon := \Big\{\big\{f(\psi^n(t)), t \le \Phi(S)\big\} : \{f(s), s \le S\} \in K_\varepsilon, n = 1,2,\ldots\Big\}$$

are uniformly continuous and, of course, uniformly bounded, so that $\hat{K}_\varepsilon$ is a compact set in $C\big([0,\Phi(S)],\mathbb{R}^d\big)$ and

$$P\Big(\big\{y^n_{\psi^n(t)}, t \le \Phi(S)\big\} \in \hat{K}_\varepsilon\Big) \ge 1-\varepsilon.$$

It only remains to observe that $y^n_{\psi^n(t)} = x^n_t$, S is arbitrary, and $\Phi(S) \to \infty$ as $S \to \infty$.

The lemma is proved.

This takes care of part of assertion (ii) of Theorem 3.1. To deal with the rest we rely on the following results due to A. V. Skorokhod (see Ch. 1, § 6 and Ch. 2, § 3 in [13]).

Lemma 5.3. *Suppose that d_1-dimensional random processes ξ^n_t $(t \ge 0,\ n = 1,2,\ldots)$ are defined on some probability spaces. Assume that for each $T > 0$ and $\varepsilon > 0$*

$$\lim_{c\to\infty}\sup_n\sup_{t\le T} P^n\big(|\xi^n_t| > c\big) = 0, \tag{5.2}$$

$$\lim_{h\downarrow 0}\sup_n \sup_{\substack{t_1,t_2\le T\\ |t_1-t_2|\le h}} P^n\big(|\xi^n_{t_1} - \xi^n_{t_2}| > \varepsilon\big) = 0. \tag{5.3}$$

Then one can choose a sequence of numbers $n' \to \infty$, a probability space, and random processes $\tilde{\xi}_t, \tilde{\xi}^{n'}_t$ defined on this probability space such that all finite-dimensional distributions of $\tilde{\xi}^{n'}_t$ coincide with the corresponding finite-dimensional distributions of $\xi^{n'}_t$ and

$$P\big(|\tilde{\xi}_t - \tilde{\xi}^{n'}_t|\big) \to 0$$

as $n' \to \infty$ for any $\varepsilon > 0$ and $t \ge 0$.

Lemma 5.4. *Suppose the assumptions of Lemma 5.3 are satisfied and ξ^n_t are defined on the same probability space. Also, suppose that d_1-dimensional Wiener processes $(w^n_t, \mathcal{F}^n_t)$ are defined on this probability space. Assume that the functions $\xi^n_t(\omega)$ are bounded on $[0,\infty)\times\Omega$ uniformly in n and that the stochastic integrals*

$$I^n_t := \int_0^t \xi^n_s\, dw^n_s$$

are defined for $t \ge 0$. Finally, let

$$\xi^n_t \to \xi^0_t, \qquad w^n_t \to w^0_t \tag{5.4}$$

in probability as $n \to \infty$ for each $t \ge 0$. Then $I^n_t \to I^0_t$ in probability as $n \to \infty$ for each $t \ge 0$.

***Remark* 5.1.** As it follows from the proof of Lemma 5.4 given in [13] we need conditions (5.2), (5.3), and (5.4) to hold only for $t,\ t_1,\ t_2$ restricted to a set of full measure in order for the assertion of the lemma to be true.

Lemma 5.5. *Let $\mathbb{R}^{2d}$-valued processes (x^i_t, w^i_t), $t \geq 0$, $i = 1, 2$, defined on perhaps different probability spaces have the same finite-dimensional distributions. Define $\mathcal{F}^i_t$ as the completion of $\sigma(x^i_s, w^i_s : s \leq t)$ and assume that w^1_t is a Wiener process with respect to $\mathcal{F}^1_t$. Also suppose that* (*a.s.*) *for all $t \geq 0$*

$$x^1_t = \int_0^t \sigma(s, x^1_s)\, dw^1_s + \int_0^t b(s, x^1_s)\, ds. \tag{5.5}$$

Then x^2_t, w^2_t have modifications (*called again x^2_t, w^2_t*) *such that w^2_t is a Wiener process with respect to $\mathcal{F}^2_t$ and* (*a.s.*) *for all $t \geq 0$*

$$x^2_t = \int_0^t \sigma(s, x^2_s)\, dw^2_s + \int_0^t b(s, x^2_s)\, ds. \tag{5.6}$$

Proof. Fix $T \in (0, \infty)$ and $\varepsilon \in (0, 1)$. Since the trajectories of (x^1_t, w^1_t) are continuous, there exists a compact set $K \subset C\big([0, T], \mathbb{R}^{2d}\big)$ such that

$$P\big((x^1_{\cdot\wedge T}, w^1_{\cdot\wedge T}) \in K\big) \geq 1 - \varepsilon.$$

Hence, there is a constant N and a continuous function $w(t)$, $t \in [0, T]$, such that $w(0) = 0$ and with probability larger than $1 - \varepsilon$ for any $s, t \in [0, T]$

$$\big|(x^1_s, w^1_s)\big| \leq N, \qquad \big|(x^1_s, w^1_s) - (x^1_t, w^1_t)\big| \leq w\big(|t - s|\big). \tag{5.7}$$

It follows that (5.7) holds for rational s, t if we replace (x^1, w^1) with (x^2, w^2). Then by continuity (x^2_t, w^2_t) is extended to all $t \in [0, T]$. The extensions coincide with the original ones (a.s.) for any t because of the stochastic continuity of the original (x^2_t, w^2_t). This is done on events whose probabilities tend to one. Because of the arbitrariness of T we may assume that (x^2_t, w^2_t) is continuous in t with probability one.

By Remark 4.3 and by the coincidence of finite dimensional distributions (and by the measurability of x^2_t due to its continuity) for any $T \in [0, \infty)$, Borel $f(t, x) \geq 0$,

$$E \int_0^T f(t, x^2_t)\, dt \leq N \|f I_{(0,T)}\|_{p,q}, \tag{5.8}$$

where N is independent of f.

Furthermore, if $\alpha(t, x)$ is a continuous $d \times d$ symmetric matrix-valued, $\beta(t, x)$ is a continuous $\mathbb{R}^d$-valued, then the distributions of

$$\left(x^i_t, \int_0^t \alpha(s, x^i_s)\, dw^i_s, \int_0^t \beta(s, x^i_s)\, ds\right), \quad i = 1, 2,$$

coincide, because the integrals can be approximated by integral sums. This coincidence also holds for $\alpha = \sigma$ and $\beta = b$ due to (5.8) and the possibility of approximation. Hence for each t with probability one (5.6) holds due to (5.5). But then with probability one it holds for all t, because both sides of (5.6) are continuous.

The lemma is proved.

***Proof of Theorem* 3.1.** Due to the possibility to use mollifiers we see that assertion (ii) implies (i). In the proof of (ii), thanks to Lemma 5.2, we need only prove the assertion concerning the convergence of finite dimensional distributions.

Having in mind Lemma 5.3 define for $M > 0$

$$\xi_t^n = \int_0^t b^n(t^n + s, x_s^n)\, ds, \qquad \xi_t^{nM} = \int_0^t b^n(t^n + s, x_s^n) I_{|b^n(t^n+s,x_s^n)|\le M}\, ds.$$

Since the derivative of ξ_t^{nM} is bounded, both conditions (5.2) and (5.3) are satisfied for ξ_t^{nM}. Furthermore,

$$P^n\left(\int_0^T |b^n(t^n + s, x_s^n)| I_{|b^n(t^n+s,x_s^n)|\ge M}\, ds > \varepsilon\right) \le \varepsilon^{-1} N \|b^n I_{|b^n|\ge M}\|_{p,q},$$

where N is independent of n and ε. Since $b^n \to b$ in the $\|\cdot\|_{p,q}$-norm, the latter quantity can be made as small as we like on the account of choosing M large enough. Therefore, Lemma 5.3 is applicable to ξ_t^n. It is, obviously, also applicable to

$$\eta_t^n = x^n + \int_0^t \sigma^n(t^n + s, x_s^n)\, dw_s^n.$$

Hence, there is a subsequence, which by common abuse of notation we identify with the original one, a probability space and random $\mathbb{R}^{2d}$-valued processes $(\tilde{x}_t^n, \tilde{w}_t^n)$, $(\tilde{x}_t^0, \tilde{w}_t^0)$ defined on this probability space such that all finite-dimensional distributions of $(\tilde{x}_t^n, \tilde{w}_t^n)$ coincide with the corresponding finite-dimensional distributions of (x_t^n, w_t^n) and

$$P\big(|(\tilde{x}_t^n, \tilde{w}_t^n) - (\tilde{x}_t^0, \tilde{w}_t^0)| \ge \varepsilon\big) \to 0 \tag{5.9}$$

as $n \to \infty$ for any $\varepsilon > 0$ and $t \ge 0$. Furthermore, for any $T \in (0, \infty)$ there exists a continuous function $w(t)$, $t \in [0, T]$, such that $w(0) = 0$ and for all $n \ge 0$, $s, t \le T$,

$$E\big|\phi(\tilde{x}_t^n) - \phi(\tilde{x}_s^n)\big| \le w\big(|t - s|\big), \tag{5.10}$$

where $\phi(x) = x/\big(1 + |x|\big)$.

For $n \ge 0$ introduce $\tilde{\mathcal{F}}_t^n$ as the completion of $\sigma(\tilde{x}_s^n, \tilde{w}_s^n, s \le t)$. It is easy to see, using Kolmogorov's continuity criterion, that $\tilde{w}_t^0$ admits a continuous modification $\hat{w}_t^0$ such that $\{\hat{w}_t^0, \tilde{\mathcal{F}}_t^0\}$ is a Wiener process.

By Lemma 5.5, for each $n \ge 1$, the process $(\tilde{x}_t^n, \tilde{w}_t^n)$ admits a continuous modification denoted by $(\hat{x}_t^n, \hat{w}_t^n)$ such that $(\hat{w}_t^n, \tilde{\mathcal{F}}_t^n)$ is a Wiener process and (a.s.) for all $t \ge 0$

$$\hat{x}_t^n = x_n + \int_0^t \sigma^n(t_n + s, \hat{x}_s^n)\, d\hat{w}_s^n + \int_0^t b^n(t_n + s, \hat{x}_s^n)\, ds. \tag{5.11}$$

In light of (5.9) and (5.10) we have

$$P\big(|(\hat{x}_t^n,\hat{w}_t^n)-(\tilde{x}_t^0,\tilde{w}_t^0)|\geq\varepsilon\big)\to 0 \tag{5.12}$$

as $n\to\infty$ for any $\varepsilon>0$ and, $t\geq 0$ and for all $n\geq 1,\ s,t\leq T$,

$$E\big|\phi(\hat{x}_t^n)-\phi(\hat{x}_s^n)\big|\leq w(|t-s|). \tag{5.13}$$

Now the fact that $\tilde{x}_t^0$ may be not measurable in t causes some problems. However, observe that, owing to (5.12), $\phi(\hat{x}_t^n)$ form a Cauchy sequence in $L_1(\Omega\times[0,T])$ and, hence, converges in that space to $\phi(\hat{x}_t^0)$, where $\hat{x}_t^0$ is measurable with respect to (ω,t). By Fubini's theorem there is a set $\mathcal{S}\subset[0,\infty)$ of full measure such that, for any $t\in\mathcal{S}$, $\hat{x}_t^0=\tilde{x}_t^0$ (a.s.). Without losing the above properties we set $\hat{x}_t^0=0$ for $t\notin\mathcal{S}$ and then, for any $s,t\geq 0$, $\hat{w}_{t+s}^0-\hat{w}_t^0$ is indepenent of $(\hat{x}_r^0,\hat{w}_r^0),r\leq t$.

Now we note that (5.13) remains valid for $n=0$ and (5.12) remains valid if we replace $(\tilde{x}_t^0,\tilde{w}_t^0)$ by $(\hat{x}_t^0,\hat{w}_t^0)$ and restrict the ranges of t,s to $t,s\in\mathcal{S}$. This is done to accommodate Remark 5.1. Then, by Lemma 5.4 for any $t\geq 0$ and continuous $d\times d$ symmetric matrix-valued $\alpha(t,x)$, we have

$$\int_0^t\alpha(s,\hat{x}_s^n)\,d\hat{w}_s^n\to\int_0^t\alpha(s,\hat{x}_s^0)\,d\hat{w}_s^0 \tag{5.14}$$

as $n\to\infty$ in probability. We want to use this to pass to the limit in the stochastic term in (5.11). But first observe that by Remark 4.3 for any $T\in[0,\infty)$, Borel $f(t,x)\geq 0$, and $n\geq 1$

$$E\int_0^T f(t,\hat{x}_t^n)\,dt\leq N\|fI_{(0,T)}\|_{p,q}, \tag{5.15}$$

where N is independent of f and n. The convergence in probability implies that (5.15) holds for $n=0$ as well with the same constant N, first for nonnegative $f\in C_0^\infty(\mathbb{R}^{d+1})$ and then, due to general measure-theoretic arguments, for any Borel nonnegative f.

We claim that on the account of (5.15), if Borel functions g^n converge to g in the $\|\cdot\|_{p,q}$-norm, then

$$E\int_0^T\big|g^n(t,\hat{x}_t^n)-g(t,\hat{x}_t^0)\big|\,dt\to 0. \tag{5.16}$$

To prove (5.16) take $\varepsilon>0$ and $g_\varepsilon\in C_0^\infty(\mathbb{R}^{d+1})$ such that

$$\|g-g_\varepsilon\|_{p,q}\leq\varepsilon.$$

For g_ε in place of g, (5.16) follows from the convergence in probability of $\hat{x}_t^n$ to $\hat{x}_t^0$ for $t\in\mathcal{S}$. After that it only remains to observe that the limit of the error of the substitution in (5.16) is less than $2N\varepsilon$ owing to (5.15). It follows, in particular, that in probability

$$\sup_{t\leq T}\left|\int_0^t b^n(t_n+s,\hat{x}_s^n)\,ds-\int_0^t b(t_0+s,\hat{x}_s^0)\,ds\right|\to 0. \tag{5.17}$$

Coming back to the stochastic part note that for any $t\geq 0$ and $c\in(0,\infty)$

$$\varlimsup_{n\to\infty} E\left|\int\limits_0^t \sigma^n(t_n+s,\hat{x}^n_s)\,d\hat{w}^n_s - \int\limits_0^t \alpha(s,\hat{x}^n_s)\,d\hat{w}^n_s\right|^2 =$$

$$= \varlimsup_{n\to\infty} E\int\limits_0^t \|\sigma^n(t_n+s,\hat{x}^n_s)-\alpha(s,\hat{x}^n_s)\|^2\,ds \le$$

$$\le N\sup_n \int\limits_0^t P(|\hat{x}^n_s|>c)\,ds + N\lim_{n\to\infty}\big\|\big(\sigma^n(t_n+\cdot,\cdot)-\alpha(\cdot,\cdot)\big)I_{[0,t]\times B_c}\big\|_{p,q} =$$

$$= N\sup_n \int\limits_0^t P(|\hat{x}^n_s|>c)\,ds + N\big\|\big(\sigma(t_0+\cdot,\cdot)-\alpha(\cdot,\cdot)\big)I_{[0,t]\times B_c}\big\|_{p,q},$$

where the constants N are independent of t and c. The last quantity also dominates

$$E\left|\int\limits_0^t \sigma(t_0+s,\hat{x}^0_s)\,d\hat{w}^0_s - \int\limits_0^t \alpha(s,\hat{x}^0_s)\,d\hat{w}^0_s\right|^2.$$

This and (5.14) show how, for any given $\varepsilon,\delta>0$, to choose c and a continuous α in order to have that

$$\varlimsup_{n\to\infty} P\left(\left|\int\limits_0^t \sigma^n(t_n+s,\hat{x}^n_s)\,d\hat{w}^n_s - \int\limits_0^t \sigma(t_n+s,\hat{x}^0_s)\,d\hat{w}^0_s\right| > \varepsilon\right) \le \delta.$$

Upon combining this with (5.17) and coming back to (5.11) we conclude that for any t (a.s.)

$$\tilde{x}^0_t = x_0 + \int\limits_0^t \sigma(t_0+s,\hat{x}^0_s)\,d\hat{w}^0_s + \int\limits_0^t b(t_0+s,\hat{x}^0_s)\,ds =: y_t.$$

In particular, this means that $\tilde{x}^0_t$ admits a continuous modification y_t. In turn, it allows us to replace in the above equation $\hat{x}^0_s$ with y_t, because for any $s\in\mathcal{S}$, $\hat{x}^0_s=\tilde{x}^0_s=y_s$ (a.s.) and therefore $\hat{x}^0_s=y_s$ for almost all (ω,s).

The theorem is proved.

6. Markov processes corresponding to σ, b. We are going to use the results in [4] applied in the case when the semicompactum E is $\mathbb{R}^{d+1}$, that is when the t-variable is considered just as one of coordinates of points $(t,x)\in\mathbb{R}^{d+1}$.

Let Ω be the set of $\mathbb{R}^{d+1}$-valued continuous function (t_0+t,x_t), $t_0\in\mathbb{R}$, defined for $t\in[0,\infty)$. For $\omega=\big\{(t_0+t,x_t),t\ge 0\big\}$, define $\mathsf{t}_t(\omega)=t_0+t$, $x_t(\omega)=x_t$, and set $\mathcal{N}_t=\sigma((\mathsf{t}_s,x_s),s\le t)$, $\mathcal{N}=\mathcal{N}_\infty$. Denote by T the set of stopping times relative to $\{\mathcal{N}_t\}$. In the following theorem we use the terminology from [3].

Theorem 6.1. *On $\mathbb{R}^{d+1}$ there exists a strong Markov process*

$$X=\big\{(\mathsf{t}_t,x_t),\infty,\mathcal{N}_t,P_{t,x}\big\}$$

such that the process

$$X_1 = \big\{(\mathtt{t}_t, x_t), \infty, \mathcal{N}_{t+}, P_{t,x}\big\}$$

is Markov and for any $(t,x) \in \mathbb{R}^{d+1}$ *there exists a* d*-dimensional Wiener process* $w_t,\ t \geq 0,$ *which is a Wiener process relative to* $\bar{\mathcal{N}_t}$*, where* $\bar{\mathcal{N}_t}$ *is the completion of* $\mathcal{N}_t$ *with respect to* $P_{t,x}$*, and such that with* $P_{t,x}$*-probability one, for all* $s \geq 0,\ \mathtt{t}_s = t + s$ *and*

$$x_s = x + \int\limits_0^s \sigma(t+u, x_u)\, dw_u + \int\limits_0^s b(t+u, x_u)\, du. \tag{6.1}$$

Proof. Define $a = \sigma^2$,

$$Lu(t,x) = \partial_t u(t,x) + \frac{1}{2} a^{ij} D_{ij} u(t,x) + b^i D_i u(t,x)$$

and introduce $\Pi_{t,x}$ as the set of probability measures on $(\Omega, \mathcal{N})$ such that $P\big((\mathtt{t}_0, x_0) = (t,x)\big) = 1$,

$$E\int\limits_0^T |b(\mathtt{t}_t, x_t)|\, dt < \infty \quad \forall T < \infty, \tag{6.2}$$

and the process

$$\eta_t(u) = u(\mathtt{t}_t, x_t) - \int\limits_0^t Lu(\mathtt{t}_s, x_s)\, ds$$

is a martingal relative to $\{\mathcal{N}_t\}$ for all $u \in C_0^\infty(\mathbb{R}^{d+1})$.

According to Lemma 4.3, if $P_{t,x} \in \Pi_{t,x}$, then the assertion of the theorem regarding (6.1) holds and (6.2) is true. Therefore, by Theorem 2 of [4] to prove the present theorem, it suffices to show that $\Pi_{t,x} \neq \varnothing$ and $\{\Pi_{t,x}\}$ is a Markov system relative to $(\mathsf{T}, \mathcal{N}_\mathtt{t})$ and $([0,\infty), \mathcal{N}_{t+})$.

That $\Pi_{t,x} \neq \varnothing$ follows from Theorem 3.1 (i). Let us prove that $\{\Pi_{t,x}\}$ is a B-system. To achieve this, as it follows from [4], it suffices to show that if $(t_n, x_n) \to (t,x)$ and $P^n \in \Pi_{t_n, x_n}$, then there exists a subsequence $n(k) \to \infty$ and $P^0 \in \Pi_{t,x}$ such that for any $f \in C_0^\infty(\mathbb{R}^{d+2})$

$$E^{n(k)} \exp\left(\int\limits_0^\infty e^{-t} f(t, \mathtt{t}_t, x_t)\, dt\right) \to E^0 \exp\left(\int\limits_0^\infty e^{-t} f(t, \mathtt{t}_t, x_t)\, dt\right),$$

where $E^{n(k)}$, E^0 are the expectation signs with respect to $P^{n(k)}, P^0$, respectively. The reader will easily derive this property from Theorem 3.1 (ii) by using Taylor's series and observing that

$$E\left(\int\limits_0^\infty e^{-t} f(t, \mathtt{t}_t, x_t)\, dt\right)^n =$$

$$= E\int\limits_0^\infty \ldots \int\limits_0^\infty e^{-t_1} f(t_1, \mathtt{t}_{t_1}, x_{t_1}) \ldots e^{-t_n} f(t_n, \mathtt{t}_{t_n}, x_{t_n})\, dt_1 \ldots dt_n.$$

What remains is to prove that for $(\mathsf{T},\mathcal{N}_t)$ and $([0,\infty),\mathcal{N}_{t+})$ the conditions 2 and 3 are satisfied of the definition of Markov system in [4]. This is done by almost literally repeating the corresponding part of the proof of Theorem 3 of [4]. One need only replace there x_t with (t_t, x_t).

The theorem is proved.

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