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MIXED PROBLEM FOR HIGHER-ORDER EQUATIONS WITH FRACTIONAL DERIVATIVE AND DEGENERATION IN BOTH VARIABLES

МІШАНА ЗАДАЧА ДЛЯ РІВНЯНЬ ВИЩОГО ПОРЯДКУ З ДРОБОВОЮ ПОХІДНОЮ, ЩО МАЄ ВИРОДЖЕННЯ ЗА ОБОМА ЗМІННИМИ

We consider an initial-boundary-value problem for a higher-order equation with fractional Riemann – Liouville derivative in a rectangular domain degenerating in both variables. The solution to the problem is constructed in the explicit form by the method of separation of variables. Uniqueness is proved by the spectral method.

Розглянуто початково-крайову задачу для рівняння вищого порядку з дробовою похідною Рімана – Ліувілля в прямокутній області, що вироджується за обома змінними. Розв'язок задачі отримано в явному вигляді методом відокремлення змінних. Єдиність доводиться за допомогою спектрального методу.

1. Introduction. In the domain $\Omega = \Omega_x \times \Omega_y,\ \Omega_x = \{x\colon 0 < x < 1\},\ \Omega_y = \{y\colon 0 < y < 1\}$, consider the equation

$$(-1)^{k+1} D_{0x}^{\alpha} u(x,y) - x^s y^m \frac{\partial^{2k} u}{\partial y^{2k}} = 0, \tag{1}$$

where $0 < \alpha < 1,\ 0 \leq m < k,\ m \notin N,\ s \in N \cup \{0\},\ k \in N,\ D_{0x}^{\alpha}$ is the operator of Riemann – Liouville fractional differentiation of order α,

$$D_{0x}^{\alpha} u(x,y) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int\limits_0^x \frac{u(\tau,y)d\tau}{(x-\tau)^{\alpha}}.$$

For equation (1), consider the problem.

Problem A. Find a solution to equation (1) from the class

$$D_{0x}^{\alpha} u(x,y) \in C(\Omega), \qquad x^{1-\alpha} u(x,y) \in C\big(\overline{\Omega}_x \times \Omega_y\big),$$
$$\frac{\partial^{k-1} u(x,y)}{\partial y^{k-1}} \in C\big(\Omega_x \times \overline{\Omega}_y\big), \qquad \frac{\partial^{2k} u(x,y)}{\partial y^{2k}} \in C(\Omega_x \times \Omega_y), \tag{2}$$

satisfying the conditions

$$\frac{\partial^j u(x,0)}{\partial y^j} = \frac{\partial^j u(x,1)}{\partial y^j} = 0, \qquad 0 < x \leq 1, \quad j = 0,1,\ldots,k-1, \tag{3}$$

$$\lim_{x\to 0} D_{0x}^{\alpha-1} u(x,y) = \varphi(y). \tag{4}$$

Here, $\varphi(y)$ is sufficiently smooth and satisfies the natural concordance conditions.

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Fractional differential equations arise in mathematical modeling of various physical processes and phenomena [1]. Second-order equations of the form (1) with partial derivatives of fractional order $\alpha \in (0,2)$ were studied in [1 – 8]. In these papers, the Cauchy problem was considered, the first, the second and mixed boundary-value problems, a fundamental solution is found, a general representation of solutions is constructed. Mixed equations and higher-order equations with a fractional derivative were studied in [9 – 12]. Degenerate fractional-order equations were studied in [1, 13]. The research will be carried out by the Fourier method. Previously, by the Fourier method, boundary-value problems for equations with a fractional derivative were studied in [6 – 9, 12].

Based on work [5], we will make some comments. Let $x^{1-\alpha}u(x,y) = u_0(x,y) \Rightarrow u = x^{\alpha-1}u_0$, from condition (4) we have

$$\lim_{x\to 0} x^{1-\alpha}u(x,y) = \frac{\varphi(y)}{\Gamma(\alpha)}. \tag{5}$$

2. Existence of a solution. We are looking for a solution in the form

$$u(x,y) = X(x)Y(y).$$

Then with respect to the variable y, taking into account condition (3), we obtain the following spectral problem:

$$\begin{gathered} Y^{(2k)}(y) = (-1)^k \lambda y^{-m} Y(y), \\ Y^{(j)}(0) = Y^{(j)}(1) = 0, \quad j = 0,1,\ldots,k-1. \end{gathered} \tag{6}$$

Notice that $\lambda = 0$ is not an eigenvalue. Using the results of [14], we can write the solution to problem (6), satisfying the conditions at the point $x = 0$, in the form

$$Y_i(y) = y^i \cdot {}_0F_{2k-1}\left[\frac{i}{2k-m}+1,\ldots,\frac{i-(i-1)}{2k-m}+1,\frac{i-(i+1)}{2k-m}+1,\ldots\right.$$
$$\left.\ldots,\frac{i-(2k-1)}{2k-m}+1,\frac{(-1)^k\lambda y^{2k-m}}{(2k-m)^{2k}}\right], \quad i = 0,1,\ldots,2k-1,$$

where

$${}_pF_q\begin{bmatrix} a_1,\ldots,a_p,x \\ b_1,\ldots,b_q \end{bmatrix} = \sum_{k=0}^{\infty} \frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{x^k}{k!}$$

is the generalized hypergeometric function

$$(a)_k = a(a+1)\ldots(a+k-1)$$

is the Pochhammer symbol.

In particular, for $k = 1$ we have ($c_0,\ldots,c_3$ are constants)

$$Y_0(t) = c_0\left(\frac{\sqrt{\lambda}y^{\frac{2-m}{2}}}{2-m}\right)^{\frac{1}{2-m}} \sum_{j=0}^{\infty} \frac{(-1)^j\left(\dfrac{2\sqrt{\lambda}y^{\frac{2-m}{2}}}{2(2-m)}\right)^{2j-\frac{1}{2-m}}}{j!\Gamma\left(j-\dfrac{1}{2-m}+1\right)} = c_1\sqrt{y}J_{-\frac{1}{2-m}}\left(\frac{2\sqrt{\lambda}y^{\frac{2-m}{2}}}{2-m}\right),$$

$$Y_1(y) = c_2 y \sum_{j=0}^{\infty} \frac{(-1)^j \left(\dfrac{\lambda y^{2-m}}{(2-m)^2}\right)^j}{j!\Gamma\left(j + \dfrac{1}{2-m} + 1\right)} = c_3\sqrt{y} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\dfrac{2\sqrt{\lambda} y^{\frac{2-m}{2}}}{2(2-m)}\right)^{2j+\frac{1}{2-m}}}{j!\Gamma\left(j + \dfrac{1}{2-m} + 1\right)} =$$

$$= c_3\sqrt{y} J_{\frac{1}{2-m}}\left(\frac{2\sqrt{\lambda} y^{\frac{2-m}{2}}}{2-m}\right),$$

where

$$J_\nu(z) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\dfrac{z}{2}\right)^{2j+\nu}}{j!\Gamma(j+\nu+1)}$$

are Bessel functions [15].

Satisfying the boundary conditions, we obtain the condition for the existence of eigenvalues

$$J_{\frac{1}{2-m}}\left(\frac{2\sqrt{\lambda}}{2-m}\right) = 0.$$

Let us get back to the general case. Because $(2k-m) \notin N$, then the system of functions $\{Y_i(y)\}_{i=0}^{i=2k-1}$ is the forms a fundamental system of solutions. Hence, the general solution of equation (6) has the form

$$Y(y) = c_0 Y_0(y) + c_1 Y_1(y) + \ldots + c_{2k-1} Y_{2k-1}(y),$$

and from the boundary conditions at the point $x = 0$, we have

$$Y(y) = c_k Y_k(y) + c_{k+1} Y_{k+1}(y) + \ldots + c_{2k-1} Y_{2k-1}(y).$$

It mean that

$$Y(y) = O(y^k), \quad y \to +0.$$

From the conditions at the point $x = 1$, we obtain the system

$$c_k Y_k(1) + c_{k+1} Y_{k+1}(1) + \ldots + c_{2k-2} Y_{2k-2}(1) + c_{2k-1} Y_{2k-1}(1) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\big(c_k Y_k(y) + c_{k+1} Y_{k+1}(y) + \ldots + c_{2k-2} Y_{2k-2}(y) + c_{2k-1} Y_{2k-1}(y)\big)_{y=1}^{(k-1)} = 0.$$

Equating to zero the main determinant of the system, one can find the eigenvalues of problem (6). But in view of the complexity of this process, we will proceed in a different way, namely: we reduce problem (6) to the integral equation using the Green function and obtain the necessary estimates for the eigenfunctions. But first, we show that $\lambda > 0$. Indeed, we have

$$\int_0^1 Y(y) Y^{(2k)}(y) dy = (-1)^k \lambda \int_0^1 y^{-m} Y^2(y) dy,$$

$$\int_0^1 \left(Y^{(k)}\right)^2 dy = \lambda \int_0^1 y^{-m} Y^2(y) dy,$$

because $\lambda = 0$ is not an eigenvalue, it follows that $\lambda > 0$. It remains to show the existence of eigenvalues and eigenfunctions of problem (6). The integral equation equivalent to problem (6) has the form

$$Y(y) = (-1)^k \lambda \int_0^1 \xi^{-m} G(y,\xi) Y(\xi) d\xi, \tag{7}$$

where

$$G(y,\xi) = -\frac{1}{(2k-1)!} \begin{cases} G_1(y,\xi), & 0 \le y \le \xi, \\ G_2(y,\xi), & \xi \le y \le 1, \end{cases}$$

is the Green function of problem (6) (see [16]). Here,

$$G_1(y,\xi) = (1-\xi)^k y^k \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^i C_{2k-1}^i C_{k-1+j}^j y^{k-i-1} \xi^{j+i},$$

$$G_2(y,\xi) = (1-y)^k \xi^k \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^i C_{2k-1}^i C_{k-1+j}^j \xi^{k-i-1} y^{j+i},$$

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

Rewrite (7) as

$$y^{-\frac{m}{2}} Y(y) = \lambda \int_0^1 \xi^{-\frac{m}{2}} \Big[(-1)^k G(y,\xi)\Big] y^{-\frac{m}{2}} \Big(\xi^{-\frac{m}{2}} Y(\xi)\Big) d\xi,$$

we introduce the notation

$$\overline{Y}(y) = y^{-\frac{m}{2}} Y(y),$$

$$\overline{G}(y,\xi) = \xi^{-\frac{m}{2}} \Big[(-1)^k G(y,\xi)\Big] y^{-\frac{m}{2}}.$$

Then we have

$$\overline{Y}(y) = \lambda \int_0^1 \overline{G}(y,\xi) \overline{Y}(\xi) d\xi. \tag{8}$$

Equation (8) is an integral equation with a continuous, in both variables, and a symmetric kernel. According to the theory of equations with symmetric kernels, equation (8) has no more than a countable number of eigenvalues and eigenfunctions. So, problem (6) has eigenvalues $\lambda_n > 0$, $n = 1, 2, \ldots$, and the corresponding eigenfunctions are $Y_n(y)$. Further, we assume that

$$\left\|Y_n(y)\right\|^2 = \int_0^1 y^{-m} Y_n^2(y)dy = 1.$$

Then, taking into account (8), we have the Bessel inequality

$$\sum_{n=0}^{\infty} \left(\frac{Y_n(y)}{\lambda_n}\right)^2 \leq \int_0^1 y^{-m} G^2(y,\xi)dy < \infty. \tag{9}$$

Now we find the conditions under which the given function $\varphi(y)$ is expanded in a series according to the eigenfunctions $Y_n(y)$. For this we use the Hilbert – Schmidt theorem.

Theorem 1. *Let the function* $\varphi(y)$ *satisfies the following conditions*:

$$\varphi(y) \in C^{2k}[0,1],$$
$$\varphi^{(i)}(0) = \varphi^{(i)}(1) = 0, \quad i = 0, 1, \ldots, k-1.$$

Then it can be expanded in a uniformly and absolutely converging series of the form

$$\varphi(y) = \sum_{n=1}^{\infty} \varphi_n Y_n(y),$$

where

$$\varphi_n = \int_0^1 y^{-m} \varphi(y) Y_n(y) dy.$$

Proof. We show the equality

$$y^{-\frac{m}{2}}\varphi(y) = \int_0^1 \overline{G}(y,\xi)\left((-1)^k \xi^{\frac{m}{2}} \frac{d^{2k}\varphi(\xi)}{d\xi^{2k}}\right)d\xi,$$

really

$$\int_0^1 \xi^{-\frac{m}{2}}\left[(-1)^k G(y,\xi)\right] y^{-\frac{m}{2}}\left((-1)^k \xi^{\frac{m}{2}} \frac{d^{2k}\varphi(\xi)}{d\xi^{2k}}\right)d\xi =$$

$$= y^{-\frac{m}{2}} \int_0^1 G(y,\xi)\frac{d^{2k}\varphi(\xi)}{d\xi^{2k}}d\xi = y^{-\frac{m}{2}}\varphi(y).$$

Those for the function $y^{-\frac{m}{2}}\varphi(y)$ the conditions of the Hilbert – Schmidt theorem are satisfied and, therefore,

$$y^{-\frac{m}{2}}\varphi(y) = \sum_{n=1}^{\infty} y^{-\frac{m}{2}} \varphi_n Y_n(y),$$

dividing by $y^{-\frac{m}{2}}$, we have

$$\varphi(y) = \sum_{n=1}^{\infty} \varphi_n Y_n(y).$$

Theorem 1 is proved.

In what follows we will assume that the function $\varphi(x)$ satisfies the conditions of Theorem 1. We proceed to solve the equation in the variable x. Taking into account condition (5), we obtain the following initial problem:

$$D_{0x}^{\alpha} X_n(x) = -\lambda_n x^s X_n(x),$$
$$\lim_{x\to 0} x^{1-\alpha} X_n(x) = \frac{\varphi_n}{\Gamma(\alpha)}, \tag{10}$$

where

$$\varphi_n = \int_0^1 \varphi(y) y^{-m} Y_n(y) dy.$$

Using the results of [17], the solution to problem (10) can be written in the form

$$X_n(x) = \frac{\varphi_n x^{\alpha-1}}{\Gamma\left(\dfrac{\alpha}{\alpha+s}\right)\ldots\Gamma\left(\dfrac{\alpha+s-1}{\alpha+s}\right)} \times$$
$$\times \sum_{m=0}^{\infty} \frac{\Gamma\left(m+\dfrac{\alpha}{\alpha+s}\right)\ldots\Gamma\left(m+\dfrac{\alpha+s-1}{\alpha+s}\right)\Gamma(m+1)}{\Gamma((\alpha+s)m+\alpha)} \frac{(-\lambda_n(\alpha+s)x^{\alpha+s})^m}{m!}.$$

This representation implies the uniqueness of the solution to problem (10).

Because $\alpha+s+1 > s+1$, then the last series converges absolutely and uniformly for fixed values of λ_n and for bounded values of x (see [18]). This means that the permutation of the series and the integral in above was legal.

In terms of special functions, the solution to problem (10) can be written in the form

$$X_n(x) = \frac{\varphi_n x^{\alpha-1}}{\Gamma\left(\dfrac{\alpha}{\alpha+s}\right)\ldots\Gamma\left(\dfrac{\alpha+s-1}{\alpha+s}\right)} \times$$
$$\times {}_{s+1}\Psi_1\left(\left(\begin{matrix}\left(1,\dfrac{\alpha}{\alpha+s}\right),\ldots,\left(1,\dfrac{\alpha+s-1}{\alpha+s}\right),(1,1), -\lambda_n(\alpha+s)^s x^{\alpha+s} \\ (\alpha+s,\alpha)\end{matrix}\right)\right),$$

where

$${}_p\Psi_q\left(\left(\begin{matrix}(\alpha_1,a_1),\ldots,(\alpha_p,a_p), z \\ (\beta_1,b_1),\ldots,(\beta_q,b_p)\end{matrix}\right)\right) = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i m + a_i)}{\prod_{i=1}^{q}\Gamma(\beta_i m + b_i)} \frac{z^m}{m!}$$

is the generalized Wright function (see [18]).

Using the results obtained in [18, 19] (Theorem 4), we obtain an asymptotic expansion of the generalized Wright function for large values λ_n and $x > \delta > 0$:

$$
{}_{s+1}\Psi_1\left(\left(\left(1, \frac{\alpha}{\alpha+s}\right), \ldots, \left(1, \frac{\alpha+s-1}{\alpha+s}\right), (1,1), -\lambda_n(\alpha+s)x^{\alpha+s} \atop (\alpha+s, \alpha)\right)\right) \sim
$$

$$
\sim H_{s+1,1}\big(\lambda_n(\alpha+s)x^{\alpha+s}\big) = H_{s+1,1}(t) =
$$

$$
= \sum_{m=1}^{s} t^{-\frac{\alpha+m-1}{\alpha+s}} S_{s+1,1}(t;m) + t^{-1} S_{s+1,1}(t;s+1),
$$

where

$$
S_{s+1,1}(t;m) =
$$

$$
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma\left(k + \dfrac{\alpha+m-1}{\alpha+s}\right)\Gamma\left(1 - \left(k + \dfrac{\alpha+m-1}{\alpha+s}\right)\right) \prod_{r=1, r\neq m}^{s} \Gamma\left(\dfrac{r-m}{\alpha+s} - k\right)}{\Gamma\left(\alpha - (\alpha+s)\left(k + \dfrac{\alpha+m-1}{\alpha+s}\right)\right)} t^{-k} =
$$

$$
= (-1)^m \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\prod_{r=1, r\neq m}^{s} \dfrac{-\pi}{(-1)^{k+1} \sin \pi \dfrac{r-m}{\alpha+s} \Gamma\left(k - \dfrac{r-m}{\alpha+s} + 1\right)}}{\dfrac{\sin \pi \dfrac{\alpha+m-1}{\alpha+s}}{\sin \pi((\alpha+s)k)\Gamma((\alpha+s)k+m)}} t^{-k} =
$$

$$
= (-1)^{m+s}\pi^{s-1} \sum_{k=0}^{\infty} \frac{(-1)^{s(k+1)}}{k!} \frac{\sin \pi((\alpha+s)k)\Gamma((\alpha+s)k+m)}{\sin \pi \dfrac{\alpha+m-1}{\alpha+s} \prod_{r=1, r\neq m}^{s} \sin \pi \dfrac{r-m}{\alpha+s} \Gamma\left(k - \dfrac{r-m}{\alpha+s} + 1\right)} t^{-k},
$$

$$
S_{s+1,1}(t;s+1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(k+1) \prod_{r=1}^{s} \Gamma\left(\dfrac{\alpha+r-1}{\alpha+s} - k - 1\right)}{\Gamma(\alpha - (\alpha+s)(k+1))} t^{-k} =
$$

$$
= (-1)^{s+1}\pi^{s-1} \sum_{k=0}^{\infty} \frac{(-1)^{k+s(k+1)}\Gamma((\alpha+s)k+s+1) \sin \pi\big((\alpha+s)k\big)}{\prod_{r=1}^{s} \Gamma\left(k + 2 - \dfrac{\alpha+r-1}{\alpha+s}\right) \sin \pi \dfrac{\alpha+r-1}{\alpha+s}} t^{-k}.
$$

Taking into account the latter, we have that there exists a number K such that, for all $n > K$ and for $x > \delta > 0$, the estimate

$$
\big|X_n(x)\big| \leq \frac{M}{\lambda_n} |\varphi_n| x^{\alpha-1}, \qquad 0 < M \quad \text{is constant.}
$$

So, the formal solution of the posed problem A has the form

$$
u(x,y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y). \tag{11}
$$

Let us show that (11) is a classical solution to equation (1). We have

$$\big|u(x,y)\big| \le \sum_{n=0}^{\infty} \big|X_n(x)\big|\big|Y_n(y)\big| \le Mx^{\alpha-1}\sum_{n=0}^{\infty}|\varphi_n|\frac{\big|Y_n(y)\big|}{\lambda_n}.$$

We apply the Cauchy – Bunyakovsky inequality

$$\sum_{n=0}^{\infty}|\varphi_n|\frac{\big|Y_n(y)\big|}{\lambda_n} \le \sqrt{\sum_{n=0}^{\infty}\varphi_n^2}\sqrt{\sum_{n=0}^{\infty}\left(\frac{Y_n(y)}{\lambda_n}\right)^2}.$$

Taking into account inequality (9) and the Bessel inequality

$$\sum_{n=0}^{\infty}\varphi_n^2 \le \int_0^1 \varphi^2(y)y^{-m}dy < \infty,$$

we obtain uniform convergence of series (11) in any closed subdomain Ω and the condition $x^{1-\alpha}u(x,y) \in C(\overline{\Omega}_x \times \Omega_y)$. Let us now turn to the proof of the legality of differentiation. We will act in the same way as above

$$\big|D_{0x}^{\alpha}u(x,y)\big| \le \sum_{n=0}^{\infty}|D_{0x}^{\alpha}X_n(x)|\big|Y_n(y)\big| \le$$

$$\le x^s\sum_{n=0}^{\infty}\lambda_n|X_n(x)|\big|Y_n(y)\big| \le Mx^{s+\alpha-1}\sum_{n=0}^{\infty}\big|\varphi_n\big|\big|Y_n(y)\big|.$$

Next, we apply the Cauchy – Bunyakovsky inequality

$$\sum_{n=0}^{\infty}\big|\varphi_n\big|\big|Y_n(y)\big| = \sum_{n=0}^{\infty}|\lambda_n\varphi_n|\left|\frac{Y_n(y)}{\lambda_n}\right| \le \sqrt{\sum_{n=1}^{\infty}\big|\lambda_n^2\varphi_n^2\big|}\sqrt{\sum_{n=0}^{\infty}\left|\frac{Y_n^2(y)}{\lambda_n^2}\right|},$$

we have

$$\varphi_n = \int_0^1 y^{-m}\varphi(y)Y_n(y)dy = \frac{(-1)^k}{\lambda_n}\int_0^1 \varphi(y)Y_n^{(2k)}(y)dy =$$

$$= \frac{(-1)^k}{\lambda_n}\left[\varphi(y)Y_n^{(2k-1)}(y)\Big|_0^1 - \int_0^1 \varphi'(y)Y_n^{(2k-1)}(y)dy\right].$$

If

$$\lim_{y\to+0} Y_n^{(2k-1)}(y) \ne \infty,$$

then

$$\varphi(0)Y_n^{(2k-1)}(0) = 0,$$

and if

$$\lim_{y\to+0} Y_n^{(2k-1)}(y) = \infty,$$

then, applying L'Hopital's rule, we get

$$\lim_{y\to+0} \frac{Y_n^{(2k-1)}(y)}{(\varphi(y))^{-1}} = \lim_{y\to+0} \frac{Y_n^{(2k)}(y)}{-(\varphi(y))^{-2}\varphi'(y)} = (-1)^{k+1}\lambda_n \lim_{y\to+0} \frac{\varphi^2(y)y^{-m}Y_n(y)}{\varphi'(y)} =$$

$$= (-1)^{k+1}\lambda_n \lim_{y\to+0} \frac{O(y^{2k})O(y^k)y^{-m}}{O(y^{k-1})} = 0,$$

whence

$$\varphi_n = \frac{(-1)^{k+1}}{\lambda_n} \int_0^1 \varphi'(y) Y_n^{(2k-1)}(y)dy = \frac{(-1)^k}{\lambda_n} \int_0^1 \varphi^{(2k)}(y) Y_n(y)dy \Rightarrow$$

$$\Rightarrow \lambda_n \varphi_n = \int_0^1 \Big[(-1)^k y^m \varphi^{(2k)}(y)\Big] Y_n(y) y^{-m} dy.$$

Hence, $\lambda_n \varphi_n$ are the Fourier coefficients of the function $(-1)^k y^m \varphi^{(2k)}(y)$. Then, by Bessel's inequality, we obtain

$$\sum_{n=0}^{\infty} \lambda_n^2 \big|\varphi_n(y)\big|^2 \le \int_0^1 y^m \big(\varphi^{(2k)}(y)\big)^2 dy. \tag{12}$$

Now, in order for the calculations made above to be legal, we impose the following restrictions on the function $\varphi(y)$:

$$\varphi^{(j)}(0) = \varphi^{(j)}(1) = 0, \qquad \varphi(y) \in C^{2k}[0,1], \quad j = 0, 1, \ldots, k-1.$$

Taking into account (9) and (12), we have that the series

$$D_{0x}^{\alpha} u(x,y) = \sum_{n=0}^{\infty} D_{0x}^{\alpha} X_n(x) Y_n(y)$$

converges uniformly in any closed subdomain Ω for $s = 0$ and converges uniformly in $\overline{\Omega}_x \times \Omega_y$ for $s = 1, 2, 3, \ldots$. The uniform convergence of the series

$$\frac{\partial^{2k} u(x,y)}{\partial y^{2k}} = \sum_{n=0}^{\infty} X_n(x) \frac{\partial^{2k} Y_n(y)}{\partial y^{2k}} = (-1)^k y^{-m} \sum_{n=0}^{\infty} \lambda_n X_n(x) Y_n(y). \tag{13}$$

Theorem 1 is proved.

Theorem 1′. *Let the function $\varphi(y)$ satisfies the following conditions*:

$$\varphi(y) \in C^{2k}[0,1], \qquad \varphi^{(j)}(0) = \varphi^{(j)}(1) = 0, \quad j = 0, 1, \ldots, k-1.$$

Then a solution to Problem A exists.

Remark. It can be seen from the construction of the solution that (see (13))

$$\frac{\partial^{2k} u(x,y)}{\partial y^{2k}} \in C\big(\Omega_x \times \overline{\Omega}_y\big). \tag{14}$$

3. Uniqueness.

Theorem 2. *If there is a solution to problem A from class* (2), (14), *then it is unique.*

Proof. Let the function $u(x,y)$ be a solution to Problem A with zero initial and boundary conditions. Consider its Fourier coefficients with respect to the system of eigenfunctions of problem (6)

$$u_n(x) = \int\limits_0^1 y^{-m} u(x,y) Y_n(y)dy,$$

it is easy to show that $u_n(x)$ is a solution to the problem

$$D_{0x}^{\alpha} u_n(x) = -\lambda_n x^s u_n(x), \qquad \lim_{x\to 0}(x^{1-\alpha}u_n(x)) = 0.$$

This problem has only a zero solution, i.e.,

$$\int\limits_0^1 y^{-m} u(x,y)Y_n(y)dy = 0 \quad \text{for all} \quad n.$$

Because $\overline{G}(y,\xi)$ symmetric, continuous,

$$\int\limits_0^1 \overline{G}^2(y,\xi)d\xi < \infty, \quad \int\limits_0^1 \overline{G}^2(y,\xi)dy < \infty, \quad \int\limits_0^1\int\limits_0^1 \overline{G}^2(y,\xi)dyd\xi < \infty,\ \lambda_n > 0 \quad \text{for all} \quad n,$$

then the conditions of Mercer's theorem are fulfilled and

$$\overline{G}(y,\xi) = \sum_{n=0}^{\infty} \frac{\overline{Y_n}(y)\overline{Y_n}(\xi)}{\lambda_n}.$$

Hence, we have

$$y^{-\frac{m}{2}}u(x,y) = \int\limits_0^1 \overline{G}(y,\xi)\left((-1)^k \xi^{\frac{m}{2}} \frac{\partial^{2k}u(x,\xi)}{\partial\xi^{2k}}\right)d\xi =$$

$$= (-1)^k \int\limits_0^1 \sum_{n=0}^{\infty} \frac{\overline{Y_n}(y)\overline{Y_n}(\xi)}{\lambda_n}\left(\xi^{\frac{m}{2}} \frac{\partial^{2k}u(x,\xi)}{\partial\xi^{2k}}\right)d\xi =$$

$$= (-1)^k \sum_{n=0}^{\infty} \frac{y^{-\frac{m}{2}}Y_n(y)}{\lambda_n} \int\limits_0^1 \xi^{-\frac{m}{2}} Y_n(\xi)\xi^{\frac{m}{2}} \frac{\partial^{2k}u(x,\xi)}{\partial\xi^{2k}}d\xi =$$

$$= (-1)^k \sum_{n=0}^{\infty} \frac{y^{-\frac{m}{2}}Y_n(y)}{\lambda_n} \int\limits_0^1 Y_n(\xi) \frac{\partial^{2k}u(x,\xi)}{\partial\xi^{2k}}d\xi =$$

$$= (-1)^k \sum_{n=0}^{\infty} \frac{y^{-\frac{m}{2}}Y_n(y)}{\lambda_n} \int\limits_0^1 Y_n^{(2k)}(\xi)u(x,\xi)d\xi =$$

$$= (-1)^k \sum_{n=0}^{\infty} \frac{y^{-\frac{m}{2}} Y_n(y)}{\lambda_n} \int_0^1 \lambda_n (-1)^k \xi^{-m} Y_n(\xi) u(x,\xi) d\xi =$$

$$= y^{-\frac{m}{2}} \sum_{n=0}^{\infty} Y_n(y) \int_0^1 \xi^{-m} Y_n(\xi) u(x,\xi) d\xi = 0 \Rightarrow u(x,y) \equiv 0.$$

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