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PSEUDO ALMOST PERIODIC SOLUTIONS IN THE ALPHA-NORM AND STEPANOV'S SENSE FOR SOME EVOLUTION EQUATIONS

ПСЕВДО-МАЙЖЕ ПЕРІОДИЧНІ РОЗВ'ЯЗКИ В АЛЬФА-НОРМІ ТА В РОЗУМІННІ СТЕПАНОВА ДЛЯ ДЕЯКИХ ЕВОЛЮЦІЙНИХ РІВНЯНЬ

Our aim is to present the concept of double-measure ergodic and double-measure pseudo almost periodic functions in Stepanov's sense. In addition, we present numerous interesting results, such as the composition theorems and completeness properties for these two spaces of the considered functions. We also establish the existence and uniqueness for the double-measure pseudo almost periodic mild solutions in Stepanov's sense for some evolution equations.

Введено поняття ергодичних функцій подвійної міри та псевдо-майже періодичних функцій подвійної міри в розумінні Степанова. Крім того, наведено багато цікавих результатів, що включають як теореми про композицію, так і властивості повноти для цих двох просторів розглянутих функцій. Встановлено також існування та єдиність псевдо-майже періодичних слабких розв'язків подвійної міри в розумінні Степанова для деяких еволюційних рівнянь.

1. Introduction and preliminaries. Roughly by 1924 – 1926, the Danish mathematician Bohr [3] pioneered the almost periodic functions theory that generalize the notion of periodicity, the so-called almost periodicity is very useful in distinct domains involving harmonic analysis, dynamical systems, physics, etc. C. Zhang [10, 11] defined the concept of pseudo almost periodicity, as a natural generalization of the notion of almost periodicity. Lately, T. Diagana [5] initiated the concept of Stepanov pseudo almost periodicity as a generalization of pseudo almost periodicity.

In this paper, we prove the existence and uniqueness of double-measure pseudo almost periodic (or (μ,ν)-PAP) solutions in Stepanov's sense for the equation

$$\begin{aligned}&\frac{d}{dt}\big[v(t)-g(t,v(t))\big]=-A\big[v(t)-g(t,v(t))\big]+f(t,v(t)) \quad \text{for} \quad t\geq\sigma,\\&v(\sigma)=v_\sigma\in X_\alpha,\end{aligned} \tag{1}$$

such that $-A$ is the generator of a semigroup $(T(t))_{t\geq 0}$ on a Banach space $(X,\|.\|)$. For $0<\alpha<1$, the domain of the operator A^α is denoted by $D(A^\alpha)$ and the Banach space $X_\alpha:=(D(A^\alpha),\|.\|_\alpha)$, where $\|x\|_\alpha=\|A^\alpha x\|$. Furthermore, we suppose that

(H0) $\exists M_\alpha,\,\omega>0$ satisfying

$$\|A^\alpha T(t)\|\leq M_\alpha\frac{e^{-\omega t}}{t^\alpha} \quad \text{for} \quad t>0.$$

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For more details on the operator A^{α} we can see [4, 9].

The research aim in this paper is to study basic properties of pseudo almost periodic functions with measure in Stepanov's sense, some composition theorems and their extensions. Specifically, the notions of (μ,ν) pseudo almost periodicity in Stepanov's sense and the double-measure pseudo almost periodic mild solutions in Stepanov's sense for equation (1) will be introduced.

Throughout this paper $(X,\|.\|)$, $(Y,\|.\|)$ be two Banach spaces. Denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ with $\mu(\mathbb{R})=+\infty$ and $\mu([a,b])<\infty$ $\forall a,b\in\mathbb{R},\ a\leq b$.

We consider the following hypothesis taken from [2].

(M0) For $\mu\ \nu\in\mathcal{M}$,

$$\limsup_{r\longrightarrow+\infty}\frac{\mu([-r,r])}{\nu([-r,r])}:=M<\infty.$$

(M1) For all $a,\ b,\ c\in\mathbb{R}$, with $0\leq a<b\leq c$, there exist $\tau_0,\ \alpha_0\geq 0$ satisfying

$$|\ \tau\ |\geq\tau_0\Longrightarrow\mu((a+\tau,b+\tau))\geq\alpha_0\mu([\tau,c+\tau]).$$

(M2) For all $\tau\in\mathbb{R}$, there exist $\beta>0$ and $I=(c,d)\subset\mathbb{R}$ satisfying

$$\mu(\{a+\tau:a\in A\})\leq\beta\mu(A)\quad\text{with}\quad A\in\mathbf{B}\quad\text{and}\quad A\cap I=\varnothing.$$

Firstly, we recall some useful definitions:

1. $AP(\mathbb{R},X)$ is the set of all continuous functions $f\in X^{\mathbb{R}}$ such that $\forall\varepsilon>0\ \exists l(\varepsilon)>0$ $\forall(a,b)\subset\mathbb{R}:b-a<\varepsilon\Rightarrow\exists\tau\in(a,b)$ satisfying

$$\|f(t+\tau)-f(t)\|<\varepsilon\quad\forall t\in\mathbb{R}.$$

Such functions are named almost periodic.

2. $\mathcal{E}(\mathbb{R},X,\mu,\nu)$ is the space of $f\in X^{\mathbb{R}}$ such that

$$\lim_{r\longrightarrow+\infty}\frac{1}{\nu([-r,r])}\int\limits_{[-r,r]}\|f(s)\|d\mu(s)=0,$$

where μ and ν are two positive measures. Such functions are named ergodic.

3. $\mathcal{E}U(\mathbb{R}\times Y,X,\mu,\nu)$ is the set of all continuous functions $f\in X^{\mathbb{R}\times Y}$ such that $f(.,y)\in$ $\in\mathcal{E}(\mathbb{R},X,\mu,\nu)$ $\forall y\in Y$ and $f:y\longmapsto f(.,y)$ is uniformly continuous on each compact $K\subset Y$, where μ and ν are two positive measures. Such functions are named ergodic uniformly.

4. $PAP(\mathbb{R},X,\mu,\nu)=AP(\mathbb{R},X)+\mathcal{E}(\mathbb{R},X,\mu,\nu)$ is the set of pseudo almost periodic functions.

5. Let $f\in X^{\mathbb{R}}$ be a function. The function given by

$$\begin{array}{ccc}f^b:\mathbb{R} & \to & L^p((0;1);X),\\ t & \mapsto & f^b(t)=f(t+.)\end{array}$$

is said to be a Bochner transform of f.

6. The boundedness in Stepanov's sense of a function $f\in L^p_{\mathrm{loc}}(\mathbb{R},X),\ p\geq 1$, is characterized by

$$\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} = \sup_{t\in\mathbb{R}}\left(\int_0^1\|f(t+s)\|^p ds\right)^{\frac{1}{p}} < \infty,$$

which define a norm on the set $BS^p(\mathbb{R},X)$ of such function, i.e.,

$$\|f\|_{BS^p} = \sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} = \sup_{t\in\mathbb{R}}\|f(t+.)\|_p.$$

7. For $f\in BS^p(\mathbb{R},X)$, $p\geq 1$, we say that f is almost periodic in Stepanov's sense (or S^p-almost periodic), if $f^b\in AP(\mathbb{R},L^p((0,1),X))$. The set of such function is denoted by $S^pAP(\mathbb{R},X)$.

8. A function $f\in X^{\mathbb{R}\times Y}$, $p\geq 1$, where $f(.,u)\in BS^p(\mathbb{R},X)$ $\forall u\in Y$, is said to be S^p-almost periodic in $t\in\mathbb{R}$ uniformly for $u\in Y$ if for all $\varepsilon>0$ and $K\subset Y$, compact, there exists a relatively dense set $P=P(\varepsilon,f,K)$ satisfying

$$\sup_{t\in\mathbb{R}}\left(\int_0^1\|f(t+s+\tau,u)-f(t+s,u)\|^p ds\right)^{\frac{1}{p}} < \varepsilon \quad \text{for all} \quad t\in\mathbb{R}, \quad \tau\in P, \quad u\in K.$$

The set of such functions is denoted by $S^pAPU(\mathbb{R}\times Y,X)$.

The following interest results are useful in the sequel:

1. $BS^p(\mathbb{R};X)$ is a Banach space under the norm $\|.\|_{BS^p}$ (see [7, 8]).
2. $(S^pAP(\mathbb{R},X),\|.\|_{BS^p})$ is a Banach space under the norm $\|.\|_{BS^p}$ (see [8]).

2. Main results. ***2.1. (μ,ν)-Ergodic functions in Stepanov's sense.*** Motivited by Diagana et al. in [6], we define the (μ,ν)-ergodic functions in Stepanov's sense and give some properties. The most important theorems and properties are contained in this subsection.

Definition 1. *For $f\in BS^p(\mathbb{R};X)$, we say that f is (μ,ν)-ergodic in Stepanov's sense (or $S^p-(\mu,\nu)$-ergodic) if $f^b\in\mathcal{E}(\mathbb{R},L^p((0;1);X),\mu,\nu)$, i.e.,*

$$\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) = 0,$$

where μ, $\nu\in\mathcal{M}$ and $p\geq 1$. The set of such functions is denoted $\mathcal{E}^p(\mathbb{R},X,\mu,\nu)$.

Theorem 1. *For $p\geq 1$ and μ, $\nu\in\mathcal{M}$ satisfy* **(M0)**, *the space $(\mathcal{E}^p(\mathbb{R},X,\mu,\nu),\|.\|_{BS^p})$ is a Banach space.*

Proof. Let f, $g\in\mathcal{E}^p(\mathbb{R},X,\mu,\nu)$ and $\lambda\in\mathbb{C}$. It is obvious to see that $\lambda f+g\in\mathcal{E}^p(\mathbb{R},X,\mu,\nu)\subset$ $\subset BS^p(\mathbb{R},X)$. It is sufficient to show that $(\mathcal{E}^p(\mathbb{R},X,\mu,\nu)$ is closed in $BS^p(\mathbb{R},X)$.

For $(f_n)_n\subset\mathcal{E}^p(\mathbb{R},X,\mu,\nu)$ satisfying $\lim_{n\to+\infty}\|f_n-f\|_{BS^p}=0$. It follows that

$$\int_{[-r,r]}\left(\int_t^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) \leq \int_{[-r,r]}\left(\int_t^{t+1}\|f(s)-f_n(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) +$$

$$+ \int\limits_{[-r,r]} \left(\int\limits_t^{t+1} \|f_n(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \leq$$

$$\leq \int\limits_{[-r,r]} \|f - f_n\|_{BS^p} d\mu(t) + \int\limits_{[-r,r]} \left(\int\limits_t^{t+1} \|f_n(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t).$$

Then

$$\frac{1}{\nu([-r,r])} \int\limits_{[-r,r]} \left(\int\limits_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \leq \frac{\mu([-r,r])}{\nu([-r,r])} \|f - f_n\|_{BS^p} +$$

$$+ \frac{1}{\nu([-r,r])} \int\limits_{[-r,r]} \left(\int\limits_t^{t+1} \|f_n(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t).$$

Therefore,

$$\lim_{r \to +\infty} \frac{1}{\nu([-r,r])} \int\limits_{[-r,r]} \left(\int\limits_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \leq M \|f - f_n\|_{BS^p} \quad \forall n \in \mathbb{N}.$$

Since $\lim_{n \to +\infty} \|f - f_n\|_{BS^p} = 0$, one can see that

$$\lim_{r \to +\infty} \frac{1}{\nu([-r,r])} \int\limits_{[-r,r]} \left(\int\limits_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) = 0.$$

Theorem 1 is proved.

Proposition 1. *The space $\mathcal{E}^p(\mathbb{R}, X, \mu, \nu)$ is translation invariant, where $\mu, \nu \in \mathcal{M}$ satisfy* **(M2)** *and $p \geq 1$.*

Proof. Suppose that $f \in \mathcal{E}^p(\mathbb{R}, X, \mu, \nu)$, we define F by $F(t) = \left(\int_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}}$. Then F belongs to $\mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$ and, from the translation invariance of $\mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$, we have

$$\frac{1}{\nu([-r,r])} \int\limits_{[-r,r]} \left(\int\limits_t^{t+1} \|f(s+a)\|^p ds \right)^{\frac{1}{p}} d\mu(t) =$$

$$= \frac{1}{\nu([-r,r])} \int\limits_{[-r,r]} F(t+a) d\mu(t) \to 0 \quad \text{as} \quad r \to +\infty.$$

Therefore, $f(.+a) \in \mathcal{E}^p(\mathbb{R}, X, \mu, \nu)$.

Theorem 2. *$\mathcal{E}(\mathbb{R}, X, \mu, \nu) \subseteq \mathcal{E}^p(\mathbb{R}, X, \mu, \nu)$, where μ, $\nu \in \mathcal{M}$ satisfy* **(M0)** *and* **(M2)**, $p \geq 1$.

Proof. Suppose that $f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. By Hölder's inequality, we get

$$\int\limits_{[-r,r]} \left(\int\limits_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) = \int\limits_{[-r,r]} \left(\int\limits_0^1 \|f(s+t)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \leq$$

$$\leq \left(\int\limits_{[-r,r]} d\mu(t) \right)^{1-\frac{1}{p}} \left(\int\limits_{[-r,r]} \left(\int\limits_0^1 \|f(s+t)\|^p ds \right) d\mu(t) \right)^{\frac{1}{p}} =$$

$$= (\mu([-r,r]))^{1-\frac{1}{p}} \left(\int\limits_{[-r,r]} \left(\int\limits_0^1 \|f(s+t)\|^{p-1} \|f(s+t)\| ds \right) d\mu(t) \right)^{\frac{1}{p}} \leq$$

$$\leq (\mu([-r,r]))^{1-\frac{1}{p}} \|f\|_\infty^{\frac{p-1}{p}} \left(\int\limits_{[-r,r]} \left(\int\limits_0^1 \|f(s+t)\| ds \right) d\mu(t) \right)^{\frac{1}{p}}.$$

Using the Fubini theorem, we have

$$\int\limits_{[-r,r]} \left(\int\limits_0^1 \|f(s+t)\| ds \right) d\mu(t) = \int\limits_0^1 \left(\int\limits_{[-r,r]} \|f(s+t)\| d\mu(t) \right) ds.$$

It follows that

$$\frac{1}{\nu([-r,r])} \int\limits_{[-r,r]} \left(\int\limits_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \leq$$

$$\leq \frac{1}{\nu([-r,r])} (\mu([-r,r]))^{1-\frac{1}{p}} \|f\|_\infty^{\frac{p-1}{p}} \left(\int\limits_0^1 \left(\int\limits_{[-r,r]} \|f(s+t)\| d\mu(t) \right) ds \right)^{\frac{1}{p}} =$$

$$= \left(\frac{\mu([-r,r])}{\nu([-r,r])} \right)^{1-\frac{1}{p}} \|f\|_\infty^{\frac{p-1}{p}} \left(\int\limits_0^1 \left(\frac{1}{\nu([-r,r])} \int\limits_{[-r,r]} \|f(s+t)\| d\mu(t) \right) ds \right)^{\frac{1}{p}}.$$

From translation invariance of $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$, we have

$$\lim_{r \longrightarrow +\infty} \frac{1}{\nu([-r,r])} \int\limits_{[-r,r]} \|f(s+t)\| d\mu(t) = 0 \quad \forall s \in [0,1].$$

Apply the Lebesgue dominated convergence, we get

$$\lim_{r\longrightarrow+\infty}\frac{1}{\nu([-r,r])}\int\limits_{[-r,r]}\left(\int\limits_t^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}d\mu(t)=0.$$

Theorem 2 is proved.

***Example* 1.** Let $p\geq 1$ and the function f given on $\mathbb{R}$ by

$$f(t)=\frac{1}{[t]}\chi_{[[t],[t]+\frac{1}{2^p[t]^p}]\cap[1,+\infty[}(t).$$

Obviously, f is not continuous on $\mathbb{R}$, then f is not ergodic. But f is ergodic in Stepanov's sense. Indeed, if we take $r>1$ and $p\geq 1$, we have

$$\frac{1}{2r}\int\limits_{-r}^{r}\left(\int\limits_t^{t+1}(f(s))^p ds\right)^{\frac{1}{p}}dt\leq\frac{1}{2r}\int\limits_1^{+\infty}\left(\int\limits_t^{t+1}(f(s))^p ds\right)^{\frac{1}{p}}dt\leq$$

$$\leq\frac{1}{2r}\int\limits_1^{+\infty}\left(\int\limits_{[t]}^{[t]+2}(f(s))^p ds\right)^{\frac{1}{p}}dt\leq$$

$$\leq\frac{2}{2r}\sum_{k=1}^{+\infty}\int\limits_k^{k+1}\left(\int\limits_k^{k+\frac{1}{2^pk^p}}(f(s))^p ds\right)^{\frac{1}{p}}dt\leq$$

$$\leq\frac{1}{2r}\sum_{k=1}^{+\infty}\frac{1}{k^2}=\frac{\pi^2}{12r}\longrightarrow 0\quad\text{as}\quad r\longrightarrow+\infty.$$

A characterization of (μ,ν)-ergodic functions in Stepanov's sense is given by the following proposition.

Proposition 2. *Let* $\mu,\nu\in\mathcal{M}$ *and* I *be a bounded interval* (*eventually* $I=\phi$). *Let* $f\in$ $\in BS^p(\mathbb{R},X)$. *Then the following statements are equivalent*:

1) $f\in\mathcal{E}^p(\mathbb{R},X,\mu,\nu)$,

2) $\lim_{r\to\infty}\dfrac{1}{\nu([-r,r]\setminus I)}\displaystyle\int_{[-r,r]\setminus I}\left(\int_t^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}d\mu(t)=0$,

3) *for any* $\varepsilon>0$, $\lim_{r\to\infty}\dfrac{\mu\left(\{t\in[-r,r]\setminus I:\left(\int_t^{t+1}\|f(t)\|^p ds\right)^{\frac{1}{p}}>\varepsilon\}\right)}{\nu([-r,r]\setminus I)}=0.$

Proof. Let $A=\nu(I)$, $B=\displaystyle\int_I\left(\int_t^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}d\mu(t)$, $C=\mu(I)$.

(i)⇒(ii). We have

$$\frac{1}{\nu([-r,r]\setminus I)}\int\limits_{[-r,r]\setminus I}\left(\int\limits_t^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}d\mu(t)=$$

$$= \frac{1}{\nu([-r,r]-A)}\left(\int\limits_{[-r,r]}\left(\int\limits_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) - B\right) =$$

$$= \frac{\nu([-r,r])}{\nu([-r,r]-A)}\left(\frac{1}{\nu([-r,r])}\int\limits_{[-r,r]}\left(\int\limits_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) - \frac{B}{\nu([-r,r])}\right).$$

(iii)⇒(ii). Let

$$A_r^{\varepsilon} = \left\{ t\in[-r,r]\setminus I \colon \int\limits_{I}\left(\int\limits_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} > \varepsilon \right\}$$

and

$$B_r^{\varepsilon} = \left\{ t\in[-r,r]\setminus I \colon \int\limits_{I}\left(\int\limits_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} \le \varepsilon \right\}.$$

Suppose that (iii) holds, that is,

$$\lim_{r\to\infty}\frac{\mu(A_r^{\varepsilon})}{\nu([-r,r]\setminus I)} = 0.$$

From the equality

$$\int\limits_{[-r,r]\setminus I}\left(\int\limits_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) = \int\limits_{A_r^{\varepsilon}}\left(\int\limits_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) + \int\limits_{B_r^{\varepsilon}}\left(\int\limits_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t)$$

we deduce that for r sufficiently large

$$\frac{1}{\nu([-r,r]\setminus I)}\int\limits_{[-r,r]\setminus I}\left(\int\limits_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) \le$$

$$\le \|f\|_{BS^p}\frac{\mu(A_r^{\varepsilon})}{\nu([-r,r]\setminus I)} + \varepsilon\frac{\mu(B_r^{\varepsilon})}{\nu([-r,r]\setminus I)} \le$$

$$\le \|f\|_{BS^p}\frac{\mu(A_r^{\varepsilon})}{\nu([-r,r]\setminus I)} + \varepsilon\frac{\mu([-r,r]\setminus I)}{\nu([-r,r]\setminus I)} \le$$

$$\le \|f\|_{BS^p}\frac{\mu(A_r^{\varepsilon})}{\nu([-r,r]\setminus I)} + \varepsilon\frac{\mu([-r,r]-C)}{\nu([-r,r]-A)} \le$$

$$\le \|f\|_{BS^p}\frac{\mu(A_r^{\varepsilon})}{\nu([-r,r]\setminus I)} + \varepsilon\frac{\mu([-r,r])}{\nu([-r,r])}\left(\frac{1-\dfrac{C}{\mu([-r,r])}}{1-\dfrac{A}{\mu([-r,r])}}\right).$$

Then, for all $\varepsilon > 0$,

$$\limsup_{r\to\infty} \frac{1}{\nu([-r,r]\setminus I)} \int\limits_{[-r,r]\setminus I} \left(\int\limits_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) < \varepsilon.$$

Therefore, (ii) holds.

(ii)⇒(iii). We have

$$\frac{1}{\nu([-r,r]\setminus I)} \int\limits_{[-r,r]\setminus I} \left(\int\limits_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \geq$$

$$\geq \frac{1}{\nu([-r,r]\setminus I)} \int\limits_{A_r^\varepsilon} \left(\int\limits_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \geq$$

$$\geq \frac{\varepsilon}{\nu([-r,r]\setminus I)} \mu_{(A_r^\varepsilon)}.$$

Proposition 2 is proved.

2.2. (μ,ν)-Pseudo almost periodic functions in Stepanov's sense.

Definition 2. *A function $f \in X^{\mathbb{R}}$ is said to be (μ,ν)-pseudo almost periodic in Stepanov's sense if it can be written in the form*

$$f = g + h,$$

where $g \in S^pAP(\mathbb{R},X)$ and $h \in \mathcal{E}^p(\mathbb{R},X,\mu,\nu)$. The set of such functions is denoted by $S^pPAP(\mathbb{R},X,\mu,\nu)$.

We have the following diagram:

$$\begin{bmatrix} AP(\mathbb{R},X) & \subset & PAP(\mathbb{R},X,\mu,\nu) & \subset & BC(\mathbb{R},X) & \subset & C(\mathbb{R},X) \\ \cap & & \cap & & \cap & & \cap \\ S^pAP(\mathbb{R},X) & \subset & S^pPAP(\mathbb{R},X,\mu,\nu) & \subset & BS^p(\mathbb{R},X) & \subset & L^p_{\mathrm{loc}}(\mathbb{R},X) \end{bmatrix}.$$

Theorem 3. *Let $\mu,\ \nu \in \mathcal{M}$ satisfy* **(M2)**. *Then the decomposition of a (μ,ν)-pseudo almost periodic function in the form $f = g + h$, where $g \in S^pAP(\mathbb{R},X)$ and $h \in \mathcal{E}^p(\mathbb{R},X,\mu,\nu)$, is unique and so*

$$S^pPAP(\mathbb{R},X,\mu,\nu) = S^pAP(\mathbb{R},X) \oplus \mathcal{E}^p(\mathbb{R},X,\mu,\nu).$$

Proof. Let the operator

$$\begin{array}{cccc} B : BS^p(\mathbb{R},X) & \to & L^\infty(\mathbb{R}, L^p([0;1];X), \\ f & \mapsto & f^b. \end{array}$$

Clearly, this operator is isometric linear. We have $S^pAP(\mathbb{R},X) = B^{-1}(AP(\mathbb{R},L^p((0;1);X)))$ and $\mathcal{E}^p(\mathbb{R},X,\mu,\nu) = B^{-1}(\mathcal{E}(\mathbb{R},L^p((0;1);X),\mu,\nu))$. Using the fact that

$$AP(\mathbb{R},L^p((0;1);X)) \bigcap \mathcal{E}(\mathbb{R},L^p((0;1);X),\mu,\nu) = \{0\}$$

and B is isometric linear, we obtain

$$B^{-1}(AP(\mathbb{R}, L^p((0;1);X))) \bigcap B^{-1}(\mathcal{E}(\mathbb{R}, L^p((0;1);X),\mu,\nu)) = \{0\}$$

and so

$$S^pAP(\mathbb{R},X) \bigcap \mathcal{E}^p(\mathbb{R},X,\mu,\nu) = \{0\}.$$

Theorem 3 is proved.

Theorem 4. *Let $\mu \in \mathcal{M}$ satisfies* **(M1)**. *Then $S^pPAP(\mathbb{R},X,\mu,\nu)$ equipped with the norm*

$$\|f\|_{S^pPAP(\mathbb{R},X,\mu,\nu)} = \|g\|_{S^pAP(\mathbb{R},X)} + \|h\|_{\mathcal{E}^p(\mathbb{R},X,\mu,\nu)},$$

where $f = g + h$, $g \in S^pAP(\mathbb{R},X)$ and $h \in \mathcal{E}^p(\mathbb{R},X,\mu,\nu)$ is a Banach space.

Proof. Let $(f_n)_n$ be a Cauchy sequence in $S^pPAP(\mathbb{R},X,\mu,\nu)$, then, for all $n \in \mathbb{N}$, there exists $(g_n,h_n) \in S^pAP(\mathbb{R},X) \times \mathcal{E}^p(\mathbb{R},X,\mu,\nu)$ such that $f_n = g_n + h_n$.

For $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, $n \geq n_0$, such that, for all m, we have

$$\|f_n - f_m\|_{BSp} = \|g_n - g_m\|_{BSp} + \|h_n - h_m\|_{BSp} < \varepsilon.$$

Then, for all $m,n \geq n_0$, we get

$$\|g_n - g_m\|_{BSp} < \varepsilon \quad \text{and} \quad \|h_n - h_m\|_{BSp} < \varepsilon.$$

Therefore, $(g_n)_n$ and $(h_n)_n$ are Cauchy sequences in the Banach spaces $S^pAP(\mathbb{R},X)$ and $\mathcal{E}^p(\mathbb{R},X,\mu,\nu)$, respectively. Then there exists $(g,h) \in S^pAP(\mathbb{R},X) \times \mathcal{E}^p(\mathbb{R},X,\mu,\nu)$ such that

$$\lim_{n\to+\infty} \|g_n - g\|_{BSp} = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|h_n - h\|_{BSp} = 0.$$

Let $f = g + h \in S^pAP(\mathbb{R},X) \oplus \mathcal{E}^p(\mathbb{R},X,\mu,\nu) = S^pPAP(\mathbb{R},X,\mu,\nu)$, then

$$\lim_{n\to+\infty} \|f_n - f\|_{BSp} = \lim_{n\to+\infty} \|g_n - g\|_{BSp} + \lim_{n\to+\infty} \|h_n - h\|_{BSp} = 0.$$

So, $(S^pPAP(\mathbb{R},X,\mu,\nu)$ is a Banach space.

Theorem 4 is proved.

Definition 3. *For $\mu,\nu \in \mathcal{M}$ and $f \in X^{\mathbb{R}\times Y}$, continuous, we say that f is uniformly (μ,ν)-pseudo almost periodic in Stepanov's sense if $f = g + h$, $g \in S^pAPU(\mathbb{R}\times Y, X)$ and $h \in \mathcal{E}^pU(\mathbb{R}\times \times Y, X, \mu,\nu)$. The set of such functions is designated by $S^pPAPU(\mathbb{R}\times Y, X,\mu,\nu)$.*

The following hypothesis is useful in the rest.

(H1) For all $1 \leq p < \infty$, there exists $L > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\| \quad \forall t \in \mathbb{R} \quad \forall x,y \in Y.$$

Theorem 5 [8]. *Let $1 \leq p < \infty$ and $f \in X^{\mathbb{R}\times Y}$ with $f(.,x) \in S^pAP(\mathbb{R},X)$ $\forall x \in Y$. Assume that f satisfies* **(H1)** *and $x \in AP(\mathbb{R},Y)$. Then $f(.,x(.)) \in S^pAP(\mathbb{R},X)$.*

Theorem 6. *Let $\mu,\nu \in \mathcal{M}$ satisfy* **(M0)**. *Assume that $x \in S^pAP(\mathbb{R},Y)$, $K = \overline{\{x(t); t \in \mathbb{R}\}}$ is a compact subset of Y and $f \in \mathcal{E}^pU(\mathbb{R}\times Y, X,\mu,\nu)$. Then $f(.,x(.)) \in \mathcal{E}^p(\mathbb{R},X,\mu,\nu)$.*

Proof. Let $f \in \mathcal{E}^p U(\mathbb{R} \times Y, X, \mu, \nu)$ and $K = \overline{\{x(t); t \in \mathbb{R}\}}$ be fixed. Then, for all $\varepsilon > 0$, there exists $\delta_{\varepsilon,K}$ such that, for all $x_1, x_2 \in K$, one has

$$\|x_1 - x_2\| \leq \delta_{\varepsilon,K} \Rightarrow \left(\int_t^{t+1} \|f(s, x_1) - f(s, x_2)\|^p \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{M} \quad \forall t \in \mathbb{R}.$$

Since K is a compact, then there exists a finite subset $\{x_1, x_2, \ldots, x_n\} \subset K$, $n \in \mathbb{N}^*$, satisfying $K \subset \bigcup_{i=1}^n B(x_i, \delta_{\varepsilon,K})$. Therefore,

$$\forall\, t \in \mathbb{R} \quad \exists i(t) = 1, \ldots, n \colon \|x(t) - x_{i(t)}\| \leq \delta,$$

$$\left(\int_t^{t+1} \|f(s, x(s))\|^p ds \right)^{\frac{1}{p}} \leq \left(\int_t^{t+1} \|f(s, x(s)) - f(s, x_{i(t)})\|^p ds \right)^{\frac{1}{p}} + \left(\int_t^{t+1} \|f(s, x_{i(t)})\|^p ds \right)^{\frac{1}{p}} \leq$$

$$\leq \frac{\varepsilon}{M} + \sum_{i=1}^n \left(\int_t^{t+1} \|f(s, x_i)\|^p ds \right)^{\frac{1}{p}}.$$

Note that, for all $i = 1, \ldots, n$, $f(., x_i) \in \mathcal{E}^p(\mathbb{R}, X, \mu, \nu)$. Hence, for r large enough,

$$\frac{1}{\nu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|f(s, x(s))\|^p ds \right)^{\frac{1}{p}} d\mu(t) \leq \frac{\varepsilon}{M} \frac{\mu([-r, r])}{\nu([-r, r])} +$$

$$+ \frac{1}{\nu([-r, r])} \sum_{i=1}^n \int_{-r}^{r} \left(\int_t^{t+1} \|f(s, x_i)\|^p ds \right)^{\frac{1}{p}} d\mu(t).$$

Consequently,

$$\limsup_{r \to +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|f(s, x(s))\|^p ds \right)^{\frac{1}{p}} d\mu(t) \leq M \frac{\varepsilon}{M} = \varepsilon.$$

Finally, we obtain

$$\lim_{r \to +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|f(s, x(s))\|^p ds \right)^{\frac{1}{p}} d\mu(t) = 0.$$

Theorem 6 is proved.

***Corollary* 1.** *Let $\mu, \nu \in \mathcal{M}$. For $x \in AP(\mathbb{R}, Y)$ and $f \in \mathcal{E}^p U(\mathbb{R} \times Y, X, \mu, \nu)$, we have that $f(., x(.)) \in \mathcal{E}^p(\mathbb{R}, X, \mu, \nu)$.*

Proof. Since $x \in AP(\mathbb{R}, Y)$, we obtain that $x \in S^p AP(\mathbb{R}, Y)$ and $K = \overline{\{x(t); t \in \mathbb{R}\}}$ is a compact subset of Y. Hence, Theorem 6 is satisfied.

Theorem 7. *Let* $\mu, \nu \in \mathcal{M}$ *satisfy* **(M2)**, $f : \mathbb{R} \times Y \to X$ *be a function such that* $f = g + h \in$ $\in S^p PAPU(\mathbb{R} \times Y, X, \mu, \nu)$ *satisfying*:

(i) $\forall x \in Y,\ g(.,x) \in S^p AP(\mathbb{R}, Y)$ *and* $h(.,x) \in \mathcal{E}^p(\mathbb{R}, Y, \mu, \nu)$,

(ii) $x = x_1 + x_2 \in PAP(\mathbb{R}, Y, \mu, \nu)$, *where* $x_1 \in AP(\mathbb{R}, Y)$ *and* $x_2 \in \mathcal{E}(\mathbb{R}, Y, \mu, \nu)$,

(iii) *f satisfies* **(H1)**.

Then $f(., x(.)) \in S^p PAP(\mathbb{R}, X, \mu, \nu)$.

Proof. Let $f = g + h$, where $g \in S^p APU(\mathbb{R} \times Y, X)$ and $h \in \mathcal{E}^p U(\mathbb{R} \times Y, X, \mu, \nu)$. Then we have

$$f(t, x(t)) = g(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + h(t, x_1(t)) =$$

$$= G(t) + F(t) + H(t).$$

From Theorem 5, it follows that $G(.) \in S^p AP(\mathbb{R}, X)$ and, by Corollary 1, we have that $H(.) \in$ $\in \mathcal{E}^p(\mathbb{R}, X, \mu, \nu)$. Now, it is sufficient to show that $F(.) \in \mathcal{E}^p(\mathbb{R}, Y, \mu, \nu)$. Indeed, for $r > 0$ large enough, we obtain

$$\frac{1}{\nu([-r,r])} \int_{-r}^{r} \left(\int_{t}^{t+1} \|F(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) \leq \frac{1}{\nu([-r,r])} \int_{-r}^{r} \left(\int_{t}^{t+1} L^p \|x(s) - x_1(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) =$$

$$= \frac{1}{\nu([-r,r])} \int_{-r}^{r} \left(\int_{t}^{t+1} L^p \|x_2(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) =$$

$$= \frac{L}{\nu([-r,r])} \int_{-r}^{r} \left(\int_{t}^{t+1} \|x_2(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t).$$

Therefore, using the fact that $x_2 \in \mathcal{E}^p(\mathbb{R}, X, \mu, \nu)$, we get

$$\lim_{r \to +\infty} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \left(\int_{t}^{t+1} \|F(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) = 0.$$

Theorem 8 [1]. *If* $u \in S^p AP(\mathbb{R},\ X)$ *and* v *defined by*

$$v(t) := \int_{-\infty}^{t} T(t-s) u(s) ds \quad \textit{for all} \quad t \in \mathbb{R},$$

then $v \in AP(\mathbb{R},\ X)$.

Theorem 9. *Let* $\mu, \nu \in \mathcal{M}$ *satisfy* **(M0)** *and* **(M2)**. *If* $u \in \mathcal{E}^p(\mathbb{R},\ X, \mu, \nu)$ *and* v *given by*

$$v(t) := \int_{-\infty}^{t} T(t-s) u(s) ds \quad for\ all\ t \in \mathbb{R},$$

then $v \in \mathcal{E}(\mathbb{R},\ X, \mu, \nu)$.

Proof. Let $v_k(t) = \int_{t-k}^{t-k+1} T(t-s)u(s)ds, \quad k = 1, 2, \ldots,$ v_k is a continuous function and satisfies

$$\|v_k(t)\| \leq \int\limits_{t-k}^{t-k+1} \|T(t-s)u(s)\|ds \leq$$

$$\leq \int\limits_{t-k}^{t-k+1} \|T(t-s)A^{\alpha}\|\|A^{-\alpha}u(s)\|ds =$$

$$= \int\limits_{k-1}^{k} \|A^{\alpha}T(\tau)\|\|A^{-\alpha}u(t-\tau)\|d\tau =$$

$$= M_{\alpha}\|A^{-\alpha}\| \int\limits_{k-1}^{k} \frac{e^{-\omega\tau}}{\tau^{\alpha}}\|u(t-\tau)\|d\tau \leq$$

$$\leq M_{\alpha}\|A^{-\alpha}\|\frac{e^{-\omega(k-1)}}{(k-1)^{\alpha}} \int\limits_{k-1}^{k} \|u(t-\tau)\|d\tau, \quad k > 1.$$

Hence, by using the Hölder inequality, we get

$$\|v_k(t)\| \leq M_{\alpha}\|A^{-\alpha}\|\frac{e^{-\omega(k-1)}}{(k-1)^{\alpha}}\left(\int\limits_{k-1}^{k} \|u(t-\tau)\|^p d\tau\right)^{\frac{1}{p}}, \quad k > 1.$$

Then, for $r > 0$ and $k > 1$, we have

$$\frac{1}{\nu([-r,r])}\int\limits_{[-r,r]} \|v_k(t)\|d\mu(t) \leq$$

$$\leq M_{\alpha}\|A^{-\alpha}\|\frac{e^{-\omega(k-1)}}{(k-1)^{\alpha}}\frac{1}{\nu([-r,r])}\int\limits_{[-r,r]}\left(\int\limits_{k-1}^{k} \|u(t-\tau)\|^p d\tau\right)^{\frac{1}{p}} d\mu(t).$$

Since $h \in \mathcal{E}^p(\mathbb{R}, X, \mu, \nu)$ which is translation invariant (Proposition 1), we deduce that the function $t \mapsto u(t-\tau)$ also ergodic in Stepanov's sense. Then

$$\lim_{r \longrightarrow +\infty} \frac{1}{\nu([-r,r])}\int\limits_{[-r,r]} \|v_k(t)\|d\mu(t) = 0,$$

which gives $v_k \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$ for each $k > 1$. Moreover,

$$\|v_k(t)\| \leq M_{\alpha}\|A^{-\alpha}\|\frac{e^{-\omega(k-1)}}{(k-1)^{\alpha}}\|u\|_{BS^p}.$$

Hence

$$\sum_{k=2}^{+\infty}\|v_k(t)\| \le M_\alpha\|A^{-\alpha}\|\sum_{k=2}^{+\infty}\left(\frac{e^{-\omega(k-1)}}{(k-1)^\alpha}\right)\|u\|_{BS^p} < \infty.$$

By using this, we conclude that the series $\sum_k v_k(t)$ converges uniformly on $\mathbb{R}$. Moreover, $v(t) := \int_{-\infty}^{t} T(t-s)u(s)ds = \sum_{k=1}^{+\infty} v_k(t)$ is continuous on $\mathbb{R}$ and

$$\|v(t)\| \le \sum_{k=1}^{+\infty}\|v_k(t)\| \le M_\alpha\|A^{-\alpha}\|\sum_{k=2}^{+\infty}\left(\frac{e^{-\omega(k-1)}}{(k-1)^\alpha}\right)\|u\|_{BS^p},$$

$$\frac{1}{\nu([-r,r])}\int\limits_{[-r,r]}\|v(t)\|d\mu(t) \le \frac{1}{\nu([-r,r])}\int\limits_{[-r,r]}\left\|v(t)-\sum_{k=1}^{n}v_k(t)\right\|d\mu(t)+$$

$$+\sum_{k=1}^{n}\frac{1}{\nu([-r,r])}\int\limits_{[-r,r]}\|v_k(t)\|d\mu(t).$$

Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \ge N$, we have

$$\sup_{t\in\mathbb{R}}\left\|v(t)-\sum_{k=1}^{n}v_k(t)\right\| < \varepsilon.$$

From **(M0)**, we obtain

$$\frac{1}{\nu([-r,r])}\int\limits_{[-r,r]}\|v(t)\|d\mu(t) \le M\varepsilon + \sum_{k=1}^{n}\frac{1}{\nu([-r,r])}\int\limits_{[-r,r]}\|v_k(t)\|d\mu(t).$$

Since $v_k \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$ for all $k \ge 1$, we get

$$\lim_{r\longrightarrow+\infty}\frac{1}{\nu([-r,r])}\int\limits_{[-r,r]}\|v(t)\|d\mu(t) = 0.$$

Consequently,

$$t \mapsto v(t) = \sum_{k=1}^{+\infty}v_k(t) \in \mathcal{E}(\mathbb{R}, X, \mu, \nu).$$

Thus, $t \mapsto v(t) := \int_{-\infty}^{t} T(t-s)u(s)ds$ is ergodic.

Theorem 9 is proved.

From Theorems 8 and 9, we get the following result.

Theorem 10. *Let $\mu, \nu \in \mathcal{M}$ satisfy* **(M0)** *and* **(M2)**. *If $u \in S^pPAP(\mathbb{R}, X, \mu, \nu)$ and v defined by*

$$v(t) := \int\limits_{-\infty}^{t} T(t-s)u(s)ds \quad \textit{for all} \quad t \in \mathbb{R},$$

then $v \in PAP(\mathbb{R}, X, \mu, \nu)$.

2.3. (μ,ν)***-Pseudo almost periodic solutions.*** In this subsection, we treat the existence and uniqueness of pseudo almost periodic solutions to the neutral equation (1). In the following, we assume that

(H1) $f \in S^p PAPU(\mathbb{R}\times X_\alpha, X, \mu,\nu)$ and there exists $L_f > 0$ such that

$$\|f(t,x) - f(t,y)\| \le L_f \|x-y\|_\alpha$$

for every $x,\ y \in X_\alpha$ and $t \in \mathbb{R}$;

(H2) $g \in PAPU(\mathbb{R}\times X_\alpha, X_\alpha, \mu,\nu)$ such that, for all bounded subset C of X_α, the function g is bounded on $\mathbb{R}\times C$ and there exists $L_g > 0$ such that

$$\|g(t,x) - g(t,y)\|_\alpha \le L_g \|x-y\|_\alpha$$

for every $x,\ y \in X_\alpha$ and $t \in \mathbb{R}$.

Definition 4. *A continuous function $v : (-\infty,+\infty) \longrightarrow X_\alpha$ is said to be a mild solution of equation* (1) *on $\mathbb{R}$, if*

$$v(t) = T(t-\sigma)\Big[v(\sigma) - g(\sigma, v(\sigma))\Big] + g(t,v(t)) + \int\limits_{\sigma}^{t} T(t-s) f(s,v(s))ds \ \textit{for any} \ \ t \ge \sigma.$$

Theorem 11. *Let $\mu,\nu \in \mathcal{M}$ satisfy* **(M2)**. *Under conditions* **(H0)**, **(H1)** *and* **(H2)**, *suppose that*

$$L_g + M_\alpha L_f \frac{\Gamma(1-\alpha)}{\omega^{1-\alpha}} < 1.$$

Then equation (1) *has a unique (μ,ν)-pseudo almost periodic mild solution, and we have*

$$v(t) = g(t,v(t)) + \int\limits_{-\infty}^{t} T(t-s) f(s,v(s))ds \quad \textit{for} \quad t\in\mathbb{R}.$$

Proof. First, suppose that $\Lambda\colon PAP(\mathbb{R}, X_\alpha,\mu,\nu) \to C(\mathbb{R}, X_\alpha)$ given by

$$\Lambda v(t) := g(t,v(t)) + \int\limits_{-\infty}^{t} T(t-s) f(s,v(s))ds \quad \text{ for } \quad t\in\mathbb{R}.$$

We can see that $\Lambda v \in C(\mathbb{R}, X_\alpha)$ is well-defined and continuous. Also, from Theorems 7, 10 and 2.26 in [6], $\Lambda v \in PAP(\mathbb{R}, X_\alpha,\mu,\nu)$, that is, $\Lambda\colon PAP(\mathbb{R}, X_\alpha,\mu,\nu) \mapsto PAP(\mathbb{R}, X_\alpha,\mu,\nu)$. It remains to prove that Λ is a strict contraction on $PAP(\mathbb{R}, X_\alpha,\mu,\nu)$.

For $u,\ v\ \in PAP(\mathbb{R}, X_\alpha,\mu,\nu)$ and $t\in\mathbb{R}$, we obtain

$$\|\Lambda u(t) - \Lambda v(t)\|_\alpha \le L_g \|u(t)-v(t)\|_\alpha + M_\alpha \int\limits_{-\infty}^{t} \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} L_f \|u(s)-v(s)\|_\alpha ds \le$$

$$\le \left(L_g + M_\alpha L_f \int\limits_{0}^{+\infty} \frac{e^{-\omega s}}{s^\alpha} ds \right) \|u-v\|_\infty.$$

Then

$$\|\Lambda u - \Lambda v\|_\infty \leq \left(L_g + M_\alpha L_f \int\limits_0^{+\infty} \frac{e^{-\omega s}}{s^\alpha} ds \right) \|u - v\|_\infty <$$

$$< \left(L_g + M_\alpha L_f \frac{\Gamma(1-\alpha)}{\omega^{1-\alpha}} \right) \|u - v\|_\infty .$$

From the Banach fixed-point theorem, we deduce that the operator Λ has a unique fixed-point, which is clearly belong to $PAP(\mathbb{R}, X_\alpha, \mu, \nu)$.

Theorem 11 is proved.

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