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ON THE PROBLEMS OF UNIQUENESS OF MEROMORPHIC MAPPINGS FROM COMPLETE KÄHLER MANIFOLDS INTO PROJECTIVE VARIETIES

ПРО ПРОБЛЕМИ ЄДИНОСТІ МЕРОМОРФНИХ ВІДОБРАЖЕНЬ ПОВНИХ МНОГОВИДІВ КАХЛЕРА У ПРОЄКТИВНІ МНОГОВИДИ

We prove the unicity theorems for meromorphic mappings of a complete Kähler manifold into projective varieties† sharing few hypersurfaces in subgeneral position without counting multiplicities, where all zeros with multiplicities greater than a certain number are omitted. We also present the uniqueness theorem in which the assumption of nondegeneracy of the mappings is no longer required. These results are extensions and generalizations of some recent results.

Доведено теорему єдиності для мероморфних відображень повного многовиду Кахлера в проєктивні многовиди, що мають кілька спільних гіперповерхонь у підзагальному положенні, без підрахунку кратностей, де всі нулі з кратністю більше певного числа пропущено. Також наведено теорему єдиності, в якій припущення про невиродженість відображень більше не вимагається. Ці результати є розширеннями та узагальненнями деяких останніх результатів.

1. Introduction. In 1926, R. Nevanlinna showed that two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$ can not have the same inverse images for five distinct values. Over the last few decades, there have been many generalizations of Nevanlinna's result to the case of meromorphic mappings from $\mathbb{C}^m$ into the complex projective space $\mathbb{P}^n(\mathbb{C})$.

We now consider the general case, where $f\colon M \to \mathbb{P}^n(\mathbb{C})$ is a meromorphic mapping of an m-dimensional complete Kähler manifold M into $\mathbb{P}^n(\mathbb{C})$.

For $\rho \geq 0$, we say that f satisfies the condition (C_ρ) if there exists a non-zero bounded continuous real-valued function h on M such that

$$\rho f^*\Omega + dd^c \log h^2 \geq \mathrm{Ric}\omega,$$

where $f^*\Omega$ is the full-back of the Fubini–Study form Ω on $\mathbb{P}^n(\mathbb{C})$ by f, $\omega = \dfrac{\sqrt{-1}}{2}\sum_{i,j} h_{i\bar{j}} dz_i \wedge \wedge d\bar{z}_j$ is Kähler form on M, $\mathrm{Ric}\omega = dd^c \log(\det(h_{i\bar{j}}))$, $d = \partial + \overline{\partial}$ and $d^c = \dfrac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$.

In 1986, Fujimoto [3] firstly gave a new type of uniqueness theorem for linearly nondegenerate meromorphic mappings of complete Kähler manifold M, whose universal covering is biholomorphic to a ball in $\mathbb{C}^m$, into $\mathbb{P}^n(\mathbb{C})$ satisfying condition (C_ρ). Since that time, uniqueness problems for meromorphic mappings over M sharing few hyperplanes as well as hypersurfaces have been studied intensively by many authors such as [5, 9] and others. Recently, Chen and Han [1, 7] obtained the uniqueness theorems for meromorphic mappings from M into smooth projective algebraic varieties V intersecting hypersurfaces located in subgeneral position. To state their result, we recall the following notation.

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Let $N \geq n$ and $q \geq N+1$. Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension k $(k \leq n)$. Let $Q_1, \ldots, Q_q$, $q \geq k+1$, be q hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. The family of hypersurfaces $\{Q_i\}_{i=1}^q$ is said to be in N-subgeneral position with respect to V if, for any $1 \leq i_1 < \ldots < i_{N+1} \leq$ $\leq q$, $V \cap \left(\bigcap_{j=1}^{N+1} Q_{i_j}\right) = \varnothing$.

When $N = n$, we say that $\{Q_i\}_{i=1}^q$ is in general position with respect to V.

Now, let d be a positive integer. We denote by $I(V)$ the ideal of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ defining V and by H_d the vector space of all homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ of degree d. Define

$$I_d(V) := \frac{H_d}{I(V) \cap H_d} \qquad \text{and} \qquad H_V(d) := \dim I_d(V).$$

Then $H_V(d)$ is called the Hilbert function of V. Each elements of $I_d(V)$ which is an equivalent class of an elements $Q \in H_d$, will be denoted by $[Q]$.

Let $f \colon M \to V$ be a meromorphic mapping. We say that f is degenerate over $I_d(V)$ if there is $[Q] \in I_d(V) \backslash \{0\}$ such that $Q(f) = 0$. Otherwise, we say that f is nondegenerate over $I_d(V)$. It is clear that if f is algebraically nondegenerate (i.e., the image of f is not contained in any hypersurfaces of $\mathbb{P}^n(\mathbb{C})$) then f is nondegenerate over $I_d(V)$ for every $d \geq 1$.

Theorem A ([1], Theorem 4.1). *Assume $V \subseteq \mathbb{P}^n(\mathbb{C})$ is an irreducible projective algebraic variety of dimension $k(\leq n)$. Let $f, g \colon B(1)(\subseteq \mathbb{C}^m) \to V$ be two algebraically nondegenerate meromorphic mappings, both satisfying the condition (C_ρ). Let $D_1, D_2, \ldots, D_q$ be q hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degrees $d_1, d_2, \ldots, d_q$, located in N-subgeneral position $(N \geq n)$ with respect to V. Suppose further that* $\limsup_{r \to 1} \dfrac{T_f(r, r_0) + T_g(r, r_0)}{\log \dfrac{1}{r-1}} < \infty$ *and f, g satisfy the following conditions*:

(i) $f^{-1}(D_j) = g^{-1}(D_j)$ *for* $j = 1, 2, \ldots, q$,

(ii) $f = g$ *on* $\cup_{j=1}^q f^{-1}(D_j)$,

(iii) $f^{-1}(D_j \cap D_{j'})$ *has dimension at most* $m-2$ *for* $1 \leq j < j' \leq q$.

Then, one has $f \equiv g$ provided, for the least common multiple d of $d_1, d_2, \ldots, d_q$,

$$q > \frac{2N-k+1}{k+1} \Big\{ H_V(d) + \frac{\rho}{d} H_V(d)(H_V(d)-1) \Big\} + \frac{2}{d}(H_V(d)-1).$$

Our purpose in this article is to extend Theorem A to general cases. In the concrete, we will consider nondegenerate meromorphic mappings over $I_d(V)$ sharing hypersurfaces without counting multiplicity, where all zeros with multiplicities more than a certain number are omitted in the first result. In the other, we omit the condition of nondegeneracy over $I_d(V)$ of the mappings.

In theorems below, we denote by ν_φ the zero divisor of meromorphic function φ. For a positive integer k or $k = +\infty$, we set

$$\nu_{\varphi, \leq k}(z) = \begin{cases} 0 & \text{if} \quad \nu_\varphi(z) > k, \\ \nu_\varphi(z) & \text{if} \quad \nu_\varphi(z) \leq k, \end{cases}$$

and $\nu_{\varphi, >k}(z)$ is similarly defined.

The following theorem is a generalization of Theorem A.

Theorem 1.1. *Let M be a complete, connected Kähler manifold whose universal covering is biholomorphic to $B(R_0) \subset \mathbb{C}^m$, where $0 < R_0 \leq \infty$. Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension k, $k \leq n$. Let $\{Q_i\}_{i=1}^q$ be hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in N-subgeneral position with respect to V with $\deg(Q_i) = d_i$, $1 \leq i \leq q$. Let d be the least common multiple of $d_i's$, i.e., $d = lcm(d_1, \ldots, d_q)$. Let $f, g \colon M \to V$ be nondegenerate meromorphic mappings over $I_d(V)$ satisfying the condition (C_ρ) for $\rho \geq 0$ with*

$$\dim \left\{ z : \nu_{Q_i(f), \leq k_i} \nu_{Q_j(f), \leq k_j}(z) > 0 \right\} \leq m - 2, \quad i \neq j,$$

where $k_1, \ldots, k_q$ are positive integers or $+\infty$. Assume that

(i) $\min\{\nu_{Q_i(f), \leq k_i}, 1\} = \min\{\nu_{Q_i(g), \leq k_i}, 1\}$ *for all* $1 \leq i \leq q$,

(ii) $f = g$ *on* $\bigcup_{i=1}^q \left\{ z : \nu_{Q_i(f), \leq k_i}(z) > 0 \right\}$.

If $q > \dfrac{(2N - k + 1)H_V(d)}{k + 1} + \sum_{i=1}^q \dfrac{H_V(d) - 1}{k_i + 1} + \dfrac{\rho(2N - k + 1)H_V(d)(H_V(d) - 1)}{d(k + 1)} + \dfrac{2(H_V(d) - 1)}{d}$, *then* $f \equiv g$.

When the assumption on linearly nondegeneracy or algebraically nondegeneracy or nondegenerate over $I_d(V)$ of the mappings are dropped, we have the following theorem.

Theorem 1.2. *Let M, V and $\{Q_i\}_{i=1}^q$ be as in Theorem* 1.1. *Let $f, g \colon M \to V$ be meromorphic mappings satisfying the condition (C_ρ) for $\rho \geq 0$ with*

$$\dim \left\{ z : \nu_{Q_i(f), \leq k_i} \nu_{Q_j(f), \leq k_j}(z) > 0 \right\} \leq m - 2, \quad i \neq j,$$

where $k_1, \ldots, k_q$ are positive integers or $+\infty$. Assume that

(i) $\min\{\nu_{Q_i(f), \leq k_i}, 1\} = \min\{\nu_{Q_i(g), \leq k_i}, 1\}$ *for all* $1 \leq i \leq q$,

(ii) $f = g$ *on* $\bigcup_{i=1}^q \left\{ z : \nu_{Q_i(f), \leq k_i}(z) > 0 \right\}$.

If $q > NH_V(d) + \sum_{i=1}^q \dfrac{H_V(d) - 1}{k_i + 1} + \dfrac{\rho N H_V(d)(H_V(d) - 1)}{d} + \dfrac{2(H_V(d) - 1)}{d}$, *then* $f \equiv g$.

2. Basic notions and auxiliary results from Nevanlinna theory. We will recall some basic notions in Nevanlinna theory due to [6, 8, 10].

2.1. Counting function. We set $||z|| = \left(|z_1|^2 + \ldots + |z_m|^2\right)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and define

$$B(r) := \{z \in \mathbb{C}^m : ||z|| < r\}, \quad S(r) := \{z \in \mathbb{C}^m : ||z|| = r\}, \quad 0 < r \leq \infty,$$

where $B(\infty) = \mathbb{C}^m$ and $S(\infty) = \varnothing$. Define

$$v_{m-1}(z) := \left(dd^c||z||^2\right)^{m-1} \quad \text{and} \quad \sigma_m(z) := d^c \log||z||^2 \wedge \left(dd^c \log||z||^2\right)^{m-1} \quad \text{on} \quad \mathbb{C}^m \setminus \{0\}.$$

A divisor E on a ball $B(R_0)$ is given by a formal sum $E = \sum \mu_\nu X_\nu$, where $\{X_\nu\}$ is a locally family of distinct irreducible analytic hypersurfaces in $B(R_0)$ and $\mu_\nu \in \mathbb{Z}$. We define the support of the divisor E by setting $\operatorname{supp}(E) = \bigcup_{\mu_\nu \neq 0} X_\nu$. Sometimes, we identify the divisor E with a function $E(z)$ from $B(R_0)$ into $\mathbb{Z}$ defined by $E(z) := \sum_{X_\nu \ni z} \mu_\nu$.

Let M, k be positive integers or $+\infty$. We define the truncated divisors $E^{[M]}$ by

$$E^{[M]} := \sum_{\nu} \min\{\mu_\nu, M\} X_\nu$$

and the truncated counting function to level M of E by

$$N^{[M]}(r, r_0; E) := \int_{r_0}^{r} \frac{n^{[M]}(t, E)}{t^{2m-1}} dt, \quad r_0 < r < R_0,$$

where

$$n^{[M]}(t, E) := \begin{cases} \int_{\mathrm{supp}(E) \cap B(t)} E^{[M]} v_{m-1} & \text{if} \quad m \geq 2, \\ \sum_{|z| \leq t} E^{[M]}(z) & \text{if} \quad m = 1. \end{cases}$$

We omit the character $^{[M]}$ if $M = +\infty$.

Let φ be a non-zero meromorphic function on $B(R_0)$. We denote by ν^0_φ (resp., ν^∞_φ) the divisor of zeros (resp., divisor of poles) of φ. The divisor of φ is defined by $\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi$.

For a positive integer M or $M = \infty$, we define the truncated divisors of ν_φ by

$$\nu^{[M]}_\varphi(z) = \min\{M, \nu_\varphi(z)\}, \quad \nu^{[M]}_{\varphi, \leq k}(z) := \begin{cases} \nu^{[M]}_\varphi(z) & \text{if} \quad \nu^{[M]}_\varphi(z) \leq k, \\ 0 & \text{if} \quad \nu^{[M]}_\varphi(z) > k. \end{cases}$$

We will write $N_\varphi(r, r_0)$ and $N^{[M]}_{\varphi, \leq k}(r, r_0)$ for $N(r, r_0; \nu^0_\varphi)$ and $N^{[M]}(r, r_0; \nu^0_{\varphi, \leq k})$, respectively.

2.2. Characteristic function. Let $f: B(R_0) \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Fix a homogeneous coordinates system $(w_0 : \cdots : w_n)$ on $\mathbb{P}^n(\mathbb{C})$. We take a reduced representation $f = (f_0 : \cdots : f_n)$, which means f_i, $0 \leq i \leq n$, are holomorphic functions and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic subset $\{f_0 = \ldots = f_n = 0\}$ of codimension at least two. Set $\|f\| = \big(|f_0|^2 + \ldots + |f_n|^2\big)^{1/2}$. Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ defined by $H = \{(\omega_0, \ldots, \omega_n) : a_0\omega_0 + \ldots + a_n\omega_n = 0\}$. We set $H(f) = a_0 f_0 + \ldots + a_n f_n$ and $\|H\| = \big(|a_0|^2 + \ldots + |a_n|^2\big)^{1/2}$. Then the pull-back of the normalized Fubini–Study metric form Ω on $\mathbb{P}^n(\mathbb{C})$ by f is given by $f^*\Omega = dd^c \log ||f||^2$.

The characteristic function of f (with respect to Fubini–Study form Ω) is defined by

$$T_f(r, r_0) := \int_{t=r_0}^{r} \frac{dt}{t^{2m-1}} \int_{B(t)} f^*\Omega \wedge v_{m-1}, \quad 0 < r_0 < r < R_0.$$

By Jensen's formula we have

$$T_f(r, r_0) = \int_{S(r)} \log ||f|| \sigma_m - \int_{S(r_0)} \log ||f|| \sigma_m, \quad 0 < r_0 < r < R_0.$$

Through this paper, we assume that the numbers r_0 and R_0 are fixed with $0 < r_0 < R_0$. By notation "$||\ P$", we means that the assertion P holds for all $r \in [r_0, R_0]$ outside a set E such that $\int_E dr < \infty$ in case $R_0 = \infty$ and $\int_E \frac{1}{R_0 - r} dr < \infty$ in case $R_0 < \infty$.

2.3. Some propositions. Let $\{Q_i\}_{i=1}^q$ be q hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the common degree d. Assume that each Q_i is defined by a homogeneous polynomial $\tilde{Q}_i \in \mathbb{C}[x_1, \ldots, x_{n+1}]$. We regard $\mathbb{C}[x_1, \ldots, x_{n+1}]$ as a complex vector space and define $\operatorname{rank}\{Q_i\}_{i\in R} = \operatorname{rank}\{\tilde{Q}_i\}_{i\in R}$ for every subset $R \subset \{1, \ldots, q\}$. It is easy to see that

$$\operatorname{rank}\{Q_i\}_{i\in R} = \operatorname{rank}\{\tilde{Q}_i\}_{i\in R} \geq n + 1 - \dim\left(\bigcap_{i\in R} Q_i\right).$$

Hence, if $\{Q_i\}_{i=1}^q$ is in N-subgeneral position, by the above equality, we have $\operatorname{rank}\{Q_i\}_{i\in R} \geq n+1$ for any subset $R \subset \{1, \ldots, q\}$ with $\sharp R = N + 1$.

Proposition 2.1 ([6], Lemma 3)**.** *Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension k, $k \leq n$. Let $\{Q_i\}_{i=1}^q$, $q > 2N - k + 1$, be q hypersurfaces of the common degree d in $\mathbb{P}^n(\mathbb{C})$ located in N-subgeneral position with respect to V. Then there are positive rational constants ω_i, $1 \leq i \leq q$, satisfying the following*:

(i) $0 \leq \omega_i \leq 1\ \forall i \in Q = \{1, \ldots, q\}$.

(ii) *Setting $\tilde{\omega} = \max_{i\in Q} \omega_i$, one gets* $\sum_{i=1}^q \omega_i = \tilde{\omega}(q - 2N + k - 1) + k + 1$.

(iii) $\dfrac{k+1}{2N-k+1} \leq \tilde{\omega} \leq \dfrac{k}{N}$.

(iv) *For $R \subset \{1, \ldots, q\}$ with its cardinality $\sharp R = N + 1$, then* $\sum_{i\in R} \omega_i \leq k + 1$.

(v) *Let $E_i \geq 1$, $1 \leq i \leq q$, be arbitrarily given numbers. For $R \subset \{1, \ldots, q\}$ with $\sharp R = N+1$, there exists subset $R^o \subset R$ such that $\sharp R^o = \operatorname{rank}\{Q_i\}_{i\in R^o} = k + 1$ and*

$$\prod_{i\in R} E_i^{\omega_i} \leq \prod_{i\in R^o} E_i.$$

The constants ω_i appearing in the above theorem are called Nochka weights and $\tilde{\omega}$ is called Nochka constant with respect to $\{Q_i\}_{i=1}^q$.

Let $\{Q_i\}_{i\in R}$ be a set of hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the common degree d. Assume that each Q_i is defined by $\sum_{I\in\mathcal{I}_d} a_{iI}x^I = 0$, where $\mathcal{I}_d = \{I = (t_1, \ldots, t_{n+1}) \in \mathbb{N}^{n+1},\ t_1 + \ldots + t_{n+1} = d\}$, $x^I = x_1^{t_1} \ldots x_{n+1}^{t_{n+1}}$ and $(x_1 : \cdots : x_{n+1})$ is homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$.

Let $f : B(R_0) \longrightarrow V \subset \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping with a reduced representation $f = (f_1 : \cdots : f_{n+1})$. We define

$$Q_i(f) = \sum_{I\in\mathcal{I}_d} a_{iI} f^I,$$

where $f^I = f_1^{t_1} \ldots f_{n+1}^{t_{n+1}}$ for $I = (t_1, \ldots, t_{n+1})$. Then $f^*Q_i = \nu_{Q_i(f)}$ as divisors.

Lemma 2.1 ([6], Lemma 4)**.** *Let $\{Q_i\}_{i\in R}$ be a set of hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the common degree d and let $f : B(R_0) \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Assume that $\bigcap_{i=1}^q Q_i \cap V = \varnothing$. Then there exist positive constants α and β such that*

$$\alpha\|f\|^d \leq \max_{i\in R} |Q_i(f)| \leq \beta\|f\|^d.$$

Lemma 2.2 ([6], Lemma 5)**.** *Let $\{Q_i\}_{i=1}^q$ be a set of hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the common degree d. Then there exist $(H_V(d) - k - 1)$ hypersurfaces $\{T_i\}_{i=1}^{H_V(d)-k-1}$ in $\mathbb{P}^n(\mathbb{C})$ such that for any subset $R \subset \{1, \ldots, q\}$ with $\sharp R = \operatorname{rank}\{Q_i\}_{i\in R} = k+1$, we get* $\operatorname{rank}\{\{Q_i\}_{i\in R} \cup \{T_i\}_{i=1}^{H_V(d)-k-1}\} = H_V(d)$.

Lemma 2.3 ([10], Lemma 3.4). *Let $\{L_i\}_{i=1}^{H_V(d)}$ be a family of hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the common degree d and let f be a meromorphic mapping of $B(R_0) \subset \mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Assume that $\{L_i\}_{i=1}^{H_V(d)}$ is linear independent. Then, for every $0 < r_0 < r < R_0$, we have*

$$T_F(r, r_0) = dT_f(r, r_0) + O(1),$$

where F is the meromorphic mapping of $B(R_0)$ into $\mathbb{P}^n(\mathbb{C})$ with the representation $F = (L_1(f) : \cdots : L_{H_V(d)}(f))$.

Proposition 2.2 ([2], Proposition 4.5). *Let $F_1, \ldots, F_{n+1}$ be meromorphic functions on $B(R_0) \subset \mathbb{C}^m$ such that they are linearly independent over $\mathbb{C}$. Then there exists an set $\{\alpha_i = (\alpha_{i1}, \ldots, \alpha_{im})\}_{i=1}^{n+1}$ with $\alpha_{ij} \geq 0$ being integers, $|\alpha_i| = \sum_{j=1}^{m} |\alpha_{ij}| \leq i$ for $1 \leq i \leq n+1$ such that the generalized Wronskians $W_{\alpha_1,\ldots,\alpha_{n+1}}(F_1, \ldots, F_{n+1}) \not\equiv 0$, where $W_{\alpha_1,\ldots,\alpha_{n+1}}(F_1, \ldots, F_{n+1}) = \det(\mathcal{D}^{\alpha_i} F_j)_{1\leq i,j\leq n+1}$.*

The set $\{\alpha_i = (\alpha_{i1}, \ldots, \alpha_{im})\}_{i=1}^{n+1}$ satisfying Proposition 2.2 is called the admissible set of F.

Let $L_1, \ldots, L_{n+1}$ be linear forms of $n+1$ variables and assume that they are linearly independent. Let $F = (F_1 : \cdots : F_{n+1}) : B(R_0) \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping and $(\alpha_1, \ldots, \alpha_{n+1})$ be an admissible set of F. Then we have following proposition.

Proposition 2.3 ([8], Proposition 3.3). *In the above situation, set $l_0 = |\alpha_1| + \ldots + |\alpha_{n+1}|$ and take t, p with $0 < tl_0 < p < 1$. Then, for $0 < r_0 < R_0$, there exists a positive constant K such that, for $r_0 < r < R < R_0$,*

$$\int\limits_{S(r)} \left| z^{\alpha_1+\ldots+\alpha_{n+1}} \frac{W_{\alpha_1,\ldots,\alpha_{n+1}}(F_1, \ldots, F_{n+1})}{L_1(F) \ldots L_{n+1}(F)} \right|^t \sigma_m \leq K \left(\frac{R^{2m-1}}{R-r} T_F(R, r_0) \right)^p,$$

where $z^\alpha = z_1^{\alpha_1} \ldots z_m^{\alpha_m}$ for $z = (z_1, \ldots, z_m)$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$.

Taking a $\mathbb{C}$-basis $\{[A_i]\}_{i=1}^{H_V(d)}$ of $I_d(V)$ with $A_i \in H_d$, we may consider $\mathbb{C}_d(V)$ as a $\mathbb{C}$-vector space $\mathbb{C}^{H_V(d)}$. We consider $[Q] \in I_d(V)$, where $Q \in \mathbb{C}[x_0, \ldots, x_n]_d$ is a homogeneous polynomial of degree d. Then

$$[Q] = \sum_{i=1}^{H_V(d)} a_i[A_i] = \sum_{i=1}^{H_V(d)} [a_i A_i]$$

with $a_i \in \mathbb{C}$, $1 \leq i \leq H_V(d)$. Denote by

$$H = (a_1 : \cdots : a_{H_V(d)}) \in \mathbb{P}^{H_V(d)-1}(\mathbb{C})$$

the hyperplane in $\mathbb{P}^{H_V(d)-1}(\mathbb{C})$ which is called the associated hyperplane of Q with respect to the basis $\{[A_i]\}_{i=1}^{H_V(d)}$.

We now consider a meromorphic mapping $f : M \to V$ and also consider the holomorphic mapping $F = (A_1(f) : \cdots : A_{H_V(d)}(f))$ of M to $\mathbb{P}^{H_V(d)-1}(\mathbb{C})$. The mapping F is said to be the associated mapping of f with respect to the basis $\{[A_i]\}_{i=1}^{H_V(d)}$. It is easy to see that $Q(f) = H(F) = a_1 A_1(f) + \ldots + a_{H_V(d)} A_{H_V(d)}(f)$.

We need the following preparation lemma.

Lemma 2.4. *Let V and $\{Q_i\}_{i=1}^q$ be as in Theorem* 1.1. *Let $f\colon B(R_0)\to V$ be a meromorphic mapping. Assume that image $F(M)$ is contained in the l-dimensional projective subspace $\mathbb{P}^l(\mathbb{C})$ of $\mathbb{P}^{H_V(d)-1}(\mathbb{C})$, but not in any subspace of dimension lower than l, where $1\le l\le H_V(d)-1$. Set $l_0=|\alpha_1|+\ldots+|\alpha_{l+1}|$ and take t,p with $0<tl_0<p<1$. Let ω_i be Nochka weights with respect to $\{Q_i\}_{i=1}^q$. For each i, we take a real number $\hat\omega_i$ satisfying $0<\hat\omega_i\le\omega_i$. Then, for $0<r_0<R_0$, there exists a positive constant K such that, for $r_0<r<R<R_0$,*

$$\int\limits_{S(r)}\left|z^{\alpha_1+\ldots+\alpha_{l+1}}\frac{W_{\alpha_1\ldots\alpha_{l+1}}(F^*)}{Q_1^{\hat\omega_1}(f)\ldots Q_q^{\hat\omega_q}(f)}\right|^t\left(\|f\|^{d(\sum_{i=1}^q\hat\omega_i-l-1)}\right)^t\sigma_m\le K\left(\frac{R^{2m-1}}{R-r}T_f(R,r_0)\right)^p,$$

where $F^*=(A_1(f)\colon\cdots\colon A_{l+1}(f))\colon M\to\mathbb{P}^l(\mathbb{C})$ is a linearly nondegenerate meromorphic mapping.

Proof. Let $Q_i,\ 1\le i\le q$, be the homogeneous polynomial in $\mathbb{C}[x_0,\ldots,x_n]$ of degree d_i defining hypersurface Q_i. Replacing Q_i by $Q_i^{d/d_i},\ i=1,\ldots,q$, if necessary, we may assume that $Q_1,\ldots,Q_q$ have the same degree of d.

Take a $\mathbb{C}$-basis $\{[A_i]\}_{i=1}^{H_V(d)}$ of $I_d(V)$, where $A_i\in\mathbb{C}[x_0,\ldots,x_n]_d$. Consider a linear equation system determining $\mathbb{P}^l(\mathbb{C})$:

$$a_{11}\omega_1+\ldots+a_{1,H_V(d)}\omega_{H_V(d)}=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{H_V(d)-1-l,1}\omega_1+\ldots+a_{H_V(d)-1-l,H_V(d)}\omega_{H_V(d)}=0.$$

Without loss of generality, assume that

$$\operatorname{rank}\,(a_{ij})_{l+2\le i\le H_V(d),l+2\le j\le H_V(d)}=H_V(d)-1-l.$$

By solving the above linear equation system, it implies that $\mathbb{P}^l(\mathbb{C})$ is determined by

$$\omega_{l+2}=b_{l+2,1}\omega_1+\ldots+b_{l+2,l,+1}\omega_{l+1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\omega_{H_V(d)}=b_{H_V(d),1}\omega_1+\ldots+b_{H_V(d),l+1}\omega_{l+1}.$$

Since $F(M)\subset\mathbb{P}^l(\mathbb{C})$, it follows that

$$A_{l+2}(f)=b_{l+2,1}A_1(f)+\ldots+b_{l+2,l+1}A_{l+1}(f),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_{H_V(d)}(f)=b_{H_V(d),1}A_1(f)+\ldots+b_{H_V(d),l+1}A_{l+1}(f).$$

Put $B=(b_{ij})_{l+2\le i\le H_V(d),1\le j\le l+1}$. Then the above linear equation system can be rewritten as follows:

$$\begin{pmatrix}A_{l+2}(f)\\ \vdots\\ A_{H_V(d)}(f)\end{pmatrix}=B\begin{pmatrix}A_1(f)\\ \vdots\\ A_{l+1}(f)\end{pmatrix}.$$

For each hypersurface Q of degree d in $\mathbb{P}^n(\mathbb{C})$, take the associated hyperplane $H: a_1\omega_1 + \ldots \ldots + a_{H_V(d)}\omega_{H_V(d)} = 0$ in $\mathbb{P}^{H_V(d)-1}(\mathbb{C})$ of Q with respect to the basis $\{[A_i]\}_{i=1}^{H_V(d)}$. We have

$$Q(f) = F(H) = a_1A_1(f) + \ldots + a_{H_V(d)}A_{H_V(d)}(f) =$$

$$= (a_1 \ldots a_{l+1})\begin{pmatrix} A_1(f) \\ \vdots \\ A_{l+1}(f)\end{pmatrix} + (a_{l+2} \ldots a_{H_V(d)})\begin{pmatrix} A_{l+2}(f) \\ \vdots \\ A_{H_V(d)}(f)\end{pmatrix} =$$

$$= ((a_1 \ldots a_{l+1}) + (a_{l+2} \ldots a_{H_V(d)})B)\begin{pmatrix} A_1(f) \\ \vdots \\ A_{l+1}(f)\end{pmatrix}.$$

Put $Q^* = H \cap \mathbb{P}^l(\mathbb{C})$. By a simple calculation, we can see that the equation of Q^* in $\mathbb{P}^l(\mathbb{C})$ is

$$((a_1 \ldots a_{l+1}) + (a_{l+2} \ldots a_{H_V(d)})B)\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{l+1}\end{pmatrix} = 0.$$

It follows that $Q^*(F^*) = H(F) = Q(f)$.

Repeating the above way for each hypersurface Q_i, we get the family hyperplanes $\{Q_i^*\}_{i=1}^q$. By the assumption, it is easy to see that $\bigcap_{i\in R} Q_i^* = \varnothing$ for any subset $R \in \{1,\ldots,q\}$ with $\sharp R = N+1$. Note that $\operatorname{rank}\{Q_i\}_{i\in R} = \operatorname{rank}\{[Q_i]\}_{i\in R} \geq \dim V + 1 = k+1$.

Since F^* is linearly nondegenerate, $\{A_i(f)\}_{i=1}^{l+1}$ is linearly independent over $\mathbb{C}$. Then, by Proposition 2.2, there exist an admissible sets $\{\alpha_1,\ldots,\alpha_l\}$ such that

$$W_{\alpha_1\cdots\alpha_{l+1}}(F^*) = \det(\mathcal{D}^{\alpha_i}A_j(f))_{1\leq i,j\leq l+1} \not\equiv 0.$$

Let z be a fixed point. Then there exists $R \subset Q = \{1,\ldots,q\}$ with $\sharp R = N+1$ such that $|Q_i(f)(z)| \leq |Q_j(f)(z)|\ \forall i \in R,\ j \notin R$. Since $\bigcap_{i\in R} Q_i \bigcap V = \varnothing$, by Lemma 2.1, there exist positive constants α and β such that $\alpha\|f\|^d \leq \max_{i\in R}|Q_i(f)| \leq \beta\|f\|^d$. We consider two cases.

***Case* 1**: $l \leq k$. We have $\operatorname{rank}\{Q_i^*\}_{i\in R} = l+1$. It implies that $\{Q_i^*\}_{i=1}^q$ are in N-subgeneral position in $\mathbb{P}^l(\mathbb{C})$. Take the Nochka weights ω_i and the Nochka constant $\tilde{\omega}$ with respect to $\{Q_j^*\}_{j=1}^q$ in $\mathbb{P}^l(\mathbb{C})$. There exists a subset $R^0 \subset R$ such that $\sharp R^0 = l+1$ and R^0 satisfies Proposition 2.1(v) with respect to number $\left\{\dfrac{\beta\|f\|^d}{Q_i(f)(z)}\right\}_{i=1}^q$. We have

$$\prod_{i\in R}\left(\frac{\beta\|f(z)\|^d}{|Q_i(f)(z)|}\right)^{\hat{\omega}_i} \leq \prod_{i\in R}\left(\frac{\beta\|f(z)\|^d}{|Q_i(f)(z)|}\right)^{\omega_i} \leq \prod_{i\in R_0}\left(\frac{\beta\|f(z)\|^d}{|Q_i(f)(z)|}\right).$$

For $R^0 = \{r_1^0,\ldots,r_{l+1}^0\} \subseteq R$, we set

$$W_{R^0} = \det\Big(D^{\alpha_i}Q^*_{r_v^0}(F^*)\ (1 \leq v \leq l+1)\Big)_{1\leq i\leq l+1}.$$

Then there exists a nonzero constant C_{R^0} such that $W_{R^0} = C_{R^0}W_{\alpha_1\ldots\alpha_{l+1}}(F^*)$. Hence, we get

$$\frac{\|f(z)\|^{d\left(\sum_{i=1}^{q}\hat{\omega}_i\right)}|W_{\alpha_1\ldots\alpha_{l+1}}(F^*)(z)|}{|Q_1^{\hat{\omega}_1}(f)(z)\ldots Q_q^{\hat{\omega}_q}(f)(z)|}\leq\frac{|W_{\alpha_1\ldots\alpha_{l+1}}(F^*)(z)|}{\alpha^{\sum_{i\notin R}\hat{\omega}_i}\beta^{\sum_{i\in R}\hat{\omega}_i}}\prod_{i\in R}\left(\frac{\beta\|f(z)\|^d}{|Q_i(f)(z)|}\right)^{\hat{\omega}_i}\leq$$

$$\leq K_0\frac{|W_{R^0}|\|f(z)\|^{d(l+1)}}{\prod\limits_{i\in R^0}|Q_i(f)(z)|},$$

where K_0 is a positive constant. Then, by the definitions of Q_i^* and F^*, we get

$$\frac{\|f(z)\|^{d\left(\sum_{i=1}^{q}\hat{\omega}_i-l-1\right)}|W_{\alpha_1\ldots\alpha_{l+1}}(F^*)(z)|}{|Q_1^{\hat{\omega}_1}(f)(z)\ldots Q_q^{\hat{\omega}_q}(f)(z)|}\leq K_0\frac{|W_{R^0}(z)|}{\prod_{i\in R^0}|Q_i(f)(z)|}=K_0\frac{|W_{R^0}(z)|}{\prod_{i\in R^0}|Q_i^*(F^*)(z)|}.$$

Put $S_{R^0}=\dfrac{W_{R^0}}{\prod_{i\in R^0}Q_i^*(F^*)}$. Then, for each $z\in\mathbb{C}^m$, we get

$$\frac{\|f(z)\|^{d\left(\sum_{i=1}^{q}\hat{\omega}_i-l-1\right)}|W_{\alpha_1\ldots\alpha_{l+1}}(F^*)(z)|}{|Q_1^{\hat{\omega}_1}(f)(z)\ldots Q_q^{\hat{\omega}_q}(f)(z)|}\leq K_0|S_{R^0}(z)|.$$

***Case* 2**: $l>k$. We have $\operatorname{rank}\{Q_i^*\}_{i\in R}=k+1$. Take the Nochka weights ω_i and the Nochka constant $\tilde{\omega}$ with respect to $\{Q_j\}_{j=1}^q$ as in Lemma 2.1. Similar to Case 1, there exists a subset $R^0\subset R$ such that $\sharp R^0=k+1$ and R^0 satisfies Proposition 2.1(v) with respect to number $\left\{\dfrac{\beta\|f\|^d}{Q_i(f)(z)}\right\}_{i=1}^q$. We also have

$$\prod_{i\in R}\left(\frac{\beta\|f(z)\|^d}{|Q_i(f)(z)|}\right)^{\hat{\omega}_i}\leq\prod_{i\in R}\left(\frac{\beta\|f(z)\|^d}{|Q_i(f)(z)|}\right)^{\omega_i}\leq\prod_{i\in R_0}\left(\frac{\beta\|f(z)\|^d}{|Q_i(f)(z)|}\right).$$

By Lemma 2.2, we can choose a family of hypersurfaces $\{T_i\}_{i=1}^{l-k}$ in $\mathbb{P}^n(\mathbb{C})$ such that $\operatorname{rank}\{\{Q_i\}_{i\in R^0}\cup\{T_i\}_{i=1}^{l-k}\}=l+1$. For $R^0=\{r_1^0,\ldots,r_{k+1}^0\}$, we set

$$W_{R^0}=\det\Big(D^{\alpha_i}Q^*_{r_v^0}(F^*)\ (1\leq v\leq k+1),D^{\alpha_i}T_j^*(F^*)\ (1\leq j\leq l-k)\Big)_{1\leq i\leq l+1}.$$

Then there exists a nonzero constant C_{R_0} such that $W_{R_0}=C_{R_0}W_{\alpha_1\cdots\alpha_{l+1}}(F^*)$. Similar to the above, we have

$$\frac{\|f(z)\|^{d\left(\sum_{i=1}^{q}\hat{\omega}_i\right)}|W_{\alpha_1\ldots\alpha_{l+1}}(F^*)(z)|}{|Q_1^{\hat{\omega}_1}(f)(z)\ldots Q_q^{\hat{\omega}_q}(f)(z)|}\leq\frac{|W_{\alpha_1\ldots\alpha_{l+1}}(F^*)(z)|}{\alpha^{\sum_{i\notin R}\hat{\omega}_i}\beta^{\sum_{i\in R}\hat{\omega}_i}}\prod_{i\in R}\left(\frac{\beta\|f(z)\|^d}{|Q_i(f)(z)|}\right)^{\hat{\omega}_i}\leq$$

$$\leq K'\frac{|W_{\alpha_1\ldots\alpha_{l+1}}(F^*)(z)|\|f(z)\|^{d(k+1)}}{\prod_{i\in R^0}|Q_i(f)(z)|}\leq$$

$$\leq K_0\frac{|W_{R^0}(z)|\|f(z)\|^{d(l+1)}}{\prod_{i\in R^0}|Q_i(f)(z)|\prod_{i=1}^{l-k}|T_i(f)(z)|},$$

where K' and K_0 are positive constants. It implies that

$$\frac{\|f(z)\|^{d\left(\sum_{i=1}^{q}\hat{\omega}_i-l-1\right)}|W_{\alpha_1\ldots\alpha_{l+1}}(F^*)(z)|}{|Q_1^{\hat{\omega}_1}(f)(z)\ldots Q_q^{\hat{\omega}_q}(f)(z)|}\le K_0\frac{|W_{R^0}(z)|}{\prod_{i\in R^0}|Q_i(f)(z)|\prod_{i=1}^{l-k}|T_i(f)(z)|}.$$

Put $S_{R^0}=\dfrac{W_{R^0}}{\prod_{i\in R^0}Q_i(f)\prod_{i=1}^{l-k}T_i(f)}=\dfrac{W_{R_0}}{\prod_{i\in R^0}Q_i^*(F^*)\prod_{i=1}^{l-k}T_i^*(F^*)}$, we also get

$$\frac{\|f(z)\|^{d\left(\sum_{i=1}^{q}\hat{\omega}_i-l-1\right)}|W_{\alpha_1\ldots\alpha_{l+1}}(F^*)(z)|}{|Q_1^{\hat{\omega}_1}(f)(z)\ldots Q_q^{\hat{\omega}_q}(f)(z)|}\le K_0|S_{R_0}(z)|.$$

We now apply Proposition 2.3, for p satisfying $0<2t\left(\sum_{s=1}^{l+1}|\alpha_s|\right)<p<1$ and, for $0<r_0<$ $<r<R<1$, we get

$$\int\limits_{S(r)}\left|z^{\alpha_1+\ldots+\alpha_{l+1}}K_0S_{R_0}\right|^{2t}\sigma_m\le K_1\left(\frac{R^{2m-1}}{R-r}dT_f(R,r_0)\right)^p.$$

Therefore,

$$\int\limits_{S(r)}\left|z^{\alpha_1+\ldots+\alpha_{l+1}}\frac{W_{\alpha_1\ldots\alpha_{l+1}}(F^*)}{Q_1^{\hat{\omega}_1}(f)(z)\ldots Q_q^{\hat{\omega}_q}(f)(z)}\right|^t\left(\|f\|^{d\left(\sum_{i=1}^{q}\hat{\omega}_i-l-1\right)}\right)^t\sigma_m\le K\left(\frac{R^{2m-1}}{R-r}T_f(R,r_0)\right)^p.$$

Lemma 2.4 is proved.

Assume that $F^*:M\to\mathbb{P}^l(\mathbb{C})$, $G^*:M\to\mathbb{P}^{l^*}(\mathbb{C})$ are linearly nondegenerate and $l\ge l^*$. By the assumptions, it is easy to see that $\mathbb{P}^l(\mathbb{C})\cap\mathbb{P}^{l^*}(\mathbb{C})=\mathbb{P}^{l'}(\mathbb{C})$ with $l'\ge 0$ and we also can see that $\{Q_j\}_{j=1}^q$ are in N-subgeneral position in $\mathbb{P}^{l'}(\mathbb{C})$. Put $h'=\min\{l',k\}$. We will consider the Nochka weights ω_i and the Nochka constant $\tilde{\omega}$ with respect to $\{Q_j\}_{j=1}^q$ in N-subgeneral position in $\mathbb{P}^{h'}(\mathbb{C})$. With this notation, we will prove the following lemma.

Lemma 2.5. *Let M, $Q_1,Q_2,\ldots,Q_q$ be as in Theorem* 1.1 *and $f,g\colon M\to\mathbb{P}^n(\mathbb{C})$ be meromorphic mappings. Let P be a holomorphic function on M and β be a positive real number such that*

$$\sum_{i=1}^{q}\frac{\omega_i}{\tilde{\omega}}\left(1-\frac{H_V(d)-1}{k_i+1}\right)(\nu_{Q_i(f)}+\nu_{Q_i(g)})-\frac{1}{\tilde{\omega}}(\nu_{W_\alpha(F^*)}+\nu_{W_\beta(G^*)})\le\beta\nu_P. \tag{2.1}$$

If $|P^\beta|\le C(\|f\|\|g\|)^\alpha$ with some positive constants C and α, then

$$q\le\frac{(2N-h'+1)H_V(d)}{h'+1}+\sum_{i=1}^{q}\frac{H_V(d)-1}{k_i+1}+\frac{\rho(2N-h'+1)H_V(d)(H_V(d)-1)}{d(h'+1)}+\frac{\alpha}{d}.$$

Proof. Since the universal covering of M is biholomorphic to $B(R_0), 0<R_0\le\infty$, by using the universal covering if necessary, we may assume that $M=B(R_0)\subset\mathbb{C}^m$. We consider the following cases.

***Case* 1**: $R_0=\infty$ or $\limsup_{r\to R_0}\dfrac{T_f(r,r_0)+T_g(r,r_0)}{\log(1/(R_0-r))}=\infty$. Integrating both sides of inequality (2.1), we get

$$\beta N_P(r) \geq \frac{1}{\tilde{\omega}}\left(\sum_{i=1}^{q}\omega_i N_{Q_i(f)}(r,r_0) - N_{W_\alpha(F^*)}(r,r_0) + \sum_{i=1}^{q}\omega_i N_{Q_i(g)}(r,r_0) - N_{W_\alpha(G^*)}(r,r_0)\right) -$$

$$-\sum_{i=1}^{q}\frac{d(H_V(d)-1)\omega_i}{\tilde{\omega}(k_i+1)}(T_f(r,r_0)+T_g(r,r_0)) + O(1). \tag{2.2}$$

Applying Lemma 2.4 to $\hat{\omega}_i = \omega_i,\ 1 \leq i \leq q$, we obtain

$$\int\limits_{S(r)}\left|z^{\alpha_1+\ldots+\alpha_{l+1}}\frac{W_{\alpha_1\ldots\alpha_{l+1}}(F^*)}{Q_1^{\omega_1}(f)(z)\ldots Q_q^{\omega_q}(f)(z)}\right|^t\left(\|f\|^{d(\sum_{i=1}^{q}\omega_i-l-1)}\right)^t\sigma_m \leq K\left(\frac{R^{2m-1}}{R-r}T_f(R,r_0)\right)^p.$$

By the concativity of the logarithmic function, we have

$$\int\limits_{S(r)}\log|z^{\alpha_1+\ldots+\alpha_{l+1}}|\sigma_m + d\left(\sum_{i=1}^{q}\omega_i - l - 1\right)\int\limits_{S(r)}\log||f||\sigma_m + \int\limits_{S(r)}\log|W_{\alpha_1\ldots\alpha_{l+1}}(F^*)|\sigma_m -$$

$$-\sum_{i=1}^{q}\omega_i\int\limits_{S(r)}\log|Q_i(f)|\sigma_m \leq \frac{pK}{t}\left(\log^+\frac{1}{R_0-r} + \log^+ T_f(r,r_0)\right).$$

By the definition of the characteristic function and since $l \leq H_V(d)-1$, we get the following estimates:

$$\Bigg\|\, d\left(\sum_{i=1}^{q}\omega_i - H_V(d)\right)T_f(r,r_0) \leq \sum_{i=1}^{q}\omega_i N_{Q_i(f)}(r,r_0) - N_{W_{\alpha_1\ldots\alpha_{l+1}}(F^*)}(r) +$$

$$+ K_1\left(\log^+\frac{1}{R_0-r} + \log^+ T_f(r,r_0)\right).$$

Using Proposition 2.1 (ii), (iii) and by simple calculation, we obtain

$$\Bigg\|\left(q - \frac{(2N-h'+1)H_V(d)}{h'+1}\right)dT_f(r,r_0) \leq \frac{1}{\tilde{\omega}}\left(\sum_{i=1}^{q}\omega_i N_{Q_i(f)}(r,r_0) - N_{W_{\alpha_1\ldots\alpha_{l+1}}(F^*)}(r,r_0)\right) +$$

$$+ K_1\left(\log^+\frac{1}{R_0-r} + \log^+ T_f(r,r_0)\right).$$

Similarly, we have

$$\Bigg\|\left(q - \frac{(2N-h'+1)H_V(d)}{h'+1}\right)dT_g(r,r_0) \leq \frac{1}{\tilde{\omega}}\left(\sum_{i=1}^{q}\omega_i N_{Q_i(g)}(r,r_0) - N_{W_{\alpha_1\ldots\alpha_{l^*+1}}(G^*)}(r,r_0)\right) +$$

$$+ K_1\left(\log^+\frac{1}{R_0-r} + \log^+ T_g(r,r_0)\right).$$

Combining these inequalities with (2.2), we get

$$\Big\| \beta N_P(r) \geq \left(q - \frac{(2N-h'+1)H_V(d)}{h'+1}\right) dT(r,r_0) - \sum_{i=1}^{q} \frac{d(H_V(d)-1)}{k_i+1} T(r,r_0) + O(1), \quad (2.3)$$

where $T(r,r_0) := T_f(r,r_0) + T_g(r,r_0)$.

On the other hand, by Jensen's formula and the definition of the characteristic function, we have the following estimates:

$$\beta N_P(r) = \beta \int\limits_{S(r)} \log |P| \sigma_n + O(1) \leq \alpha \int\limits_{S(r)} (\log ||f|| + \log ||g||) \sigma_n + O(1) =$$

$$= \alpha(T_f(r,r_0) + T_g(r,r_0)) + o(T_f(r,r_0) + T_g(r,r_0)). \quad (2.4)$$

Together (2.3) with (2.4), we obtain

$$\Big\| \left(q - \frac{(2N-h'+1)H_V(d)}{h'+1}\right) dT(r,r_0) - \sum_{i=1}^{q} \frac{d(H_V(d)-1)}{k_i+1} T(r,r_0) \leq$$

$$\leq \alpha T(r,r_0) + o(T(r,r_0))$$

for every r outside a Borel finite measure set. Letting $r \to \infty$, we deduce that

$$q - \frac{(2N-h'+1)H_V(d)}{h'+1} - \sum_{i=1}^{q} \frac{H_V(d)-1}{k_i+1} \leq \frac{\rho(2N-h'+1)H_V(d)(H_V(d)-1)}{d(h'+1)} + \frac{\alpha}{d}$$

with $\rho = 0$.

***Case* 2**: $R_0 < \infty$ and $\limsup_{r \to R_0} \dfrac{T_f(r,r_0) + T_g(r,r_0)}{\log(1/(R_0 - r))} < \infty$. It suffices to prove the theorem in the case where $B(R_0) = B(1)$. Suppose that

$$q > \frac{(2N-h'+1)H_V(d)}{h'+1} + \sum_{i=1}^{q} \frac{H_V(d)-1}{k_i+1} + \frac{\rho(2N-h'+1)H_V(d)(H_V(d)-1)}{d(h'+1)} + \frac{\alpha}{d}.$$

Then, by Proposition 2.1(iii), we get

$$q > 2N - h' + 1 + \frac{(l+1)-(h'+1)}{\tilde{\omega}} + \sum_{i=1}^{q} \frac{\omega_i}{\tilde{\omega}} \frac{H_V(d)-1}{k_i+1} + \frac{\rho H_V(d)(H_V(d)-1)}{d\tilde{\omega}} + \frac{\alpha}{d}.$$

Combining the above inequality with Proposition 2.1(ii), we obtain

$$\sum_{i=1}^{q} \omega_i \left(1 - \frac{H_V(d)-1}{k_i+1}\right) - (l+1) - \frac{\alpha\tilde{\omega}}{d} > \frac{\rho H_V(d)(H_V(d)-1)}{d}. \quad (2.5)$$

Put

$$t = \frac{\rho}{d\left(\sum_{i=1}^{q} \hat{\omega}_i - (l+1) - \dfrac{\alpha\tilde{\omega}}{d}\right)},$$

$t^* = \dfrac{t(l+1)}{l^*+1}$ and $\omega_i^* = \dfrac{t\hat{\omega}_i}{t^*}$, where $\hat{\omega}_i := \omega_i\left(1 - \dfrac{H_V(d)-1}{k_i+1}\right)$, $1 \leq i \leq q$. It follows from our assumption $l \geq l^*$ that $t^* \geq t$ and $\omega_i^* \leq \hat{\omega}_i$ for all i. It is easy to see that

$$t^* = \frac{\rho}{d\left(\sum_{i=1}^{n} \omega_i^* - (l^*+1) - \dfrac{\alpha\tilde{\omega}t}{dt^*}\right)}.$$

Since (2.5), we have

$$d\left(\sum_{i=1}^{n} \hat{\omega}_i - (l+1) - \frac{\alpha\tilde{\omega}}{d}\right) > \rho H_V(d)(H_V(d)-1)$$

and so

$$d\left(\sum_{i=1}^{n} \omega_i^* - (l^*+1) - \frac{\alpha\tilde{\omega}t}{dt^*}\right) > \rho H_V(d)(H_V(d)-1) = \frac{t}{t^*} =$$

$$= \rho H_V(d)(H_V(d)-1)\frac{l^*+1}{l+1}.$$

Therefore, we get

$$2t\left(\sum_{s=1}^{l+1} |\alpha_s|\right) < \frac{2\rho}{\rho H_V(d)(H_V(d)-1)} \frac{l(l+1)}{2} \leq 1$$

and

$$2t^*\left(\sum_{s=1}^{l^*+1} |\alpha_s|\right) < \frac{2\rho}{\rho H_V(d)(H_V(d)-1)} \frac{l+1}{l^*+1} t \frac{l^*(l^*+1)}{2} \leq 1.$$

Then we may choose a positive numbers p and p^* such that

$$0 < 2t\left(\sum_{s=1}^{l+1} |\alpha_s|\right) < p < 1 \quad \text{and} \quad 0 < 2t^*\left(\sum_{s=1}^{l^*+1} |\alpha_s|\right) < p^* < 1.$$

Put $\phi = z^{\alpha_1+\ldots+\alpha_{l+1}} \dfrac{W_{\alpha_1\ldots\alpha_{l+1}}(F^*)}{Q_1^{\hat{\omega}_1}(f)\ldots Q_q^{\hat{\omega}_q}(f)}$ and $\psi = z^{\beta_1+\ldots+\beta_{l^*+1}} \dfrac{W_{\beta_1\ldots\beta_{l^*+1}}(G^*)}{Q_1^{\omega_1^*}(g)\ldots Q_q^{\omega_q^*}(g)}$. We deduce that $\phi^t\psi^{t^*}P^{\beta\tilde{\omega}t}$ is holomorphic and hence $u := \log|\phi^t\psi^{t^*}P^{t\beta\tilde{\omega}}|$ is plurisubharmonic on $B(1)$.

We now write the given Kähler metric form as $\omega = \dfrac{\sqrt{-1}}{2\pi}\sum_{i,j} h_{i\bar{j}} dz_i \wedge d\bar{z}_j$. From the assumption that both f and g satisfy condition (C_ρ), there are continuous plurisubharmonic functions u_1, u_2 on $B(1)$ such that

$$e^{u_1}\det(h_{i\bar{j}})^{\frac{1}{2}} \leq \|f\|^\rho, \quad e^{u_2}\det(h_{i\bar{j}})^{\frac{1}{2}} \leq \|g\|^\rho.$$

Therefore,

$$e^{u+u_1+u_2}\det(h_{i\bar{j}}) \leq e^u\|f\|^\rho\|g\|^\rho = |\phi|^t|\psi|^{t^*}|P|^{t\beta\tilde{\omega}}\|f\|^\rho\|g\|^\rho \leq$$

$$\leq C|\phi|^t|\psi|^{t^*}\|f\|^{\rho+t\alpha\tilde{\omega}}\|g\|^{\rho+t\alpha\tilde{\omega}} \leq$$

$$\leq C|\phi|^t|\psi|^{t^*}\|f\|^{td\left(\sum_{i=1}^{q}\hat{\omega}_i - l - 1\right)}\|g\|^{t^*d\left(\sum_{i=1}^{q}\omega_i^* - l^* - 1\right)} \tag{2.6}$$

with some positive constant C. Note that the volume form on $B(1)$ is given by

$$dV := c_m \det(h_{i\bar{j}}) v_m, \quad \text{where} \quad c_m = \left(\frac{\sqrt{-1}}{2}\right)^m.$$

By the result of S. T. Yau [11] and L. Karp [4], we have necessarily $\int_{B(1)} e^{u+u_1+u_2} dV = \infty$, because $B(1)$ has infinite volume with respect to the given complete Kähler metric (cf. [4], Theorem B).

Now, from inequality (2.6), we have

$$\int\limits_{B(1)} e^{u+u_1+u_2} dV \leq C \int\limits_{B(1)} |\phi|^t |\psi|^{t^*} \|f\|^{td\left(\sum_{i=1}^q \hat{\omega}_i - l - 1\right)} \|g\|^{t^* d\left(\sum_{i=1}^q \omega_i^* - l^* - 1\right)} v_m.$$

Thus, by the Hölder inequality and by noticing that

$$v_m = (dd^c \|z\|^2)^m = 2m \|z\|^{2m-1} \sigma_m \wedge d\|z\|,$$

we obtain

$$\int\limits_{B(1)} e^{u+u_1+u_2} dV \leq$$

$$\leq C \left(\int\limits_{B(1)} |\phi|^{2t} \|f\|^{2td\left(\sum_{i=1}^q \hat{\omega}_i - l - 1\right)} v_m \right)^{\frac{1}{2}} \left(\int\limits_{B(1)} |\psi|^{2t^*} \|g\|^{2t^* d\left(\sum_{i=1}^q \hat{\omega}_i - l^* - 1\right)} v_m \right)^{\frac{1}{2}} \leq$$

$$\leq C \left(2m \int\limits_0^1 r^{2m-1} \left(\int\limits_{S(r)} |\phi|^{2t} \|f\|^{2td(\sum_{i=1}^q \hat{\omega}_i - l - 1)} \sigma_m \right) dr \right)^{\frac{1}{2}} \times$$

$$\times \left(2m \int\limits_0^1 r^{2m-1} \left(\int\limits_{S(r)} |\psi|^{2t^*} \|f\|^{2t^* d(\sum_{i=1}^q \omega_i^* - l^* - 1)} \sigma_m \right) dr \right)^{\frac{1}{2}}.$$

Applying Lemma 2.4 to $\hat{\omega}_i$ and ω_i^*, we get

$$\int\limits_{S(r)} |\phi|^{2t} \|f\|^{2td\left(\sum_{i=1}^q \hat{\omega}_i - l - 1\right)} \sigma_m \leq K_1 \left(\frac{R^{2m-1}}{R-r} dT_f(R, r_0) \right)^p$$

and

$$\int\limits_{S(r)} |\psi|^{2t^*} \|g\|^{2t^* d\left(\sum_{i=1}^q \omega_i^* - l^* - 1\right)} \sigma_m \leq K_1 \left(\frac{R^{2m-1}}{R-r} dT_g(R, r_0) \right)^{p^*}$$

outside a subset $E \subset [0,1]$ such that $\int_E \frac{1}{1-r} dr \leq +\infty$. Choosing $R = r + \frac{1-r}{eT_f(r, r_0)}$, we have $T_f(R, r_0) \leq 2T_f(r, r_0)$. Hence, the above inequality implies that

$$\int\limits_{S(r)} |\phi|^{2t}\|f\|^{2td\left(\sum_{i=1}^{q}\hat{\omega}_i-l-1\right)}\sigma_m \le \frac{K_2}{(1-r)^p}(T_f(r,r_0))^{2p} \le \frac{K_2}{(1-r)^p}\left(\log\frac{1}{1-r}\right)^{2p},$$

since $\lim_{r\to R_0}\sup\dfrac{T_f(r,r_0)+T_g(r,r_0)}{\log(1/(R_0-r))} < \infty$. It implies that

$$\int\limits_0^1 r^{2m-1}\left(\int\limits_{S(r)} |\phi|^{2t}\|f\|^{2td\left(\sum_{i=1}^{q}\hat{\omega}_i-l-1\right)}\sigma_m\right)dr \le \int\limits_0^1 r^{2m-1}\frac{K_2}{(1-r)^p}\left(\log\frac{1}{1-r}\right)^{2p}dr < \infty.$$

Similarly,

$$\int\limits_0^1 r^{2m-1}\left(\int\limits_{S(r)} |\psi|^{2t^*}\|g\|^{2t^*d\left(\sum_{i=1}^{q}\omega_i^*-l^*-1\right)}\sigma_m\right)dr \le \int\limits_0^1 r^{2m-1}\frac{K_2}{(1-r)^p}\left(\log\frac{1}{1-r}\right)^{2p}dr < \infty.$$

Hence, we conclude that $\displaystyle\int_{B(1)} e^{u+u_1+u_2}dV < \infty$, which is a contradiction.

Lemma 2.5 is proved.

Lemma 2.6. *Let $M, V, \{Q_j\}_{j=1}^q$ in Theorem 1.1 and $f, g\colon M \to \mathbb{P}^n(\mathbb{C})$ be meromorphic mappings. Suppose that images $F^*(M)$ and $G^*(M)$ are contained in $\mathbb{P}^l(\mathbb{C})$ and $\mathbb{P}^{l^*}(\mathbb{C})$ respectively with $l^* \le l$. Then $f \equiv g$, if*

$$q > \frac{(2N-h'+1)H_V(d)}{h'+1} + \sum_{i=1}^{q}\frac{H_V(d)-1}{k_i+1} + \frac{\rho(2N-h'+1)H_V(d)(H_V(d)-1)}{d(h'+1)} + \frac{\alpha}{d},$$

where $h' = \min\{k, l'\}$ as in Lemma 2.5 and $\alpha = 2(H_V(d)-1)$.

Proof. Assume that $f \not\equiv g$, we may choose distinct indices i_0 and j_0 such that

$$P := f_{i_0}g_{j_0} - f_{j_0}g_{i_0} \not\equiv 0.$$

By the assumptions, $P = 0$ on $\bigcup_{i=1}^q\Big(\{z : \nu_{Q_i(f),\le k_i}(z) > 0\}\bigcup\{z : \nu_{Q_i(g),\le k_i}(z) > 0\}\Big)$. Therefore, we get $\nu_P(z) \ge \sum_{i=1}^q \min\{1, \nu_{Q_i(f),\le k_i}(z)\}$. It implies that

$$\nu_P(z) \ge \frac{1}{H_V(d)-1}\sum_{i=1}^{q}\min\{H_V(d)-1, \nu_{Q_i(f),\le k_i}(z)\}. \tag{2.7}$$

Similarly, we also obtain

$$\nu_P(z) \ge \frac{1}{H_V(d)-1}\sum_{i=1}^{q}\min\{H_V(d)-1, \nu_{Q_i(g),\le k_i}(z)\}. \tag{2.8}$$

Next, by usual arguments in the Nevanlinna theory, we have

$$\sum_{i=1}^{q}\omega_i\nu_{Q_i(f)}(z) - \nu_{W_{\alpha_1\cdots\alpha_{l+1}}(F^*)}(z) \le \sum_{i=1}^{q}\omega_i\min\{l, \nu_{Q_i(f)}(z)\} \le$$

$$\leq \sum_{i=1}^{q} \omega_i \min\{H_V(d)-1, \nu_{Q_i(f)}(z)\} \leq$$

$$\leq \sum_{i=1}^{q} \omega_i \min\{H_V(d)-1, \nu_{Q_i(f),\leq k_i}(z)\} + \sum_{i=1}^{q} \omega_i \min\{H_V(d)-1, \nu_{Q_i(f),>k_i}(z)\} \leq$$

$$\leq \sum_{i=1}^{q} \tilde{\omega} \min\{H_V(d)-1, \nu_{Q_i(f),\leq k_i}(z)\} + \sum_{i=1}^{q} \omega_i \frac{H_V(d)-1}{k_i+1} \nu_{Q_i(f)}(z).$$

This follows from (2.7) that

$$\sum_{i=1}^{q} \omega_i \left(1 - \frac{H_V(d)-1}{k_i+1}\right) \nu_{Q_i(f)}(z) - \nu_{W_{\alpha_1\ldots\alpha_{l+1}}(F^*)}(z) \leq \tilde{\omega}(H_V(d)-1)\nu_P(z).$$

Similarly, from (2.8), we get

$$\sum_{i=1}^{q} \omega_i \left(1 - \frac{H_V(d)-1}{k_i+1}\right) \nu_{Q_i(g)}(z) - \nu_{W_{\beta_1\ldots\beta_{l^*+1}}(G^*)}(z) \leq \tilde{\omega}(H_V(d)-1)\nu_P(z).$$

Since two above inequalities, we obtain

$$\sum_{i=1}^{q} \frac{\omega_i}{\tilde{\omega}} \left(1 - \frac{H_V(d)-1}{k_i+1}\right) (\nu_{Q_i(f)} + \nu_{Q_i(g)}) - \frac{1}{\tilde{\omega}} (\nu_{W_\alpha(F^*)} + \nu_{W_\beta(G^*)}) \leq \beta \nu_P,$$

where $\beta := 2(H_V(d)-1)$. Then $\alpha = \beta$ and we have $|P^\beta| \leq C(\|f\| \cdot \|g\|)^\alpha$ with some positive constants C. Applying Lemma 2.5, we get

$$q \leq \frac{(2N-h'+1)H_V(d)}{h'+1} + \sum_{i=1}^{q} \frac{H_V(d)-1}{k_i+1} + \frac{\rho(2N-h'+1)H_V(d)(H_V(d)-1)}{d(h'+1)} + \frac{\alpha}{d},$$

which contradicts the assumption.

Lemma 2.6 is proved.

3. Proofs of main theorems. ***Proof of Theorem* 1.1.** By the assumption on nondegeneracy over $I_d(V)$ of the mappings f and g, it is easy to see that $h' = k$. Applying Lemma 2.6, we get $f \equiv g$ if

$$q > \frac{(2N-k+1)H_V(d)}{k+1} + \sum_{i=1}^{q} \frac{H_V(d)-1}{k_i+1} + \frac{\rho(2N-k+1)H_V(d)(H_V(d)-1)}{d(k+1)} + \frac{\alpha}{d}.$$

***Proof of Theorem* 1.2.** Since the assumption, for all $1 \leq h' \leq k \leq n$, we have $N \geq \geq \dfrac{2N-h'+1}{h'+1}$. Therefore,

$$q > NH_d(V) + \sum_{i=1}^{q} \frac{H_d(V)-1}{k_i+1} + \frac{\rho N H_d(V)(H_d(V)-1)}{d} + \frac{\alpha}{d} \geq$$

$$\geq \frac{(2N-h'+1)H_d(V)}{h'+1} + \sum_{i=1}^{q} \frac{H_d(V)-1}{k_i+1} + \frac{\rho(2N-h'+1)H_d(V)(H_d(V)-1)}{h'+1} + \frac{\alpha}{d}.$$

Applying Lemma 2.6, we obtain $f \equiv g$.

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