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SIGNLESS LAPLACIAN DETERMINATION OF A FAMILY OF DOUBLE STARLIKE TREES

БЕЗЗНАКОВЕ ЛАПЛАСІВСЬКЕ ВИЗНАЧЕННЯ СІМ'Ї ПОДВІЙНО ЗІРКОПОДІБНИХ ДЕРЕВ

Two graphs are said to be Q-cospectral if they have the same signless Laplacian spectrum. A graph is said to be DQS if there are no other nonisomorphic graphs Q-cospectral with it. A tree is called double starlike if it has exactly two vertices of degree greater than 2. Let $H_n(p,q)$ with $n \geq 2,\ p \geq q \geq 2$ denote the double starlike tree obtained by attaching p pendant vertices to one pendant vertex of the path P_n and q pendant vertices to the other pendant vertex of P_n. In this paper, we prove that $H_n(p,q)$ is DQS for $n \geq 2,\ p \geq q \geq 2$.

Два графи називаються Q-коспектральними, якщо вони мають однакові беззнакові лапласівські спектри. Граф називається DQS, якщо не існує інших неізоморфних графів, що є Q-коспектральними по відношенню до нього. Дерево називається подвійно зіркоподібним, якщо воно має рівно дві вершини степеня більшого за 2. Нехай $H_n(p,q)$ з $n \geq 2,\ p \geq q \geq 2$ є подвійно зіркоподібним деревом, одержаним за допомогою додавання p висячих вершин до однієї висячої вершини шляху P_n та q висячих вершин до іншої висячої вершини P_n. У цій роботі запропоновано доведення того, що $H_n(p,q)$ є DQS для $n \geq 2,\ p \geq q \geq 2$.

1. Introduction. All graphs considered here are simple and undirected. All notions on graphs that are not defined here can be found in [4].

Let $G = (V,E)$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$, where $v_1, v_2, \ldots, v_n$ are indexed in the nonincreasing order of degrees, i.e., $d_1 \geq d_2 \geq \ldots \geq d_n$, where $d_i = d_i(G) = d_G(v_i)$ is the degree of the vertex v_i, for $i = 1, \ldots, n$. We denote the degree sequence of G by $\deg(G) = (d_1, d_2, \ldots, d_n)$. The number of triangles of G is denoted by $t(G)$.

For $v \in V(G)$, the graph $G - v$ is an induced subgraph of G obtained from G by deleting the vertex v and all edges incident with it. For two disjoint graphs G and H, let $G \cup H$ denotes the disjoint union of G and H.

The line graph of a graph G, denoted by G^L, has the edges of G as its vertices, and two vertices of G^L are adjacent if and only if the corresponding edges in G have a common vertex.

We denote by P_n, C_n, and $K_{1,n-1}$ the path, the cycle and the star of order n, respectively.

The *adjacency matrix* A_G of G is a square matrix of order n, whose (i,j)-entry is 1 if v_i and v_j are adjacent in G and 0 otherwise.

The *degree matrix* D_G of G is a diagonal matrix of order n defined as $D_G = \operatorname{diag}(d_1, \ldots, d_n)$. The matrices $L_G = D_G - A_G$ and $Q_G = D_G + A_G$ are called the *Laplacian matrix* and the *signless Laplacian matrix* of G, respectively. The multiset of eigenvalues of Q_G (resp., L_G, A_G) is called the *Q-spectrum* (resp., *L-spectrum*, *A-spectrum*) of G. Since A_G, L_G and Q_G are real symmetric, their eigenvalues are real numbers. Moreover, L_G and Q_G are positive semidefinite, and so their eigenvalues are nonnegative. We use $q_1(G) \geq q_2(G) \geq \ldots \geq q_n(G)$ to denote the Q-spectrum of G.

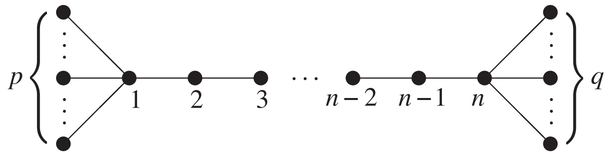

Fig. 1. Double starlike tree $H_n(p, q)$.

Two graphs are Q-cospectral (resp., L-cospectral, A-cospectral) if they have the same Q-spectrum (resp., L-spectrum, A-spectrum). A graph G is said to be DQS (resp., DLS, DAS) if there is no other nonisomorphic graph Q-cospectral (resp., L-cospectral, A-cospectral) with G. Obviously, if a graph is DLS, then it is not necessary DQS. For example, consider, $K_{1,3}$ that is Q-cospectral with $K_3 \cup K_1$.

The problem "which graphs are determined by their spectra?" originates from chemistry. Günthard and Primas [6] raised this question in the context of Hückels theory. Since this problem is generally very difficult, van Dam and Haemers [14] proposed a more modest problem, that is, "Which trees are determined by their spectra?".

A tree is called *starlike* if it has exactly one vertex of degree greater than 2. We denote by $T(l_1, l_2, \ldots, l_\Delta)$ the starlike tree with maximum degree Δ such that

$$T(l_1, l_2, \ldots, l_\Delta) - v = P_{l_1} \cup P_{l_2} \cup \ldots \cup P_{l_\Delta}, \tag{1.1}$$

where v is the vertex of degree Δ, $l_1, l_2, \ldots, l_\Delta$ are any positive integers. A starlike tree with maximum degree 3 is called a T-shape tree. A tree is called *double starlike* if it has exactly two vertices of degree greater than 2. Denote by $H_n(p, q)$ the tree obtained by attaching p pendant vertices (vertices of degree 1) to one pendant vertex of P_n and q pendant vertices to the other pendant vertex of P_n (see Fig. 1). It is clear that $H_n(p, q)$ is double starlike if and only if $n \geq 2$ and $p \geq q \geq 2$.

Note that:

Liu et al., proved that double starlike trees $H_n(p, p),\ n \geq 2,\ p \geq 1$, are DLS [9];

Lu and Liu proved that double starlike trees $H_n(p, q)$ are DLS [10];

$H_1(2, 1)$ is the star $K_{1,3}$ which is not DQS, since it is Q-cospectral with $K_3 \cup K_1$ [14];

$H_1(p, q) \cong K_{1,p+q}$, which is DQS when $p + q \neq 3$ [8];

$H_n(1, 1) \cong P_{n+2}$, which is DQS [14];

Omidi and Vatandoost proved that starlike trees with maximum degree 4 are DQS [12];

Bu and Zhou proved that starlike trees whose maximum degree exceed 4 are DQS [3];

if $p \geq 1$, then $H_n(p, 1)$ is a starlike tree, which is DQS [3, 12];

Omidi gave all T-shape trees, which are DQS [11].

In this paper, we prove that for $n \geq 2,\ p \geq q \geq 2$ the double starlike tree $H_n(p, q)$ is DQS.

2. Preliminaries. Some useful established results about the spectrum are presented in this section, will play an important role throughout this paper.

Lemma 2.1 [13, 14]. *For any graph, the following can be determined by its adjacency and Laplacian spectrum*:

(i) *the number of vertices*;

(ii) *the number of edges*.

For any graph, the following can be determined by its adjacency spectrum:

(iii) *the number of closed walks of any length;*

(iv) *being regular or not and the degree of regularity.*

For any graph, the following can be determined by its Laplacian spectrum:

(v) *the number of components;*

(vi) *the sum of the squares of degrees of vertices.*

For any graph, the following can be determined by its signless Laplacian spectrum:

(vii) *the number of bipartite components;*

(viii) *the sum of the squares of degrees of vertices.*

Lemma 2.2 [1]. *Let G be a graph with n vertices, m edges, t triangles and vertex degrees $d_1, d_2, \ldots, d_n$. Let $T_k = \sum_{i=1}^{n} q_i(G)^k$, then*

$$T_0 = n, \qquad T_1 = \sum_{i=1}^{n} d_i = 2m, \qquad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \qquad T_3 = 6t + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

Lemma 2.3 [7]. *Let G be a graph with $V(G) \neq \varnothing$ and $E(G) \neq \varnothing$. Then*

$$d_1 + 1 \leq \mu_1(G) \leq \max\left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} \;\middle|\; v_i v_j \in E(G) \right\},$$

where $\mu_1(G)$ denotes the spectral radius of L_G and m_i denotes the average of the degrees of the vertices adjacent to vertex v_i in G.

Lemma 2.4 [4]. *Let G be a graph. Then $q_n(G) \geq 0$ with equality if and only if G is a bipartite graph. Bedsides, the multplicity of 0 as the signless Laplacian eigenvalue of G, is the number of bipartite components of G. Moreover, $\mathrm{Spec}_Q(G) = \mathrm{Spec}_L(G)$ if and only if G is a bipartite graph.*

Lemma 2.5 [16]. *If two graphs G and H are Q-cospectral, then their line graphs G^L and H^L are A-cospectral. The converse is true if G and H have the same number of vertices and edges.*

Lemma 2.6 (Interlacing theorem [14]). *Suppose that N is a symmetric $n \times n$ matrix with eigenvalues $a_1 \geq a_2 \geq \ldots \geq a_n$. Then the eigenvalues $b_1 \geq b_2 \geq \ldots \geq b_m$ of a principal submatrix of N of size m satisfy $a_i \geq b_i \geq a_{n-m+i}$ for $i = 1, 2, \ldots, m$.*

Lemma 2.7 [5]. *If G is a graph of order n, then $q_2(G) \geq d_2(G) - 1$ and $q_3(G) \geq d_3(G) - \sqrt{2}$.*

A connected graph with n vertices is said to be unicyclic if it has n edges. If the girth of a unicyclic graph is odd, then this unicyclic graph is said to be odd unicyclic.

Lemma 2.8 [2]. *Let G be a graph Q-spectral with a tree of order n. Then G is either a tree or a union of a tree of order f and c odd unicyclic graphs, and $n = 4^c f$.*

Lemma 2.9 [15]. *Let G be a graph of order n and $v \in V(G)$. Then, for $i = 1, 2, \ldots, n-1$,*

$$q_{i+1}(G) - 1 \leq q_i(G - v) \leq q_i(G),$$

where the right equality holds if and only if v is an isolated vertex.

3. Main results. We first bound the second largest and the third largest signless Laplacian eigenvalues of $H = H_n(p,q)$ with $n \geq 2$ and $p \geq q \geq 2$.

Lemma 3.1. *For a double starlike tree $H = H_n(p,q)$ with $n \geq 2$ and $p \geq q \geq 2$ we have*

(i) $q_2(H) \leq q + 3 + \dfrac{1}{q+2}$;

(ii) $q_3(H) < 4$.

Proof. Let u and v be the vertices in $H = H_n(p,q)$ of degree $p+1$ and $q+1$, respectively.

(i) By Lemma 2.9 we have

$$q_2(H) - 1 \leq q_1(H-u) < q_1(H).$$

It follows from Lemmas 2.3 and 2.4 that

$$q_1(H-u) \leq q + 2 + \frac{1}{q+2},$$

from which we conclude that

$$q_2(H) \leq q_2(H-u) + 1 \leq q + 3 + \frac{1}{q+2}.$$

(ii) Let N_{uv} be the $(p+q+n-2)\times(p+q+n-2)$ principal submatrix of the signless Laplacian matrix of H formed by removing the rows and the columns corresponding to u and v. In this case, the largest eigenvalue of N_{uv} is less than 4. By Lemma 2.6 we have

$$q_3(H) < 4. \tag{3.1}$$

Lemma 3.1 is proved.

Proposition 3.1. *Let G be a connected graph Q-cospectral with $H_n(p,q)$, $n \geq 2$ and $p \geq q \geq$ ≥ 2. Then*:

(i) $d_1(G) \leq p+1$;

(ii) $d_2(G) \leq q+4$;

(iii) $d_3(G) \leq 5$.

Proof. (i) By Lemma 2.1, G has $n+p+q$ vertices and $n+p+q-1$ edges. It follows from Lemma 2.4 that G has only one bipartite component. Therefore, G is a tree, since G is connected. By Lemmas 2.3 and 2.4, we have $d_1(G) + 1 \leq q_1(G) = \mu_1(G) \leq p + 2 + \dfrac{1}{p+2}$ and so $\Delta = d_1(G) \leq p+1$.

(ii) By Lemma 3.1(i) we obtain

$$q_2(G) = q_2(H_n(p,q)) \leq q + 3 + \frac{1}{q+2}.$$

By Lemma 2.7 $d_2(G) \leq q+4$.

(iii) By Lemma 3.1(ii) we get $q_3(G) = q_3(H_n(p,q)) < 4$. It follows from Lemma 2.7 that $d_3(G) \leq 5$.

Proposition 3.1 is proved.

Proposition 3.2. *Let G be a connected graph Q-cospectral with $H_n(p,q)$, $n \geq 2$ and $p \geq q \geq \geq 2$. Then G is a double starlike tree with the degree set*

$$\deg(G) = \Bigg(p+1, q+1, \underbrace{2,\ldots,2}_{n-2}, \underbrace{1,\ldots,1}_{p+q}\Bigg).$$

Proof. It follows from Lemma 2.1 that G is bipartite with $n+p+q$ vertices and $n+ + p + q - 1$ edges. So, G is a tree. Denote by n_k the number of vertices of degree k in G for $k \in \{1,2,3,\ldots,d_1(G)\}$. By Lemma 2.1(i), (ii), (viii) and Lemma 2.2 we have the following equations:

$$\sum_{k=1}^{d_1(G)} n_k = n+p+q,$$

$$\sum_{k=1}^{d_1(G)} kn_k = 2(n+p+q-1),$$

$$\sum_{k=1}^{d_1(G)} k^2 n_k = (p+1)^2 + (q+1)^2 + p + q + 4(n-2),$$

$$\sum_{k=1}^{d_1(G)} k^3 n_k = (p+1)^3 + (q+1)^3 + p + q + 8(n-2).$$

By Lemma 2.5 $H_n(p,q)^L$ and G^L are A-cospectral. In addition, by Lemma 2.1(iii) they have the same number of triangles (six times the number of closed walks of length 3). In other words,

$$\sum_{k=1}^{d_1(G)} \binom{k}{3} n_k = \binom{p+1}{3} + \binom{q+1}{3}.$$

Let us denote by n_i the number of vertices of degree i in G for $i = 1,2,3,4,5$. Therefore,

$$n_1 + n_2 + n_3 + n_4 + n_5 = n + p + q - 2,$$

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 = 2(n+p+q-1) - x,$$

$$n_1 + 4n_2 + 9n_3 + 16n_4 + 25n_5 = (p+1)^2 + p + q + (q+1)^2 + 4(n-2) - y,$$

$$n_1 + 8n_2 + 27n_3 + 64n_4 + 125n_5 = (p+1)^3 + p + q + (q+1)^3 + 8(n-2) - w,$$

$$6n_3 + 24n_4 + 60n_5 = (p+1)p(p-1) + (q+1)q(q-1) - z,$$

where

$$x = d_1 + d_2, \qquad y = d_1^2 + d_2^2, \qquad w = d_1^3 + d_2^3, \qquad z = d_1(d_1^2 - 3d_1 + 2) + d_2(d_2^2 - 3d_2 + 2).$$

It follows that

$$n_5 = \frac{12(2x - 3y + w - z)}{12} = 0, \qquad n_4 = \frac{-48(2x - 3y + w - z)}{12} = 0,$$

and so

$$n_1 = p + q, \qquad n_2 = n - 2, \qquad n_3 = 0, \qquad d_1 = p + 1, \qquad d_2 = q + 1.$$

This means that G is a tree having exactly two vertices of degree more than 2. So, G is a double starlike tree with the degree set $\deg(G) = \Big(p+1, q+1, \underbrace{2, \ldots, 2}_{n-2}, \underbrace{1, \ldots, 1}_{p+q}\Big)$.

Proposition 3.2 is proved.

Proposition 3.3. *There is no connected graph Q-spectral with $H_n(p,q)$, $n \geq 2$ and $p \geq q \geq 2$.*

Proof. Let G be a connected graph Q-cospectral with $H_n(p,q)$ for $n \geq 2$ and $p \geq q \geq 2$. By Proposition 3.2, G is a double starlike tree with $\deg(G) = \Big(p+1, q+1, \underbrace{2, \ldots, 2}_{n-2}, \underbrace{1, \ldots, 1}_{p+q}\Big)$. Now we show that $G \cong H_n(p,q)$. We consider the following two cases:

Case 1. *The two vertices of degree greater than* 2 *are adjacent in* G.

Suppose that there exist $q - a$ (resp., $p - b$) pendant vertices adjacent to the vertices of degree $p + 1$ in G, where a and b are nonnegative integers satisfying that

$$0 \leq b \leq p, \quad 0 \leq a \leq q.$$

Let $m^i_{G^L}$ be the number of vertices of degree i in G^L. Therefore,

$$m^1_{G^L} = a + b, \qquad m^{p+q}_{G^L} = 1, \qquad m^{p+1}_{G^L} = b, \qquad m^p_{G^L} = p - b, \qquad m^q_{G^L} = p - a,$$

$$m^{q+1}_{G^L} = a, \qquad m^2_{G^L} = n - 2 - a - b, \qquad m^k_{G^L} = 0, \quad k \notin \{1, 2, p, q, p+1, q+1, p+q\}.$$

Note that $H_n(p,q)^L$ and G^L are A-cospectral. By Lemma 2.1(ii), (iii), $H_n(p,q)^L$ and G^L have the same number of edges and the same number of closed walks of length 4. Moreover, they have the same number of 4-cycles. Lemma 2.1 implies that $H_n(p,q)^L$ and G^L have the same number of induced paths of length 2, that is,

$$\binom{p+1}{2} + \binom{q+1}{2} + p\binom{p}{2} + q\binom{q}{2} + (n-3)\binom{2}{2} =$$

$$= \binom{p+q}{2} + b\binom{p+1}{2} + (p-b)\binom{p}{2} +$$

$$+a\binom{q+1}{2} + (q-a)\binom{p}{2} + (n-2-a-b)\binom{2}{2}.$$

It follows that $(p-1)(q-1) + b(p-1) + a(q-1) = 0$, a contradiction, since $0 \leq b \leq p$, $0 \leq a \leq q$ and $p \geq q \geq 2$.

Case 2. *The two vertices of degree greater than* 2 *are non-adjacent in* G *as shown in Fig.* 2.

Suppose that there exist $q - a$ (resp., $p - b$) pendant vertices adjacent to the vertices of degree $p + 1$ in G, where a and b are nonnegative integers satisfying that $0 \leq b \leq p$, $0 \leq a \leq q$. We show that $a = b = 0$. Then, by counting the number of vertices in G and $H_n(p,q)$, we obtain

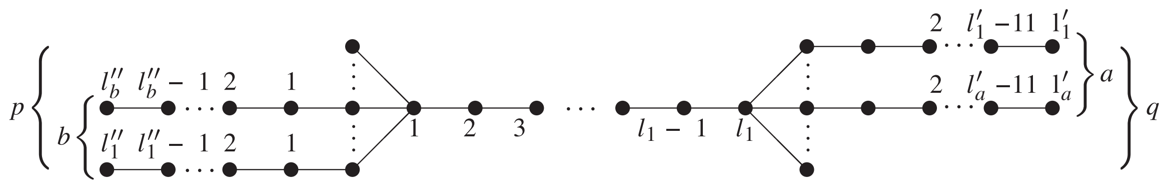

Fig. 2. The graph G in Case 2 of Proposition 3.3.

$$l+\sum_{j=1}^{a} l'_j+\sum_{l=1}^{b} l''_j+p+q=n+p+q.$$

Hence $l+a+b=n$. On the other hand,

$$m^1_{G^L}=a+b,\qquad m^{p+1}_{G^L}=b+1,\qquad m^p_{G^L}=p-b,\qquad m^{q+1}_{G^L}=a+1,\qquad m^q_{G^L}=q-a,$$

$$m^2_{G^L}=n-3-a-b,\qquad m^k_{G^L}=0,\quad k\notin\{1,2,p,q,p+1,q+1\}.$$

Note that $H_n(p,q)^L$ and G^L are A-cospectral. By Lemma 2.1, $H_n(p,q)^L$ and G^L have the same number of edges and the same number of closed walks of length 4. Moreover, they have the same number of 4-cycles. Lemma 2.5 implies that $H_n(p,q)^L$ and G^L have the same number of induced paths of length 2, that is,

$$\binom{p+1}{2}+\binom{q+1}{2}+p\binom{p}{2}+q\binom{q}{2}+(n-3)\binom{2}{2}=$$

$$=(b+1)\binom{p+1}{2}+(p-b)\binom{p}{2}+$$

$$+(q-a)\binom{q}{2}+(a+1)\binom{q+1}{2}+$$

$$+(p-a)\binom{p}{2}+(n-3-a-b)\binom{2}{2}.$$

By a simple computation $a(q-1)+b(p-1)=0$ and so $a=b=0$. Therefore, $l=n$. This means that $G\cong H_n(p,q)$.

Proposition 3.3 is proved.

Lemma 3.2. *Let G be a disconnected graph Q-cospectral with $H_n(p,q)$, $n\ge 2$ and $p\ge q\ge 2$. Then G has no triangles.*

Proof. By Lemma 2.4 G has one bipartite component. We show that $t(G)=0$. Suppose not and so $t(G)\ge 1$. Obviously, $t(G)\le 2$, otherwise, since G is disconnected, it follows from Lemma 2.8 that G has at least three odd unicyclic components each of them has a triangle. So $q_1(G),q_2(G),q_3(G)\ge 4$, which is a contradiction to Lemma 3.1(ii). First suppose that $t(G)=1$. Consider the following two cases:

Case 1. Let $d_n(G)<1$, i.e., $d_n(G)=0$. Since G has only one bipartite component, one may deduce that G has only one isolated vertex.

Subcase 1.1. By Lemma 2.8 $G=K_1\cup H_1$, where H_1 is an odd unicyclic graph consisting of a triangle. By Lemmas 2.5 and 2.1(iii), we have

$$t(G^L) = t(H_n(p,q)^L) = \binom{p+1}{3} + \binom{q+1}{3}.$$

Let us denote by n_i, the number of vertices of degree i for $i = 1,2,3,4,5$. Therefore,

$$n_1 + n_2 + n_3 + n_4 + n_5 = n + p + q - 3,$$

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 = 2(n + p + q - 1) - x,$$

$$n_1 + 4n_2 + 9n_3 + 16n_4 + 25n_5 = (p+1)^2 + p + q + (q+1)^2 + 4(n-2) - y,$$

$$n_1 + 8n_2 + 27n_3 + 64n_4 + 125n_5 = (p+1)^3 + p + q + (q+1)^3 + 8(n-2) - w - 6,$$

$$6n_3 + 24n_4 + 60n_5 = (p+1)p(p-1) + (q+1)q(q-1) - z,$$

where

$$x = d_1 + d_2, \qquad y = d_1^2 + d_2^2, \qquad w = d_1^3 + d_2^3, \qquad z = d_1(d_1^2 - 3d_1 + 2) + d_2(d_2^2 - 3d_2 + 2).$$

By a simple computation $n_4 = \dfrac{-48(2x - 3y + w - z + 6)}{12} < 0$, a contradiction, since $2x - 3y + w - z = 0$.

Subcase 1.2. Let $t(G) = 2$. By a similar argument and by using the previous notations we obtain $G = K_1 \cup H_1 \cup H_2$, and so

$$n_1 + n_2 + n_3 + n_4 + n_5 = n + p + q - 3,$$

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 = 2(n + p + q - 1) - x,$$

$$n_1 + 4n_2 + 9n_3 + 16n_4 + 25n_5 = (p+1)^2 + p + q + (q+1)^2 + 4(n-2) - y,$$

$$n_1 + 8n_2 + 27n_3 + 64n_4 + 125n_5 = (p+1)^3 + p + q + (q+1)^3 + 8(n-2) - w - 12,$$

$$6n_3 + 24n_4 + 60n_5 = (p+1)p(p-1) + (q+1)q(q-1) - z.$$

By a simple computation $n_4 = \dfrac{-48(2x - 3y + w - z + 12)}{12} < 0$, a contradiction, since $2x - 3y + w - z = 0$.

Case 2. Let $d_n(G) \geq 1$. By Lemma 2.8 if $t(G) = 1$, then $G = Y \cup T$, where Y and T are a connected graph consisting of a triangle and a tree, respectively. Consider the following two subcases:

Subcase 2.1. By using the previous notations

$$n_1 + n_2 + n_3 + n_4 + n_5 = n + p + q - 2,$$

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5e = 2(n + p + q - 1) - x,$$

$$n_1 + 4n_2 + 9n_3 + 16n_4 + 25n_5 = (p+1)^2 + p + q + (q+1)^2 + 4(n-2) - y,$$

$$n_1 + 8n_2 + 27n_3 + 64d + 125n_5 = (p+1)^3 + p + q + (q+1)^3 + 8(n-2) - w - 6,$$

$$6n_3 + 24n_4 + 60n_5 = (p+1)p(p-1) + (q+1)q(q-1) - z.$$

By a simple computation $n_4 = \dfrac{-48(2x-3y+w-z+6)}{12} < 0$, a contradiction, since $2x-3y+ + w - z = 0$.

Subcase 2.2. Let $t(G)=2$. By Lemma 2.8, we have $G = T \cup Y_1 \cup Y_2$, where Y_1 and Y_2 are connected graphs consisting of a triangle and T is a tree with at least two vertices. By using the previous notations, we have

$$n_1+n_2+n_3+n_4+n_5 = n+p+q-2,$$

$$n_1+2n_2+3n_3+4n_4+5n_5 = 2(n+p+q-1)-x,$$

$$n_1+4n_2+9n_3+16n_4+25n_5 = (p+1)^2+p+q+(q+1)^2+4(n-2)-y,$$

$$n_1+8n_2+27n_3+64n_4+125n_5 = (p+1)^3+p+q+(q+1)^3+8(n-2)-w-12,$$

$$6n_3+24n_4+60n_5 = (p+1)p(p-1)+(q+1)q(q-1)-z,$$

where

$$x=d_1+d_2, \qquad y=d_1^2+d_2^2, \qquad w=d_1^3+d_2^3, \qquad z=d_1(d_1^2-3d_1+2)+d_2(d_2^2-3d_2+2).$$

By a simple computation $n_4 = \dfrac{-48(2x-3y+w-z+12)}{12} < 0$, a contradiction.

Lemma 3.2 is proved.

Proposition 3.4. *There is no disconnected graph Q-spectral with $H_n(p,q)$, $n \geq 2$ and $p \geq q \geq 2$.*

Proof. Suppose by the contrary that G is a disconnected graph Q-spectral with $H_n(p,q)$, $n \geq 2$ and $p \geq q \geq 2$. By Lemma 3.2, $t(G)=0$. Similar to Proposition 3.2 we have the following two cases:

Case 1. Let $d_n(G)=0$. By Lemma 2.8 if $s=1$, then $G = Y \cup T$, where Y is a connected graph consisting of a unique cycle of order at least 5 and $T = K_1$. On the other hand, Lemma 2.8 implies that $H_n(p,q)$ is either $K_{1,3}$ or P_4, a contradiction. So, let $s=2$. In this case $G = = Y_1 \cup Y_2 \cup K_1$, where Y_1 and Y_2 are connected graph consisting of an unique cycle of order at least 5. It is clear that $|V(G)| = 16$ and $|E(G)| = 15$. Since $\mathrm{Spec}_Q(G) = \mathrm{Spec}_Q(H_n(p,q))$, so $|V(G)| = |V(H_n(p,q))| = n+p+q$. Therefore, $n+p+q=16$, that is, $p+q=16-n$. Applying the previous notations, we get

$$n_1+n_2+n_3+n_4+n_5 = 13,$$

$$n_1+2n_2+3n_3+4n_4+5e = 2(15)-x,$$

$$n_1+4n_2+9n_3+16n_4+25n_5 = (16-n-q)^2+(q+1)^2+16-n+4(n-2)-y,$$

$$n_1+8n_2+27n_3+64n_4+125n_5 = (16-n-q)^3+(q+1)^3+16-n+8(n-2)-w,$$

$$6n_3+24n_4+60n_5 = q(q-1)(q+1)+(16-n-q)(17-n-q)(15-n-q)-z.$$

(Note that the degree of one of vertices is $d_n(G)=0$ and the two others degrees are d_1, d_2. We can subtract these three vertices from 16. So, the number of vertices of degrees 1, 2, 3, 4 and 5 is 13.)

Then

$$n_5 = \frac{12(2x - 3y + w - z + 710 + 3n^2 + n(6q - 89) + 3q(q - 31))}{12} =$$
$$= 710 + 3n^2 + n(6q - 89) + 3q(q - 31). \quad (3.2)$$

We know that $n \geq 2$, $q \geq p \geq 2$ and $n + p + q = 16$. By substituting $(n, q) \in \{(2, 12), \ldots, (11, 3)\}$ in (3.2), we will have a contradiction.

If $s \geq 3$, then $q_1(G), q_2(G), q_3(G) \geq 4$, a contradiction to (3.1).

Case 2. Let $d_n(G) \geq 1$. Applying the previous notations

$$n_1 + n_2 + n_3 + n_4 + n_5 = n + p + q - 2,$$

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5e = 2(n + p + q - 1) - x,$$

$$n_1 + 4n_2 + 9n_3 + 16n_4 + 25n_5 = (p + 1)^2 + p + q + (q + 1)^2 + 4(n - 2) - y,$$

$$n_1 + 8n_2 + 27n_3 + 64n_4 + 125n_5 = (p + 1)^3 + p + q + (q + 1)^3 + 8(n - 2) - w,$$

$$6n_3 + 24n_4 + 60n_5 = (p + 1)p(p - 1) + (q + 1)q(q - 1) - z.$$

By solving this system of equations, we obtain that

$$n_3 = n_4 = n_5 = 0.$$

Since G is dis-connected, it follows from Lemma 2.8 that G consists of at least one odd unicyclic graph that the order of its odd cycle is greater than or equal to 5. This means that G has at least one vertex of degree 3 or 4 or 5.

Proposition 3.4 is proved.

Combining Propositions 3.3 and 3.4, we have the following main result.

Theorem 3.1. *Any double starlike tree $H_n(p, q)$, $n \geq 2$ and $p \geq q \geq 2$, is DQS.*

Note that if uv is an edge of $H = H_n(p, q)$, then the degree of uv as a vertex of the line graph $H_n(p, q)^L$ is $d_H(u) + d_H(v) - 2$. Since

$$\deg(H) = \left(p + 1, q + 1, \underbrace{2, \ldots, 2}_{n-2}, \underbrace{1, \ldots, 1}_{p+q}\right),$$

it is easy to see that the degree sequence of H^L is

$$\deg(H^L) = \left(p + 1, q + 1, \underbrace{p, \ldots, p}_{p}, \underbrace{q, \ldots, q}_{q}, \underbrace{2, \ldots, 2}_{n-3}\right).$$

***Corollary* 3.1.** *Let G be a graph such that G^L is A-cospectral with $H_n(p, q)^L$. If $|V(G)| = |V(H_n(p, q))|$, then $G^L \cong H_n(p, q)^L$.*

Proof. Let G be a graph such that G^L is A-cospectral with $H_n(p, q)^L$. Therefore, by Lemma 2.1(i), G and $H_n(p, q)$ have the same number of edges. Hence, by Lemma 2.5

$$\mathrm{Spec}_Q(G) = \mathrm{Spec}_Q(H_n(p, q)),$$

since $|V(G)| = |V(H_n(p, q))|$. In the other hand, Theorem 3.1 implies that $G \cong H_n(p, q)$. Therefore, $G^L \cong H_n(p, q)^L$.

Corollary 3.1 is proved.

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