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T. Chtioui (Univ. Sfax, Tunisia),
S. Mabrouk (Univ. Gafsa, Tunisia),
A. Makhlouf[1] (Univ. de Haute Alsace, IRIMAS, Mulhouse, France)

HOM – JORDAN – MALCEV – POISSON ALGEBRAS

АЛГЕБРИ ХОМА – ЙОРДАНА – МАЛЬЦЕВА – ПУАССОНА

We provide and study a Hom-type generalization of Jordan – Malcev – Poisson algebras called Hom – Jordan – Malcev – Poisson algebras. We show that they are closed under twisting by suitable self-maps and give a characterization of admissible Hom – Jordan – Malcev – Poisson algebras. In addition, we introduce the notion of pseudo-Euclidian Hom – Jordan – Malcev – Poisson algebras and describe its T^*-extension. Finally, we generalize the notion of Lie – Jordan – Poisson triple system to the Hom setting and establish its relationships with Hom – Jordan – Malcev – Poisson algebras.

Введено та досліджено узагальнення типу Хома для алгебр Йордана – Мальцева – Пуассона, які називаються алгебрами Хома – Йордана – Мальцева – Пуассона. Показано, що всі ці алгебри замкнені щодо скруту відповідними самовідображеннями. Дано характеристику допустимих алгебр Хома – Йордана – Мальцева – Пуассона. Крім того, введено поняття псевдоевклідової алгебри Хома – Йордана – Мальцева – Пуассона та описано її T^*-розширення. Насамкінець узагальнено поняття потрійної системи Лі – Йордана – Пуассона до постановки Хома і встановлено її зв'язки з алгеброю Хома – Йордана – Мальцева – Пуассона.

1. Introduction. Nonassociative algebras become an important research field due to their importance in various problems related to physics and other branches of mathematics. The first instances of nonassociative Hom-algebras appeared in the study of quasi-deformations of Lie algebras of vector fields. Hom – Lie algebras were first introduced by Hartwig, Larsson and Silvestrov in order to describe q-deformations of Witt and Virasoro algebras using σ-derivations (see [4]). The corresponding associative type objects, called Hom-associative algebras were introduced by Makhlouf and Silvestrov in [7]. Hom-alternative, Hom – Jordan and Hom-flexible algebras were introduced first in [6] and then considered as sa well as Hom – Malcev algebras in [11].

Poisson algebras form an important class of nonassociative algebras. They are used in many fields of mathematics and physics. They play a fundamental role in Poisson geometry, quantum groups, deformation theory, Hamiltonian mechanics and topological field theories. Poisson algebras are generalized in many ways. If we omit the commutativity of the associative structure, we get the class of noncommutative Poisson algebras. Another way to generalize this class is to replace the associative structure by a Jordan product and the Lie-bracket by a Malcev one. Hence, we obtain a new class of algebras called Jordan – Malcev – Poisson algebras (JMP algebras) which are defined by a triple $A\ (A,[,],\circ)$ consisting of a linear space equipped with a Malcev bracket and a Jordan structure satisfying the Leibniz rule:

$$[x,y\circ z]=[x,y]\circ z+y\circ[x,z].$$

They were introduced by Ait Ben Haddou, Benayadi and Boulmane in [3]. Such algebras can be described in terms of a single bilinear operation, called admissible JMP algebras. This class contains

[1] Corresponding author, e-mail: abdenacer.makhlouf@uha.fr.

alternative algebras. In the particular case, where $(A, \circ)$ is associative commutative, $(A, [,], \circ)$ becomes a Malcev – Poisson algebra, a concept introduced first by Shestakov in [9].

The purpose of this paper is to study a twisted generalization of JMP algebras, called Hom – JMP algebras and some other related algebraic structures (admissible Hom – JMP algebras). Next, we introduce and study pseudo-Euclidian Hom – JMP algebras, which are Hom – JMP algebras endowed with symmetric invariant nondegenerate bilinear forms. We provide a twist construction and extend the T^*-extension theory to this class of nonassociative Hom-algebras. We also construct a generalized Hom-triple systems called Hom – Lie – Jordan – Poisson triple systems from admissible Hom – JMP algebras.

This paper is organized as follows. In Section 1, we summarize the definitions and some key constructions of Hom – JMP algebras. In Section 2, we study and highlight relationships between Hom – JMP algebras and admissible Hom – JMP algebras. In addition, it is shown that admissible Hom – JMP algebras are power Hom-associative. In Section 3, we introduce the notion of pseudo-Euclidian Hom – JMP algebras and describe its T^*-extension. Section 4 is devoted to the study of a Hom version of Lie – Jordan – Poisson triple system algebras and to provide its connection with admissible Hom – JMP algebras.

1. Definitions and preliminary results. ***1.1. Basic definitions.*** In this section, we introduce Hom – JMP algebras as a generalization of both Hom – Poisson algebras, Malcev – Poisson algebras and JMP algebras. We show that Hom – JMP algebras are closed under suitable twisting by weak morphisms.

Let us begin with the basic definitions regarding Hom-algebras. We work over a fixed commutative field $\mathbb{K}$ of characteristic 0.

Definition 1.1. *Let (A, μ, α) be a Hom-algebra.*

1. *The Hom-associator $as_A : A^{\otimes 3} \to A$ is defined as*

$$as_A(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z)).$$

2. *The Hom-algebra A is called a Hom – Jordan algebra if it is commutative and satisfies the Hom – Jordan identity*

$$as_A(x^2, \alpha(y), \alpha(x)) = 0.$$

3. *The Hom – Jacobiator $J_A : A^{\otimes 3} \to A$ is defined as*

$$J_A(x, y, z) = \circlearrowleft_{x,y,z} \mu(\mu(x, y), \alpha(z)),$$

where $\circlearrowleft_{x,y,z}$ denotes the cyclic summation over x, y, z.

4. *A Hom – Malcev algebra is a Hom-algebra $(A, [,], \alpha)$ such that $[,]$ is skewsymmetric and the Hom – Malcev identity*

$$J_A(\alpha(x), \alpha(y), [x, z]) = [J_A(x, y, z), \alpha^2(x)]$$

is satisfied for all $x, y, z \in A$.

5. *A Hom-flexible algebra is a Hom-algebra (A, μ, α) satisfying*

$$as_A(x, y, z) + as_A(z, y, x) = 0,$$

or equivalently, $as_A(x, y, x) = 0$ for all $x, y, z \in A$.

6. *A Hom-alternative algebra is a Hom-algebra* (A, μ, α) *satisfying*

$$as_A(x, y, z) + as_A(y, x, z) = as_A(x, y, z) + as_A(x, z, y) = 0,$$

or equivalently, $as_A(x, x, y) = as_A(x, y, y) = 0$ *for all* $x, y, z \in A$.

Note that any Hom-alternative algebra is Hom-flexible.

Let (A, μ, α) be a Hom-algebra. Define the cyclic Hom-associator S_A by

$$S_A(x, y, z) = \circlearrowleft_{x,y,z} as_A(x, y, z).$$

Let us recall the definition of a JMP algebra [5].

Definition 1.2. *A JMP algebra* $(A, \{,\}, \circ)$ *consists of a Malcev algebra* $(A, \{,\})$ *and a Jordan algebra* $(A, \circ)$ *such that the Leibniz identity*

$$\{x, y \circ z\} = \{x, y\} \circ z + y \circ \{x, z\}$$

is satisfied for all $x, y, z \in A$.

In a JMP algebra $(A, \{,\}, \circ)$, the bracket $\{,\}$ is called the Poisson bracket, and $\circ$ is called the Jordan product. The Leibniz identity says that $\{x, -\}$ is a derivation with respect to the Jordan product.

Hom – Poisson algebras were first introduced in [8] by Makhlouf and Silvestrov. We now define the Hom-type generalization of a JMP algebra.

Definition 1.3. *A Hom – JMP algebra* $(A, \{,\}, \circ, \alpha)$ *consists of a Hom – Malcev algebra* $(A, \{,\}, \alpha)$ *and a Hom – Jordan algebra* $(A, \circ, \alpha)$ *such that the Hom – Leibniz identity*

$$\{\alpha(x), y \circ z\} = \{x, y\} \circ \alpha(z) + \alpha(y) \circ \{x, z\}$$

is satisfied for any $x, y, z \in A$.

In a Hom – JMP algebra $(A, \{,\}, \circ, \alpha)$, the operations $\{,\}$ and $\circ$ are called the Hom – Poisson bracket and the Hom – Jordan product, respectively. By the skewsymmetry of the Hom – Poisson bracket $\{,\}$, the Hom – Leibniz identity is equivalent to

$$\{x \circ y, \alpha(z)\} = \{x, z\} \circ \alpha(y) + \alpha(x) \circ \{y, z\}.$$

A JMP algebra is exactly a Hom – JMP algebra with identity twisting map.

Let $(A, \{,\}_A, \circ_A, \alpha_A)$ and $(B, \{,\}_B, \circ_B, \alpha_B)$ be two Hom – JMP algebras. A weak morphism $f\colon A \to B$ is a linear map such that

$$f\{,\}_A = \{,\}_B f^{\otimes 2} \quad \text{and} \quad f \circ_A = \circ_B f^{\otimes 2}.$$

A morphism $f\colon A \to B$ is a weak morphism such that $f\alpha_A = \alpha_B f$.

Note that a quadruple $(A, \{,\}, \circ, \alpha)$ is said multiplicative if and only if the twisting map $\alpha\colon A \to A$ is a morphism.

The following result says that Hom – JMP algebras are closed under twisting by weak self-morphisms.

Theorem 1.1. *Let* $(A, \{,\}, \circ, \alpha)$ *be a Hom – JMP algebra and* $\beta\colon A \to A$ *be a weak morphism. Then*

$$A_\beta = (A, \{,\}_\beta = \beta\{,\}, \circ_\beta = \beta\circ, \beta\alpha)$$

is also a Hom – JMP algebra. Moreover, if A *is multiplicative and* β *is a morphism, then* A_β *is a multiplicative Hom – JMP algebra.*

Proof. In [11] the author proved that $(A, \{,\}_\beta, \beta\alpha)$ is a Hom – Malcev algebra and $(A, \circ_\beta, \beta\alpha)$ is Hom – Jordan algebra.

It remains to show the Hom – Leibniz identity. Let $x, y, z \in A$, we know that

$$\{\alpha(x), y \circ z\} = \{x, y\} \circ \alpha(z) + \alpha(y) \circ \{x, z\}.$$

Now applying β^2 to the previous identity, we obtain

$$\{\beta^2\alpha(x), \beta^2(y) \circ \beta^2(z)\} = \{\beta^2(x), \beta^2(y)\} \circ \beta^2\alpha(z) + \beta^2\alpha(y) \circ \{\beta^2(x), \beta^2(z)\},$$

that is,

$$\{\beta\alpha(x), y \circ_\beta z\}_\beta = \{x, y\}_\beta \circ_\beta \beta\alpha(z) + \beta\alpha(y) \circ_\beta \{x, z\}_\beta.$$

Therefore, A_β is a Hom – JMP algebra.

Theorem 1.1. is proved.

***Corollary* 1.1.** *Let $(A, \{,\}, \circ)$ be a JMP algebra and $\alpha\colon A \to A$ be a JMP morphism. Then $(A, \{,\}_\alpha = \alpha\{,\}, \circ_\alpha = \alpha\circ, \alpha)$ is a multiplicative Hom – JMP algebra.*

1.2. Admissible Hom – JMP algebras. Let $(A, \cdot, \alpha)$ be a Hom-algebra. One can define the two following new products:

$$[x, y] = x \cdot y - y \cdot x \quad \text{and} \quad x \circ y = \frac{1}{2}(x \cdot y + y \cdot x) \quad \text{for all} \quad x, y, z \in A.$$

We will denote by A^- (respectively, A^+) the algebra A with multiplication $[-,-]$ (respectively, $\circ$).

Lemma 1.1 [11]. *Let $(A, \cdot, \alpha)$ be a Hom-flexible algebra. Then we have*

$$2S_A = J_{A^-}.$$

Lemma 1.2 [7]. *A Hom-algebra $(A, \cdot, \alpha)$ is flexible if and only if*

$$[\alpha(x), y \circ z] = [x, y] \circ \alpha(z) + \alpha(y) \circ [x, z].$$

Lemma 1.3. *Let $(A, \cdot, \alpha)$ be a Hom-flexible algebra. Then*

$$J_{A^-}(x^2, \alpha(y), \alpha(x)) = 0 \quad \forall x, y \in A, \quad \text{where} \quad x^2 = x \cdot x = x \circ x.$$

Proof. *Let $x, y \in A$. Then we have*

$$[[x^2, \alpha(y)], \alpha^2(x)] + [[\alpha(y), \alpha(x)], \alpha(x^2)] + \underbrace{[[\alpha(x), x^2], \alpha^2(y)]}_{=0} =$$

$$= 2[\alpha^2(x), [y, x] \circ \alpha(x)] + 2[[y, x], \alpha(x)] \circ \alpha^2(x) =$$

$$= 2\Big\{\alpha^2(x).([y, x].\alpha(x)) + \alpha^2(x).(\alpha(x).[y, x]) - ([y, x].\alpha(x)).\alpha^2(x) - (\alpha(x).[y, x]).\alpha^2(x) +$$

$$+ ([y, x].\alpha(x)).\alpha^2(x) - (\alpha(x).[y, x]).\alpha^2(x) + \alpha^2(x).([y, x].\alpha(x)) - \alpha^2(x).(\alpha(x).[y, x])\Big\} = 0.$$

Lemma 1.3 is proved.

The following result gives a characterization of Hom-flexible algebras.

Proposition 1.1. *A Hom-algebra* $(A, \cdot, \alpha)$ *is flexible if and only if*

$$as_A(x,y,z) = \frac{1}{4}J_{A^-}(x,y,z) + \frac{1}{4}[\alpha(y),[z,x]] + as_{A^+}(x,y,z) \quad \textit{for all} \quad x,y,z \in A. \tag{1.1}$$

Proof. If $(A, \cdot, \alpha)$ is Hom-flexible, then, by Lemma 1.1, we have

$$J_{A^-}(x,y,z) + [\alpha(y),[z,x]] + 4as_{A^+}(x,y,z) =$$

$$= 2S_A(x,y,z) + \alpha(y)(zx) - \alpha(y)(xz) - (zx)\alpha(y) + (xz)\alpha(y) + (xy)\alpha(z) +$$

$$+ (yx)\alpha(z) + \alpha(z)(xy) + \alpha(z)(yx) - \alpha(x)(yz) - \alpha(x)(zy) - (yz)\alpha(x) - (zy)\alpha(x) =$$

$$= 2S_A(x,y,z) - as_A(y,z,x) + as_A(y,x,z) - as_A(z,x,y) + as_A(x,z,y) + as_A(x,y,z) -$$

$$- as_A(z,y,x) = 4as_A(x,y,z).$$

Conversely, suppose that Eq. (1.1) holds, then

$$as_A(x,y,x) = \frac{1}{4}J_{A^-}(x,y,x) + \frac{1}{4}[\alpha(y),[x,x]] + as_{A^+}(x,y,x) =$$

$$= \frac{1}{4}\Big([[x,y],\alpha(x)] + [[y,x],\alpha(x)] + [[x,x],\alpha(y)]\Big) + (x \circ y) \circ \alpha(x) - \alpha(x) \circ (y \circ x) =$$

$$= 0 \ \ (\text{since} \circ \text{is commutative}).$$

Hence, A is Hom-flexible.

***Corollary* 1.2.** *Let* $(A, \cdot, \alpha)$ *be a Hom-flexible algebra. Then*

$$as_A(x^2, \alpha(y), \alpha(x)) = as_{A^+}(x^2, \alpha(y), \alpha(x)).$$

Proof. Straightforward.

Definition 1.4. *A Hom-algebra* $(A, \cdot, \alpha)$ *is said to be an admissible Hom – JMP algebra if* $(A, [,], \circ, \alpha)$ *is a Hom – JMP algebra.*

***Remark* 1.1.** Given a Hom – JMP algebra $(A, \{,\}, \circ, \alpha)$, then the vector space A endowed with the morphism α and the product, defined by $x \cdot y := \dfrac{1}{2}\{x,y\} + x \circ y$, is an admissible Hom – JMP algebra.

Proposition 1.2. *Let* $(A, \cdot, \alpha)$ *be an admissible Hom – JMP algebra and* $\beta \colon A \to A$ *be a weak morphism. Then*

$$A_\beta = (A, \cdot_\beta, \beta\alpha)$$

is also an admissible Hom – JMP algebra, where $x \cdot_\beta y = \beta(x) \cdot \beta(y)$*. Moreover, if* A *is multiplicative and* β *is a morphism, then* A_β *is a multiplicative admissible Hom – JMP algebra.*

Proof. Straightforward.

***Example* 1.1.** Every Hom-alternative algebra is an admissible Hom – JMP algebra.

***Remark* 1.2.** Note that not all admissible Hom – JMP algebras are Hom-alternative algebras. Indeed, let A be the three-dimensional algebra defined with respect to of basis $\{e_1, e_2, e_3\}$ by

$\star$	e_1	e_2	e_3
e_1	0	e_2	$-e_3$
e_2	$-e_2$	0	e_1
e_3	e_3	$-e_1$	0

According to [3], A is an admissible JMP algebra. Consider the morphism $\alpha\colon A \to A$ defined by

$$\alpha(e_1) = e_1, \quad \alpha(e_2) = \lambda e_2, \quad \alpha(e_3) = \frac{1}{\lambda} e_3, \quad \lambda \in \mathbb{K}\backslash\{0\}.$$

Then, in view of Proposition 1.2, $(A, \star_\alpha, \alpha)$ is an admissible Hom – JMP algebra. On the other hand,

$$as_A(e_2, e_3, e_3) = (e_2 \star_\alpha \star e_3) \star_\alpha \alpha(e_3) - \alpha(e_2) \star_\alpha (e_3 \star_\alpha e_3) = -\frac{1}{\lambda^2} e_3 \neq 0.$$

Then A is not a Hom-alternative algebra.

***Remark* 1.3.** Every admissible Hom – JMP algebra is Hom-flexible.

Theorem 1.2. *Let $(A, \cdot, \alpha)$ be a Hom-flexible and Hom – Malcev admissible algebra. Then A is an admissible Hom – JMP algebra if and only if $(A, \cdot, \alpha)$ satisfies the identity*

$$R_{\alpha^2(x)} L_{x^2} \alpha = L_{\alpha(x^2)} R_{\alpha(x)} \alpha \quad \forall x \in A, \tag{1.2}$$

where L_x (respectively, R_x) is the left multiplication (respectively, the right multiplication) by x in the algebra $(A, \cdot, \alpha)$.

Proof. Since $(A, \cdot, \alpha)$ is Hom-flexible and due to Corollary 1.2, we have

$$as_A(x^2, \alpha(y), \alpha(x)) = as_{A^+}(x^2, \alpha(y), \alpha(x)).$$

The rest of the proof follows immediately.

***Example* 1.2.** Consider the five-dimensional Hom-algebra $(A, \cdot, \alpha)$ with respect to a basis $\{e_1, \ldots, e_5\}$ and multiplication table

$\cdot$	e_1	e_2	e_3	e_4	e_5
e_1	0	$e_5 + \frac{1}{2} e_4$	0	$\frac{\nu}{2} e_1$	0
e_2	$e_5 - \frac{1}{2} e_4$	0	0	$-\frac{\nu^{-1}}{2} e_2$	0
e_3	0	0	0	$\frac{\lambda}{2} e_3$	0
e_4	$-\frac{\nu}{2} e_1$	$\frac{\nu^{-1}}{2} e_2$	$-\frac{\lambda}{2} e_3$	$-e_5$	0
e_5	0	0	0	0	0

where $\alpha\colon A \to A$ is given by

$$\alpha(e_1) = \nu e_1, \quad \alpha(e_2) = \nu^{-1} e_2, \quad \alpha(e_3) = \lambda e_3,$$

$$\alpha(e_4) = e_4, \quad \alpha(e_5) = e_5.$$

In [11], D. Yau proved that $(A, \cdot, \alpha)$ is Hom-flexible and Hom – Malcev admissible algebra. In addition, by a direct calculation, we can easily verify that it satisfies condition (1.2). Hence, using Theorem 1.2, we conclude that $(A, \cdot, \alpha)$ is an admissible Hom – JMP algebra.

Now we prove that multiplicative admissible Hom – JMP algebras are power Hom-associative. The power associativity of admissible JMP algebras is shown in [3] (Proposition 2.2). Let us begin by recalling the definition of a power Hom-associative algebra.

Definition 1.5 [10]. *Let $(A, \cdot, \alpha)$ be a Hom-algebra, $x \in A$, and n be a positive integer.*

1. *The nth Hom-power $x^n \in A$ is defined by*

$$x^1 = x, \qquad x^n = x^{n-1} \cdot \alpha^{n-2}(x), \quad n \geq 2.$$

2. *A is called nth power Hom-associative if*

$$x^n = \alpha^{n-i-1}(x^i) \cdot \alpha^{i-1}(x^{n-i})$$

for all $x \in A$ and $i \in \{1, \ldots, n-1\}$.

3. *A is called power Hom-associative if A is nth power Hom-associative for all $n \geq 2$.*

If the twisting map α is the identity map, then the nth power Hom-associativity becomes

$$x^n = x^i \cdot x^{n-i}. \tag{1.3}$$

The class of power Hom-associative algebras contains multiplicative right Hom-alternative algebras and noncommutative Hom – Jordan algebras. Other results for power Hom-associative algebras can be found in [10].

A well-known result of Albert [1] says that an algebra $(A, \cdot)$ is power associative if and only if it is third and fourth power associative, i.e., the condition (1.3) holds for $n = 3,\ 4$. Moreover, for (1.3) to hold for $n = 3,\ 4$, it is necessary and sufficient that

$$(x \cdot x) \cdot x = x \cdot (x \cdot x) \quad \text{and} \quad ((x \cdot x) \cdot x) \cdot x = (x \cdot x) \cdot (x \cdot x) \quad \text{for all} \quad x \in A.$$

The Hom-versions of these statements are also proved in [10]. More precisely, a multiplicative Hom-algebra $(A, \cdot, \alpha)$ is power Hom-associative if and only if it is third and fourth power Hom-associative, which in turn is equivalent to

$$x^3 = x^2 \cdot \alpha(x) = \alpha(x) \cdot x^2 \quad \text{and} \quad x^4 = x^3 \cdot \alpha^2(x) = \alpha(x^2) \cdot \alpha(x^2) \tag{1.4}$$

for all $x \in A$.

The following result is the Hom-version of Proposition 2.2 in [3].

Theorem 1.3. *Every multiplicative admissible Hom – MJP algebra is power Hom-associative.*

Proof. As discussed above, by a result in [10], it is sufficient to prove the two identities in (1.4). The Hom-flexibility implies that

$$0 = as_A(x, x, x) = x^2 \cdot \alpha(x) - \alpha(x) \cdot x^2,$$

which proves the first identity in (1.4). To prove the second equality in (1.4), since $(A, \circ, \alpha)$ is Hom – Jordan and due to Corollary 1.2, we get

$$0 = as_A(x^2, \alpha(x), \alpha(x)) = (x^2 \cdot \alpha(x)) \cdot \alpha^2(x) - \alpha(x^2) \cdot (\alpha(x) \cdot \alpha(x)) =$$

$$= x^3 \cdot \alpha^2(x) - \alpha(x^2) \cdot \alpha(x^2).$$

We have proved the second identity in (1.4).

Theorem 1.3 is proved.

2. Pseudo-Euclidian Hom – JMP algebras. In this section, we extend the notion of pseudo-Euclidian JMP algebra to Hom – JMP algebras and provide some properties. Let $(A, \cdot, \alpha)$ be a Hom-algebra and $B\colon A \times A \to \mathbb{K}$ be a bilinear form. Then B is called:

(i) symmetric if $B(x, y) = B(y, x)\ \forall x, y \in A$;

(ii) nondegenerate if $B(x, y) = 0\ \forall y \in A \Rightarrow x = 0$ and if $B(x, y) = 0\ \forall x \in A \Rightarrow y = 0$;

(iii) invariant if $B(x \cdot y, z) = B(x, y \cdot z)\ \forall x, y, z \in A$.

In this case, $(A, \cdot, \alpha, B)$ will be called a pseudo-Euclidian Hom-algebra if, in addition,

$$B(\alpha(x), y) = B(x, \alpha(y)) \quad \forall x, y \in A.$$

Definition 2.1. *Let $(A, \{,\}, \circ, \alpha)$ be a Hom – JMP algebra and $B\colon A \times A \to \mathbb{K}$ be a symmetric, nondegenerate and invariant bilinear form on A. We say that $(A, \{,\}, \circ, \alpha, B)$ is a pseudo-Euclidian Hom – JMP algebra if $(A, \{,\}, \alpha, B)$ and $(A, \circ, \alpha, B)$ are pseudo-Euclidian Hom-algebras.*

Definition 2.2. *A Hom – JMP algebra $(A, \{,\}, \circ, \alpha)$ is called Hom – pseudo-Euclidian if there exists (B, γ), where B is a symmetric and nondegenerate bilinear form on A and $\gamma\colon A \to A$ is an homomorphism such that*

$$B(\alpha(x), y) = B(x, \alpha(y)),\ \ B(\{x, y\}, \gamma(z)) = B(\gamma(x), \{y, z\}),\ \ B(x \circ y, \gamma(z)) = B(\gamma(x), y \circ z)$$

for all $x, y, z \in A$.

***Remark* 2.1.** Note that we recover pseudo-Euclidian Hom – JMP algebras when $\gamma = id$.

Let $(A, \{,\}, \circ, B)$ be a pseudo-Euclidian JMP algebra. We denote by $\mathrm{Aut}_S(A, B)$ the set of symmetric automorphisms of A with respect to B, that is automorphisms $\beta\colon A \to A$ such that $B(\beta(x), y) = B(x, \beta(y))\ \forall x, y \in A$.

Proposition 2.1. *Let $(A, \{,\}, \circ, B)$ be a pseudo-Euclidian JMP algebra and $\alpha \in \mathrm{Aut}_S(A, B)$. Then $A_\alpha = (A, \{,\}_\alpha, \circ_\alpha, \alpha, B_\alpha)$ is a pseudo-Euclidian Hom – JMP algebra, where, for any $x, y \in A$,*

$$\{x, y\}_\alpha = \{\alpha(x), \alpha(y)\}, \quad x \circ_\alpha y = \alpha(x) \circ \alpha(y), \quad B_\alpha(x, y) = B(\alpha(x), y).$$

Proof. By Corollary 1.1, $(A, \{,\}_\alpha, \circ_\alpha, \alpha)$ is a Hom – JMP algebra. The bilinear form B_α is nondegenerate since B is nondegenerate and α bijective. Now, let $x, y, z \in A$, then

$$B_\alpha(\{x, y\}_\alpha, z) = B(\alpha(\{\alpha(x), \alpha(y)\}), z) = B(\{\alpha(x), \alpha(y)\}, \alpha(z)) =$$

$$= B(\alpha(x), \{\alpha(y), \alpha(z)\}) = B_\alpha(x, \{y, z\}_\alpha)$$

and

$$B_\alpha(x \circ_\alpha y, z) = B(\alpha(\alpha(x) \circ \alpha(y)), z) =$$

$$= B(\alpha(x) \circ \alpha(y), \alpha(z)) = B(\alpha(x), \alpha(y) \circ \alpha(z)) = B_\alpha(x, y \circ_\alpha z).$$

On the other hand,

$$B_\alpha(x, y) = B(\alpha(x), y) = B(x, \alpha(y)) = B(\alpha(y), x) = B_\alpha(y, x)$$

and

$$B_\alpha(\alpha(x), y) = B(\alpha(\alpha(x)), y) = B(\alpha(x), \alpha(y)) = B_\alpha(x, \alpha(y)).$$

Proposition 2.1 is proved.

The following result allows to obtain new pseudo-Euclidian Hom – JMP algebras starting from multiplicative pseudo-Euclidian Hom – JMP algebras.

Proposition 2.2. *Let* $(A, \{,\}, \circ, \alpha, B)$ *be a pseudo-Euclidian Hom – JMP algebra. For any* $n \geq$ ≥ 0, *the quadruple*

$$A_n = \Big(A, \{,\}_n = \alpha^n\{,\}, \circ_n = \alpha^n \circ, \alpha^{n+1}, B_n\Big),$$

where B_n *is defined for* $x, y \in A$ *by* $B_n(x, y) = B(\alpha^n(x), y)$, *determines a pseudo-Euclidian Hom – JMP algebra.*

Proof. Straightforward.

We provide here a construction of pseudo-Euclidian Hom – JMP algebra from an arbitrary Hom – JMP algebra (not necessarily pseudo-Euclidian).

Let $(A, \{,\}, \circ)$ be a JMP-algebra and A^* be the dual vector space of the underlying vector space of A. On the vector space $P = A \oplus A^*$, we define the following bracket $\{,\}_P$ and multiplication $\circ_P$ by

$$\{x + f, y + g\}_P := \{x, y\} + f\ ad_y - g\ ad_x,$$

$$(x + f) \circ_P (y + g) := x \circ y + f\ L_y + g\ L_x \quad \forall (x, f), (y, g) \in P,$$

where $ad_x(y) = \{x, y\}$ and $L_x(y) = x \circ y$. Moreover, we consider the bilinear form B defined on P by

$$B(x + f, y + g) = f(y) + g(x) \quad \forall (x, f), (y, g) \in P.$$

$(P, \{,\}_P, \circ_P, B)$ is a pseudo-Euclidian JMP algebra called the T^*-extension of A by means of A^*.

Proposition 2.3. *Let* $(A, \{,\}, \circ)$ *be a JMP algebra and* $\alpha \in \mathrm{Aut}(A)$. *Then the endomorphism* $\beta = \alpha +^t \alpha$ *of* P *is an automorphism of* P *if and only if* $\mathrm{Im}(\alpha^2 - Id) \subseteq Z_J(A) \cap Z_M(A)$, *where* $Z_J(A)$ *is the center of* $(A, \circ)$ *and* $Z_M(A)$ *is the center of* $(A, \{,\})$. *Hence, if* $\mathrm{Im}(\alpha^2 - Id) \subseteq$ $\subseteq Z_J(A) \cap Z_M(A)$ *then* $(P, \{,\}_{P,\beta}, \circ_{P,\beta}, B_\beta)$ *is a regular pseudo-Euclidian Hom – JMP algebra.*

Proof. Let $x, y \in A$ and $f, g \in A^*$. Then

$$\beta(\{x + f, y + g\}_P) = \beta(\{x, y\} + f ad_y - g ad_x) =$$

$$= \alpha(\{x, y\}) + f ad_y \alpha - g ad_x \alpha,$$

and

$$\{\beta(x + f), \beta(y + g)\}_P = \{\alpha(x) + f\alpha, \alpha(y) + g\alpha\}_P =$$

$$= \{\alpha(x), \alpha(y)\} + f\alpha ad_{\alpha(y)} - g\alpha ad_{\alpha(x)}.$$

Then $\beta(\{x + f, y + g\}_P) = \{\beta(x + f), \beta(y + g)\}_P$ if and only if

$$f ad_y \alpha - g ad_x \alpha = f\alpha ad_{\alpha(y)} - g\alpha ad_{\alpha(x)} \quad \forall x, y \in A,$$

that is, for all $z \in A$,

$$f(\{y, \alpha(z)\}) - g(\{x, \alpha(z)\}) = f(\alpha\{\alpha(y), z\}) - g(\alpha\{\alpha(x), z\}).$$

Hence, β is an automorphism of $(P, \{,\}_P)$ if and only if $f(\{x, \alpha(y)\}) = f(\alpha\{\alpha(x), y\})$ $\forall f \in A^*$ $\forall x, y \in A$, which is equivalent to $\{x, \alpha(y)\} = \alpha(\{\alpha(x), y\})$ $\forall x, y \in A$.

As a consequence, β is an automorphism of $(P, \{,\}_P)$ if and only if $\{\alpha^2(x) - x, \alpha(y)\} = 0$ $\forall x, y \in A$, i.e., $\mathrm{Im}(\alpha^2 - id) \subseteq Z_M(A)$, since $\alpha \in \mathrm{Aut}(A)$.

Similarly, β is an automorphism of $(P, \circ_P)$ if and only if $\mathrm{Im}(\alpha^2 - id) \subseteq Z_J(A)$. Then $\beta \in$ $\in \mathrm{Aut}(P)$ if and only if $\mathrm{Im}(\alpha^2 - id) \subseteq Z_M(A) \cap Z_J(A)$.

In the following we show that β is symmetric with respect to B. Indeed, let $x, y \in A$ and $f, g \in A^*$, then

$$B(\beta(x+f), y+g) = B(\alpha(x) + f\alpha, y+g) = f(\alpha(y)) + g(\alpha(x)) =$$

$$= B(x+f, \alpha(y) + g\alpha) = B(x+f, \beta(y+g)).$$

The last assertion is a consequence of the previous calculations and Proposition 2.1.

Proposition 2.3 is proved.

***Corollary* 2.1.** *Let $(A, \{,\}, \circ)$ be a JMP algebra and $\theta \in \mathrm{Aut}(A)$ such that $\theta^2 = id$ (θ is an involution). Then $(P, \{,\}_{P,\beta}, \circ_{P,\beta}, B_\beta)$ is a regular pseudo-Euclidian Hom–JMP algebra, where $\beta = \theta +^t \theta$.*

3. Hom – Lie – Jordan – Poisson triple system. In this section, we generalize the notion of Lie – Jordan – Poisson triple system introduced in [3] to the Hom setting. We provide the relationships of this class of algebras with admissible Hom – JMP algebras. Finally, we endow it with a symmetric nondegenerate invariant bilinear form and give some key constructions.

Definition 3.1. *A Hom – Lie triple system is a triple $(L, [,,], \alpha = (\alpha_1, \alpha_2))$ that satisfies the following conditions*:

(i) $[x, y, z] = -[y, x, z]$ (*left skewsymmetry*),

(ii) $[x, y, z] + [y, z, x] + [z, x, y] = 0$ (*ternary Jacobi identity*),

(iii) $[\alpha_1(x), \alpha_2(y), [u, v, w]] = [[x, y, u], \alpha_1(v), \alpha_2(w)] + [\alpha_1(u), [x, y, v], \alpha_2(w)] + [\alpha_1(u), \alpha_2(v), [x, y, w]]$

for all $u, v, w, x, y, z \in L$.

A particular situation, interesting for our setting, occurs when twisting maps α_i are all equal, that is, $\alpha_1 = \alpha_2 = \alpha$, and $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$ for all $x, y, z \in L$. The Hom – Lie triple system $(L, [.,.,.], \alpha)$ is said to be multiplicative.

Definition 3.2. *A Hom – Lie – Jordan – Poisson triple system is a quadruple $(A, \{,,\}, \circ, \alpha)$ such that*

(i) *$(A, \circ, \alpha)$ is a Hom – Jordan algebraz*,

(ii) *$(A, \{,,\}, \alpha)$ is a multiplicative Hom – Lie triple system*,

(iii) *$\{\alpha(x), \alpha(y), z \circ t\} = \{x, y, z\} \circ \alpha(t) + \alpha(z) \circ \{x, y, t\}$ $\forall x, y, z, t \in A$.*

In the case where $(A, \circ, \alpha)$ is a commutative Hom-associative algebra, the quadruple $(A, \{,,\}, \circ, \alpha)$ is called a Hom – Lie – Poisson triple system.

Lemma 3.1 [2]. *Let $(A, \{,\}, \alpha)$ be a Hom – Malcev algebra. Then $(A, \{,,\}, \alpha^2)$ is a multiplicative Hom – Lie triple system*, *where*

$$\{x, y, z\} = 2\{\{x, y\}, \alpha(z)\} - \{\{y, z\}, \alpha(x)\} - \{\{z, x\}, \alpha(y)\} \quad \forall x, y, z \in A.$$

Proposition 3.1. *Let $(A, \{,\}, \circ, \alpha)$ be a Hom – JMP algebra. Then the quadruple $(A, \{,,\}, \circ_\alpha, \alpha^2)$ is a Hom – Lie – Jordan – Poisson triple system*, *where*

$$\{x, y, z\} = 2\{\{x, y\}, \alpha(z)\} - \{\{y, z\}, \alpha(x)\} - \{\{z, x\}, \alpha(y)\} \quad \forall x, y, z \in A, \tag{3.1}$$

$$x \circ_\alpha y = \alpha(x) \circ \alpha(y) \quad \forall x, y \in A. \tag{3.2}$$

Proof. Let $x, y, z, t \in A$, we have

$$\{\alpha^2(x), \alpha^2(y), z \circ_\alpha t\} = 2\{\{\alpha^2(x), \alpha^2(y)\}, \alpha(z \circ_\alpha t)\} -$$

$$- \{\{\alpha^2(y), z \circ_\alpha t\}, \alpha^3(x)\} - \{\{z \circ_\alpha t, \alpha^2(x)\}, \alpha^3(y)\} =$$

$$= \alpha\Big(2\{\{\alpha(x), \alpha(y)\}, \alpha(z) \circ \alpha(t)\} - \{\{\alpha(y), z \circ t\}, \alpha^2(x)\} - \{\{z \circ t, \alpha(x)\}, \alpha^2(y)\}\Big) =$$

$$= \alpha\Big(2\{\{x, y\}, \alpha(z)\} \circ \alpha^2(t) + 2\alpha^2(z) \circ \{\{x, y\}, \alpha(t)\} - \{\{y, z\} \circ \alpha(t), \alpha^2(x)\} -$$

$$- \{\alpha(z) \circ \{y, t\}, \alpha^2(x)\} - \{\alpha(z) \circ \{t, x\}, \alpha^2(y)\} - \{\{z, x\} \circ \alpha(t), \alpha^2(y)\}\Big) =$$

$$= \alpha\Big(2\{\{x, y\}, \alpha(z)\} \circ \alpha^2(t) + 2\alpha^2(z) \circ \{\{x, y\}, \alpha(t)\} - \{\alpha(y), \alpha(z)\} \circ \{\alpha(t), \alpha(x)\} -$$

$$- \{\{y, z\}, \alpha(x)\} \circ \alpha^2(t) - \{\alpha(z), \alpha(x)\} \circ \{\alpha(y), \alpha(t)\} - \alpha^2(z) \circ \{\{y, t\}, \alpha(x)\} -$$

$$- \{\alpha(z), \alpha(y)\} \circ \{\alpha(t), \alpha(x)\} - \alpha^2(z) \circ \{\{t, x\}, \alpha(y)\} -$$

$$- \{\alpha(z), \alpha(x)\} \circ \{\alpha(t), \alpha(y)\} - \{\{z, x\}, \alpha(y)\} \circ \alpha^2(t)\Big) =$$

$$= \alpha\Big(\alpha^2(z) \circ (\{\{x, y\}, \alpha(t)\} - \{\{y, t\}, \alpha(x)\} - \{\{t, x\}, \alpha(y)\}) +$$

$$+ (2\{\{x, y\}, \alpha(z)\} - \{\{y, z\}, \alpha(x)\} - \{\{z, x\}, \alpha(y)\}\circ)\alpha^2(t)\Big) =$$

$$= \alpha^2(z) \circ_\alpha \{x, y, t\} + \{x, y, z\} \circ_\alpha \alpha^2(t).$$

Proposition 3.1 is proved.

Let $(A, \{,,\}, \alpha)$ be a multiplicative Hom – Lie triple system and $B\colon A \times A \to \mathbb{K}$ be a symmetric nondegenerate bilinear form. We say that B is invariant if

$$B(L(x, y)(z), t) = -B(z, L(x, y)(t)) \quad \forall x, y, z, t \in A,$$

where $L(x, y)(z) = \{x, y, z\}$. In this case $(A, \{,,\}, \alpha, B)$ will be called a pseudo-Euclidian Hom – Lie triple system if, in addition,

$$B(\alpha(x), y) = B(x, \alpha(y)) \quad \forall x, y \in A.$$

Definition 3.3. *Let $(A, \{,,\}, \circ, \alpha)$ be a Hom – Lie – Jordan – Poisson triple system and $B\colon A \times \times A \to \mathbb{K}$ be a bilinear form on A. We say that $(A, \{,,\}, \circ, \alpha, B)$ is a pseudo-Euclidian Hom – Lie – Jordan – Poisson triple system if $(A, \{,,\}, \alpha, B)$ is a pseudo-Euclidian Hom – Lie triple system and $(A, \circ, \alpha, B)$ is a pseudo-Euclidian Hom – Jordan algebra.*

Definition 3.4. *A Hom – Lie – Jordan – Poisson triple system $(A, \{,,\}, \circ, \alpha)$ is called Hom – pseudo-Euclidian if there exists (B, γ), where B is a symmetric and nondegenerate bilinear form on A and $\gamma\colon A \to A$ is a homomorphism such that*

$$B(\alpha(x), y) = B(x, \alpha(y)),$$

$$B(L(x, y)(z), \gamma(t)) = -B(\gamma(z), L(x, y)(t)),$$

$$B(x \circ y, \gamma(z)) = B(\gamma(x), y \circ z)$$

for all $x, y, z, t \in A$.

***Corollary* 3.1.** *Let* $(A, \{,\}, \circ, \alpha, B)$ *be a pseudo-Euclidian Hom – JMP algebra. Then the* 6-*uplet* $(A, \{,,\}, \circ_\alpha, \alpha^2, B, \alpha)$ *is a Hom – pseudo-Euclidian Hom – Lie – Jordan – Poisson triple system, where* $\{,,\}$ *and* $\circ_\alpha$ *are defined in* (3.1) *and* (3.2).

Proof. Let $x, y, z, t \in A$, we have

$$B(\{x,y,z\},\alpha(t)) = B(2\{\{x,y\},\alpha(z)\} - \{\{y,z\},\alpha(x)\} - \{\{z,x\},\alpha(y)\},\alpha(t)) =$$

$$= B(2\{\{x,y\},\alpha(z)\},\alpha(t)) - B(\{\{y,z\},\alpha(x)\},\alpha(t)) - B(\{\{z,x\},\alpha(y)\},\alpha(t)) =$$

$$= -B(\alpha(z), 2\{\{x,y\},\alpha(t)\}) - B(\{y,z\},\{\alpha(x),\alpha(t)\}) - B(\{z,x\},\{\alpha(y),\alpha(t)\}) =$$

$$= -B(\alpha(z), 2\{\{x,y\},\alpha(t)\}) - B(\{\alpha(y),\alpha(z)\},\{x,t\}) - B(\{\alpha(z),\alpha(x)\},\{y,t\}) =$$

$$= -B(\alpha(z), 2\{\{x,y\},\alpha(t)\}) - B(\alpha(z),\{\{x,t\},\alpha(y)\}) - B(\alpha(z),\{\{t,y\},\alpha(x)\}) =$$

$$= -B(\alpha(z), 2\{\{x,y\},\alpha(t)\} - \{\{x,t\},\alpha(y)\} - \{\{t,y\},\alpha(x)\}) =$$

$$= -B(\alpha(z),\{x,y,t\}).$$

Corollary 3.1 is proved.

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