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POSITIVE SOLUTIONS OF A THREE-POINT BOUNDARY-VALUE PROBLEM FOR p-LAPLACIAN DYNAMIC EQUATION ON TIME SCALES

ДОДАТНІ РОЗВ'ЯЗКИ ТРИТОЧКОВОЇ КРАЙОВОЇ ЗАДАЧІ ДЛЯ ДИНАМІЧНОГО РІВНЯННЯ ІЗ p-ЛАПЛАСІАНОМ НА ЧАСОВИХ ШКАЛАХ

We consider a three-point boundary-value problem for p-Laplacian dynamic equation on time scales. We show the existence at least three positive solutions of the boundary-value problem by using the Avery and Peterson fixed point theorem. The conditions we used here differ from those in the majority of papers as we know. The interesting point is that the nonlinear term f involves the first derivative of the unknown function. As an application, an example is given to illustrate our results.

Розглядається триточкова крайова задача для динамічного рівняння із p-лапласіаном на часових шкалах. За допомогою теореми Ейвери та Петерсона про нерухому точку доведено існування принаймні трьох додатних розв'язків такої крайової задачі. Умови, які використовуються тут, відрізняються від умов, які використано у більшості відомих нам робіт. Цікавим моментом є те, що нелінійний член f містить першу похідну невідомої функції. Як застосування наведено приклад для ілюстрації отриманих результатів.

1. Introduction. This paper is concerned with the existence of positive solutions of the p-Laplacian dynamic equation on time scales

$$\big(\phi_p(u^{\Delta}(t))\big)^{\nabla} + g(t)f\big(t, u(t), u^{\Delta}(t)\big) = 0, \quad t \in [0,T]_{\mathbb{T}}, \tag{1.1}$$

$$u(0) - B_0\big(u^{\Delta}(\nu)\big) = 0, \quad u^{\Delta}(T) = 0, \tag{1.2}$$

or

$$u^{\Delta}(0) = 0, \qquad u(T) + B_1\big(u^{\Delta}(\nu)\big) = 0, \tag{1.3}$$

where $\phi_p(s)$ is p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, with $(\phi_p)^{-1} = \phi_q$ and $1/p + 1/q = 1$, $\nu \in (0, \rho(T))_{\mathbb{T}}$. Some basic knowledge and definitions about time scales, which can be found in [7, 8]. As we know, when the nonlinear term f is involved in the first-order derivative, difficulties arise immediately. In this work, we use a fixed point theorem because of Avery and Peterson to overcome the difficulties.

Throughout the paper, we will suppose that the following conditions are satisfied:

(H_1) $\mathbb{T}$ is a time scales with $0,\ T \in \mathbb{T},\ \nu \in (0, \rho(T))_{\mathbb{T}}$;

(H_2) let $\zeta \geq \min\left\{t \in \mathbb{T} : t \geq \dfrac{T}{2}\right\}$, and there exists $\tau \in \mathbb{T}$ such that $\zeta < \tau < T$ holds;

(H_3) $f : [0,T]_{\mathbb{T}} \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous, and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$;

(H_4) $g : \mathbb{T} \to \mathbb{R}^+$ is left dense continuous $\big($i.e., $g \in C_{ld}(\mathbb{T}, \mathbb{R}^+)\big)$, and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$;

(H$_5$) $B_0(\upsilon)$ and $B_1(\upsilon)$ are both continuous odd functions defined on $\mathbb{R}$ and satisfy that there exist $A, B > 0$ such that

$$B\upsilon \le B_j(\upsilon) \le A\upsilon, \qquad \upsilon \ge 0 \quad j = 0, 1.$$

In [3], Anderson established the existence of multiple positive solutions to the nonlinear second-order three-point boundary-value problem (BVP) on time scale $\mathbb{T}$ given by

$$u^{\Delta\nabla}(t) + f(t, u(t)) = 0, \quad t \in (0, T) \subset \mathbb{T},$$

$$u(0) = 0, \quad au(\eta) = u(T).$$

He employed the Leggett – Williams fixed point theorem in an appropriate cone to guarantee the existence of at least three positive solutions to this nonlinear problem.

Anderson et al. [4] studied the time scale, delta-nabla dynamic equation

$$(g(u^{\Delta}))^{\nabla} + c(t)f(u) = 0 \quad \text{for} \quad a < t < b$$

with boundary conditions

$$u(a) - B_0\big(u^{\Delta}(\nu)\big) = 0 \quad \text{and} \quad u^{\Delta}(b) = 0.$$

They established the existence result of at least one positive solution by a fixed point theorem of cone expansion and compression of functional type.

In [9], Dogan investigated the following p-Laplacian dynamic equation on time scales:

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + a(t)f\big(t, u(t), u^{\Delta}(t)\big) = 0, \quad t \in [0, T]_{\mathbb{T}},$$

$$u(0) - B_0(u^{\Delta}(0)) = 0, \quad u^{\Delta}(T) = 0,$$

where $\phi_p(u) = |u|^{p-2}u$ for $p > 1$. We proved the existence of triple positive solutions for the one-dimensional p-Laplacian BVP by using the Leggett – Williams fixed point theorem.

In [10], Dogan studied the existence of positive solutions of the p-Laplacian dynamic equation on time scales

$$\big(\phi_p(y^{\Delta}(t))\big)^{\nabla} = -w(t)f\big(t, y(t), y^{\Delta}(t)\big), \quad t \in [0, T]_{\mathbb{T}},$$

$$y(0) - B_0(y^{\Delta}(\nu)) = 0, \quad y^{\Delta}(T) = 0,$$

or

$$y^{\Delta}(0) = 0, \quad y(T) + B_1(y^{\Delta}(\nu)) = 0.$$

We proved the existence at least three positive solutions of the BVP by using the Avery and Peterson fixed point theorem.

In [11], Guo considered the following one-dimensional p-Laplacian three-point BVP on time scales

$$\big(\varphi_p(u^{\Delta}(t))\big)^{\nabla}+h(t)f\big(t,u(t),u^{\Delta}(t)\big)=0,\quad t\in(0,T)_{\mathbb{T}},$$

$$u(0)-\beta u^{\Delta}(0)=\gamma u^{\Delta}(\eta),\quad u^{\Delta}(T)=0.$$

He established existence criteria for at least three positive solutions by using a fixed point theorem for operators on a cone.

In [12], He investigated the existence of positive solutions of the p-Laplacian dynamic equation on a time scale

$$\big[\phi_p(u^{\Delta}(t))\big]^{\nabla}+a(t)f(u(t))=0,\quad t\in[0,T]_{\mathbb{T}},$$

satisfying the boundary conditions

$$u(0)-B_0\big(u^{\Delta}(\nu)\big)=0,\quad u^{\Delta}(T)=0,$$

or

$$u^{\Delta}(0)=0,\quad u(T)+B_1\big(u^{\Delta}(\nu)\big)=0,$$

where $\phi_p(s)$ is p-Laplacian operator, i.e., $\phi_p(s)=|s|^{p-2}s,\ p>1,\ (\phi_p)^{-1}=\phi_q,\ 1/p+1/q=1,$ $\nu\in(0,\rho(T))_{\mathbb{T}}$. By using a new double fixed point theorem due to Avery et al. [5] in a cone, he proved that there exists at least double positive solutions of BVP.

In [19], Sun and Li studied the one-dimensional p-Laplacian BVP on time scales

$$\big(\varphi_p(u^{\Delta}(t))\big)^{\Delta}+h(t)f(u^{\sigma}(t))=0,\quad t\in[a,b],$$

$$u(a)-B_0(u^{\Delta}(a))=0,\quad u^{\Delta}(\sigma(b))=0,$$

where $\varphi_p(u)$ is p-Laplacian operator, i.e., $\varphi_p(u)=|u|^{p-2}u,\ p>1$. They found some new results for the existence of at least single, twin or triple positive solutions of the above problem by using Krasnosel'skii's fixed point theorem, new fixed point theorem because of Avery and Henderson and Leggett – Williams fixed point theorem.

Sun et al. [20] considered the eigenvalue problem for the following one-dimensional p-Laplacian three-point BVP on time scales

$$\big(\varphi_p(u^{\Delta}(t))\big)^{\nabla}+\lambda h(t)f(u(t))=0,\quad t\in(0,T)_{\mathbb{T}},$$

$$u(0)-\beta u^{\Delta}(0)=\gamma u^{\Delta}(\eta),\quad u^{\Delta}(T)=0.$$

They established some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the BVP.

In [23], Wang considered the existence of three positive solutions to the following BVPs for p-Laplacian dynamic equations on time scales:

$$\left[\phi_p(u^{\Delta}(t))\right]^{\nabla} + a(t)f(u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}},$$

$$u^{\Delta}(0) = 0, \quad u(T) + B_1(u^{\Delta}(\eta)) = 0,$$

or

$$u(0) - B_0(u^{\Delta}(\eta)) = 0, \quad u^{\Delta}(T) = 0.$$

He established the existence result for at least three positive solutions by using the Leggett – Williams fixed point theorem.

In recent years, there has been much current attention focused on study of positive solutions of BVPs on time scales. When the nonlinear term f does not depend on the first-order derivative, nonlinear BVPs on time scales have been studied extensively in the literature (see [1, 3, 4, 12 – 23]). However, there are few papers dealing with the existence of positive solutions for BVPs on time scales when the nonlinear term f is involved in the first-order derivative explicitly (see [9, 11]).

Compared with [9] and [11], in this paper, we remark that our boundary conditions are entirely different from those used in [9, 11]. Dogan [9] studied the existence of positive solutions of a two-point BVP on time scales by using Leggett – Williams fixed point theorem. Here we study the existence of positive solutions of a three-point BVP on time scales by using Avery and Peterson fixed point theorem. Guo [11] studied the existence of positive solutions for p-Laplacian three-point BVPs on time scales by using the Avery and Peterson fixed point theorem. His method was the same as ours. But the assumptions we used in the paper are different from those in [11]. We have defined that Banach space E and the cone P are different from [11]. Guo [11] took $t \in [0,T]$. We have taken $t \in [0, \sigma(T)]$ instead of $t \in [0,T]$.

Compared with [10], in this paper, we are concerned with same problem. So the papers all look the same and both papers seem to achieve similar results. But here we have replaced $\nu \in (0,T)_{\mathbb{T}}$ with $\nu \in (0, \rho(T))_{\mathbb{T}}$. We have also replaced $t \in [0,T]_{\mathbb{T}}$ with $t \in [0, \sigma(T)]_{\mathbb{T}}$. Therefore, Lemmas 2.1 and 2.2 and their proofs are different from Lemmas 2.2 and 2.3 in [10]. We define that the cones P and P_1 are different from the published paper [10]. Moreover, example is slightly different from the published paper [10].

Motivated by works mentioned above, in this paper, we shall show that the BVP (1.1) and (1.2) has a least three positive solutions by using the the fixed point theorem due to Avery and Peterson. The interesting point is that the nonlinear term f is involved with the first-order derivative explicitly. Our results are new for the special cases of difference equations and differential equations as well as in the general time scale setting.

This paper is organized as follows. In Section 2, we state some definitions, notations, lemmas and prove several preliminary results. In Sections 3 and 4, by defining an appropriate Banach space and cones, we impose the growth conditions on f which allow us to apply the fixed point theorem in finding existence of three positive solutions of (1.1), (1.2) (respectively, (1.1), (1.3)). In Section 5, we give an example to demonstrate our results.

2. Preliminaries and lemmas. In this section, we list the following well-known definitions which can be found in [7, 8].

Definition 2.1. *A time scale* $\mathbb{T}$ *is a closed nonempty subset of* $\mathbb{R}$. *For* $t < \sup \mathbb{T}$ *and* $r > \inf \mathbb{T}$, *define the forward jump operator* σ *and the backward jump operator* ρ *as, respectively,*

$$\sigma(t) = \inf\{\tau \in \mathbb{T} \colon \tau > t\} \in \mathbb{T},$$

$$\rho(r) = \sup\{\tau \in \mathbb{T} \colon \tau < r\} \in \mathbb{T}$$

for all $t, r \in \mathbb{T}$. *If* $\sigma(t) > t$, t *is said to be right scattered, and if* $\sigma(t) = t$, t *is said to be right dense* (*rd*). *If* $\rho(t) < t$, t *is said to be left scattered, and if* $\rho(t) = t$, t *is said to be left dense* (*ld*). *A function* f *is left dense continuous, ld-continuous,* f *is continuous at each left dense point in* $\mathbb{T}$ *and its right-hand sided limits exist at each right dense points in* $\mathbb{T}$.

Definition 2.2. *For* $x \colon \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}$ (*assume* t *is not left scattered if* $t = \sup \mathbb{T}$), *we define the delta derivative of* $x(t)$, $x^{\Delta}(t)$, *to be the number* (*when it exists*) *with the property that, for each* $\epsilon > 0$, *there is a neighborhood* U *of* t *such that*

$$\left|x(\sigma(t)) - x(s) - x^{\Delta}(t)(\sigma(t) - s)\right| < \epsilon|\sigma(t) - s|$$

for all $s \in U$. *For* $x \colon \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}$ (*assume* t *is not right scattered if* $t = \inf \mathbb{T}$), *we define the nabla derivative of* $x(t)$, $x^{\nabla}(t)$, *to be the number* (*when it exists*) *with the property that, for each* $\epsilon > 0$, *there is a neighborhood* V *of* t *such that*

$$\left|x(\rho(t)) - x(s) - x^{\nabla}(t)(\rho(t) - s)\right| < \epsilon|\rho(t) - s|$$

for all $s \in V$.

If $\mathbb{T} = \mathbb{R}$, then $x^{\Delta}(t) = x^{\nabla}(t) = x'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $x^{\Delta}(t) = x(t+1) - x(t)$ is the forward difference operator while $x^{\nabla}(t) = x(t) - x(t-1)$ is the backward difference operator.

Definition 2.3. *If* $F^{\Delta}(t) = f(t)$, *then we define the delta integral by*

$$\int\limits_a^t f(s)\Delta s = F(t) - F(a).$$

If $\Phi^{\nabla}(t) = f(t)$, *then we define the nabla integral by*

$$\int\limits_a^t f(s)\nabla s = \Phi(t) - \Phi(a).$$

We provide some background materials from the theory of cones in Banach spaces.

Definition 2.4. *Let* E *be a real Banach space. A nonempty, closed, convex set* $P \subset E$ *is a cone if it satisfies the following two conditions*:

(i) $x \in P$, $\lambda \geq 0$ *imply* $\lambda x \in P$;

(ii) $x \in P$, $-x \in P$ *imply* $x = 0$.

Every cone $P \subset E$ *induces an ordering in* E *given by* $x \leq y$ *if and only if* $y - x \in P$.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then, for positive real numbers $r_1,\ r_2,\ r_3,$ and r_4 we define the following sets:

$$P(\gamma, r_4) = \{x \in P : \gamma(x) < r_4\},$$

$$P(\gamma, \alpha, r_2, r_4) = \{x \in P : r_2 \leq \alpha(x),\ \gamma(x) \leq r_4\},$$

$$P(\gamma, \theta, \alpha, r_2, r_3, r_4) = \{x \in P : r_2 \leq \alpha(x),\ \theta(x) \leq r_3,\ \gamma(x) \leq r_4\},$$

$$R(\gamma, \psi, r_1, r_4) = \{x \in P : r_1 \leq \psi(x),\ \gamma(x) \leq r_4\}.$$

Let the Banach space $E = C^1_{ld}([0, \sigma(T)]_{\mathbb{T}} \to \mathbb{R})$ with the norm

$$\|u\| = \max\left\{ \sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} |u(t)|,\ \sup_{t\in[0,T]_{\mathbb{T}}} \left|u^{\Delta}(t)\right| \right\},$$

and define the cone $P \subset E$ by

$$P = \left\{u \in E : u(t) \geq 0,\ t \in [0, \sigma(T)]_{\mathbb{T}};\ u^{\Delta\nabla}(t) \leq 0,\ u^{\Delta}(t) \geq 0,\ t \in [0, T]_{\mathbb{T}},\ u^{\Delta}(T) = 0\right\}.$$

We note that $u(t)$ is a solution to the BVP (1.1), (1.2) if and only if

$$u(t) = \int_0^t \phi_q\left(\int_s^T g(r) f\big(r, u(r), u^{\Delta}(r)\big)\nabla r\right)\Delta s +$$

$$+ B_0\left(\phi_q\left(\int_{\nu}^T g(r) f\big(r, u(r), u^{\Delta}(r)\big)\nabla r\right)\right).$$

Define the operator $F : P \to E$ by

$$(Fu)(t) = \int_0^t \phi_q\left(\int_s^T g(r) f\big(r, u(r), u^{\Delta}(r)\big)\nabla r\right)\Delta s +$$

$$+ B_0\left(\phi_q\left(\int_{\nu}^T g(r) f\big(r, u(r), u^{\Delta}(r)\big)\nabla r\right)\right).$$

Lemma 2.1. *If* $u \in P$, *then*:
(i) $u(t) \geq \dfrac{t}{\sigma(T)} u(\sigma(T))$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$;
(ii) $tu(s) \geq su(t)$ *for* $t, s \in [0, \sigma(T)]_{\mathbb{T}}$ *with* $s \leq t$.

Proof. (i) Since $u^{\Delta\nabla}(t) \leq 0$, it follows that $u^{\Delta}(t)$ is nonincreasing. Thus, for $0 < t < \sigma(T)$,

$$u(t) - u(0) = \int_0^t u^{\Delta}(s)\Delta s \geq t u^{\Delta}(t)$$

and

$$u(\sigma(T)) - u(t) = \int\limits_t^{\sigma(T)} u^\Delta(s)\Delta s \le (\sigma(T) - t)u^\Delta(t)$$

from which we have

$$u(t) \ge \frac{tu(\sigma(T)) + (\sigma(T) - t)u(0)}{\sigma(T)} \ge \frac{t}{\sigma(T)} u(\sigma(T)).$$

(ii) If $t = s$, then the conclusion holds. If $s < t$, since $u(t)$ is concave, nonnegative on $[0, \sigma(T)]_{\mathbb{T}}$ and $u^\Delta(\sigma(T)) = 0$, hence, we get

$$\frac{u(t) - u(0)}{t} \le \frac{u(s) - u(0)}{s}.$$

Thus,

$$tu(s) \ge su(t) + (t - s)u(0) \ge su(t).$$

Lemma 2.1 is proved.

Lemma 2.2. *For any $u \in P$, there exists a real number $M > 0$ such that* $\sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} u(t) \le$ $\le M \sup_{t\in[0,T]_{\mathbb{T}}} u^\Delta(t)$, *where* $M = \max\left\{1, \dfrac{\sigma(T)}{T}(B + T)\right\}$.

Proof. Because $u(t) = u(0) + \int\limits_0^t u^\Delta(t)\Delta t$ and $u^\Delta(t) \ge u^\Delta(T) = 0$, we obtain

$$u(T) = \sup_{t\in[0,T]_{\mathbb{T}}} u(t) \le \sup_{t\in[0,T]_{\mathbb{T}}} \left\{ B_0(u^\Delta(\nu)) + \int\limits_0^t u^\Delta(t)\Delta t \right\} \le$$

$$\le (B + T) \sup_{t\in[0,T]_{\mathbb{T}}} u^\Delta(t).$$

From Lemma 2.1, we have

$$\sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} u(t) = u(\sigma(T)) \le \frac{\sigma(T)}{T} u(T) \le \frac{\sigma(T)}{T}(B + T) \sup_{t\in[0,T]_{\mathbb{T}}} u^\Delta(t).$$

Lemma 2.2 is proved.

The next theorem from Theorem 1.3 in [17] is stated in context of $\mathbb{T} \subseteq \mathbb{R}$. The proof is, therefore, omitted.

Theorem 2.1 (Arzela – Ascoli theorem on $\mathbb{T}$). *Let $D \subseteq C([a, b]_{\mathbb{T}}; \mathbb{R})$. Then D is relatively compact if and only if it is bounded and equicontinuous.*

Lemma 2.3. *$F : P \to P$ is completely continuous.*

Proof. Firstly, we verify that $F\colon P \to P$. From (H$_3$), it is obvious that $(Fu)(t) \geq 0$ for $t \in [0,T]_{\mathbb{T}} \subset t \in [0,\sigma(T)]_{\mathbb{T}}$ and $(Fu)^{\Delta}(T) = 0$. Moreover,

$$(Fu)^{\Delta}(t) = \phi_q\left(\int\limits_t^T g(r)f\big(r,u(r),u^{\Delta}(r)\big)\nabla r\right) \geq 0$$

is continuous and nonincreasing in $[0,T]_{\mathbb{T}}$,

$$\left(\int\limits_t^T g(r)f\big(r,u(r),u^{\Delta}(r)\big)\nabla r\right)^{\nabla} = -g(t)f(t,u(t),u^{\Delta}(t)) \leq 0, \quad t \in [0,T]_{\mathbb{T}}.$$

In addition, $\phi_q(u)$ is a monotone increasing continuously differentiable function for $u > 0$. If

$$\int\limits_t^T g(r)f\big(r,u(r),u^{\Delta}(r)\big)\nabla r > 0 \quad \text{for} \quad t \in [0,T]_{\mathbb{T}},$$

we find $(Fu)^{\Delta\nabla}(t) \leq 0$ for $t \in [0,T]_{\mathbb{T}}$. If

$$\int\limits_t^T g(r)f\big(r,u(r),u^{\Delta}(r)\big)\nabla r = 0 \quad \text{for} \quad t \in [0,T]_{\mathbb{T}},$$

then $(Fu)^{\Delta\nabla}(t) = 0$ for $t \in [0,T]_{\mathbb{T}}$.

Secondly, we prove that F maps a bounded set into itself. Suppose that $c > 0$ is a constant and

$$u \in \overline{P_c} = \left\{u \in P\colon \|u\| = \max\left\{\sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} |u(t)|,\ \sup_{t\in[0,T]_{\mathbb{T}}} \left|u^{\Delta}(t)\right|\right\} \leq c\right\}.$$

Notice that $f(t,u,v)$ is continuous, so there exists a constant $C > 0$ such that $f(t,u,v) \leq \phi_p(C)$ for $(t,u,v) \in [0,T]_{\mathbb{T}} \times [0,c] \times [0,c]$. From here, $t \in [0,T]_{\mathbb{T}}$,

$$\left|\phi_q\left(\int\limits_t^T g(r)f\big(r,u(r),u^{\Delta}(r)\big)\nabla r\right)\right| < +\infty \tag{2.1}$$

and

$$\left|\int\limits_0^t \phi_q\left(\int\limits_s^T g(r)f\big(r,u(r),u^{\Delta}(r)\big)\nabla r\right)\Delta s + \right.$$

$$\left. + B_0\left(\phi_q\left(\int\limits_{\nu}^T g(r)f\big(r,u(r),u^{\Delta}(r)\big)\nabla r\right)\right)\right| < +\infty. \tag{2.2}$$

Consequently, F maps a bounded set into a bounded set.

Thirdly, if $t_1, t_2 \in [0,T]_{\mathbb{T}}$ and $t_1 < t_2$, then we have

$$\big|(Fu)(t_1) - (Fu)(t_2)\big| = \left| \int\limits_{t_1}^{t_2} \phi_q\left(\int\limits_{s}^{T} g(r) f\big(r, u(r), u^{\Delta}(r)\big) \nabla r \right) \Delta s \right| \leq$$

$$\leq \left| \int\limits_{t_1}^{t_2} \phi_q\left(\int\limits_{0}^{T} g(r) f\big(r, u(r), u^{\Delta}(r)\big) \nabla r \right) \Delta s \right| \leq$$

$$\leq C|t_1 - t_2| \phi_q\left(\int\limits_{0}^{T} g(r) \nabla r \right).$$

Therefore, by Theorem 2.1, we see that $F\overline{P_c}$ is relatively compact.

We next claim that $F\colon \overline{P_c} \to P$ is continuous. Suppose that $\{u_n\}_{n=1}^{\infty} \subset \overline{P_c}$ and $\lim_{n\to\infty} \|u_n - u_0\| \to 0$. This means that

$$\lim_{n\to\infty} |u_n - u_0| \to 0 \qquad \text{and} \qquad \lim_{n\to\infty} |u_n^{\Delta} - u_0^{\Delta}| \to 0.$$

Since $\big\{(Fu_n)(t)\big\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $[0,T]_{\mathbb{T}}$, there exists a uniformly convergent subsequence in $\big\{(Fu_n)(t)\big\}_{n=1}^{\infty}$. Let $\big\{(Fu_{n(m)})(t)\big\}_{m=1}^{\infty}$ be a subsequence which converges to $w(t)$ uniformly on $[0,T]_{\mathbb{T}}$. Examine that

$$(Fu_n)(t) = \int\limits_{0}^{t} \phi_q\left(\int\limits_{s}^{T} g(r) f(r, u_n(r), u_n^{\Delta}(r)) \nabla r \right) \Delta s +$$

$$+ B_0\left(\phi_q\left(\int\limits_{\nu}^{T} g(r) f(r, u_n(r), u_n^{\Delta}(r)) \nabla r \right)\right).$$

From (2.1) and (2.2), inserting $u_{n(m)}$ into the above and then letting $m \to \infty$, we find

$$w(t) = \int\limits_{0}^{t} \phi_q\left(\int\limits_{s}^{T} g(r) f(r, u_0(r), u_0^{\Delta}(r)) \nabla r \right) \Delta s +$$

$$+ B_0\left(\phi_q\left(\int\limits_{\nu}^{T} g(r) f(r, u_0(r), u_0^{\Delta}(r)) \nabla r \right)\right).$$

From the definition of F, we know that $w(t) = Fu_0(t)$ on $[0,T]_{\mathbb{T}}$. This shows that each subsequence of $\big\{(Fu_n)(t)\big\}_{n=1}^{\infty}$ uniformly converges to $(Fu_0)(t)$. So, the sequence $\big\{(Fu_n)(t)\big\}_{n=1}^{\infty}$ uniformly converges to $(Fu_0)(t)$. This means that F is continuous at $u_0 \in \overline{P_c}$. Therefore, F is continuous on $\overline{P_c}$ since u_0 is arbitrary. Thus, F is completely continuous.

Lemma 2.3 is proved.

The following fixed point theorem due to Avery and Peterson is fundamental in the proofs our main results.

Theorem 2.2 [6]. *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda u) \leq \lambda\psi(u)$ for $0 \leq \lambda \leq 1$ such that for some positive numbers h and r_4,*

$$\alpha(u) \leq \psi(u) \qquad \textit{and} \qquad \|u\| \leq h\gamma(u)$$

for all $u \in \overline{P(\gamma, r_4)}$. Suppose $F : \overline{P(\gamma, r_4)} \to \overline{P(\gamma, r_4)}$ is completely continuous and there exist positive real numbers r_1, r_2, and r_3 with $r_1 < r_2$ such that:

(S$_1$) $\{u \in P(\gamma, \theta, \alpha, r_2, r_3, r_4) : \alpha(u) > r_2\} \neq \varnothing$ *and* $\alpha(Fu) > r_2$ *for* $u \in P(\gamma, \theta, \alpha, r_2, r_3, r_4)$;

(S$_2$) $\alpha(Fu) > r_2$ *for* $u \in P(\gamma, \alpha, r_2, r_4)$ *with* $\theta(Fu) > r_3$;

(S$_3$) $0 \notin R(\gamma, \psi, r_1, r_4)$ *and* $\psi(Fu) < r_1$ *for all* $u \in R(\gamma, \psi, r_1, r_4)$ *with* $\psi(u) = r_1$.

Then F has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, r_4)}$ such that

$\gamma(u_i) \leq r_4$ *for* $i = 1, 2, 3$;

$r_2 < \alpha(u_1)$;

$r_1 < \psi(u_2)$ *with* $\alpha(u_2) < r_2$;

$\psi(u_3) < r_1$.

3. Solutions of (1.1) and (1.2) in a cone. Let $\zeta \in \mathbb{T}$ be such that $0 < \nu < \zeta < T$, and define the nonnegative continuous convex functionals γ and θ, nonnegative continuous concave functional α, and nonnegative continuous functional ψ, respectively, on P by

$$\alpha(u) = \inf_{t \in [\zeta, T]_{\mathbb{T}}} u(t) = u(\zeta),$$

$$\psi(u) = \inf_{t \in [\zeta, T]_{\mathbb{T}}} u(t) = u(\zeta),$$

$$\gamma(u) = \sup_{t \in [0, T]_{\mathbb{T}}} u^{\Delta}(t) = u^{\Delta}(0),$$

$$\theta(u) = \sup_{t \in [\tau, T]_{\mathbb{T}}} u^{\Delta}(t) = u^{\Delta}(\tau).$$

In view of Lemma 2.2, we find

$$\sup_{t \in [0, \sigma(T)]_{\mathbb{T}}} u(t) \leq M \sup_{t \in [0, T]_{\mathbb{T}}} u^{\Delta}(t) = M\gamma(u) \quad \text{for all} \quad u \in P.$$

We also see that $\theta(\lambda u) = \lambda\theta(u)$ for $\lambda \in [0, 1]$.

For notational convenience, we denote λ_0, μ and δ by

$$\lambda_0 = \phi_q\left(\int\limits_0^T g(r)\nabla r\right),$$

$$\mu = (\zeta + B)\phi_q\left(\int\limits_\zeta^T g(r)\nabla r\right),$$

$$\delta = (\zeta + A)\phi_q\left(\int\limits_0^T g(r)\nabla r\right).$$

Theorem 3.1. *Assume that there exist constants* $r_1,\ r_2,\ r_4$ *such that* $0 < r_1 < \dfrac{\zeta}{T}r_2 < \dfrac{\zeta\mu}{T\lambda_0}r_4$, $\lambda_0\zeta > \mu$, *and suppose that* f *satisfies the following conditions*:

(A_1) $f(t,u,v) \leq \phi_p\left(\dfrac{r_4}{\lambda_0}\right)$ *for* $(t,u,v) \in [0,T]_{\mathbb{T}} \times [0, Mr_4] \times [-r_4, r_4]$;

(A_2) $f(t,u,v) > \phi_p\left(\dfrac{r_2}{\mu}\right)$ *for* $(t,u,v) \in [\zeta,T]_{\mathbb{T}} \times [r_2, Mr_4] \times [-r_4, r_4]$;

(A_3) $f(t,u,v) < \phi_p\left(\dfrac{r_1}{\delta}\right)$ *for* $(t,u,v) \in [0,T]_{\mathbb{T}} \times \left[0, \dfrac{T}{\zeta}r_1\right] \times [-r_4, r_4]$.

Then the BVP (1.1), (1.2) *has at least three positive solutions* $u_1,\ u_2$, *and* u_3 *such that*

$$\|u_i\| \leq r_4 \quad \textit{for} \quad i = 1,2,3, \quad r_2 < u_1(\zeta), \quad r_1 < u_2(\zeta) \quad \textit{and} \quad u_2(\zeta) < r_2 \quad \textit{with} \quad u_3(\zeta) < r_1. \tag{3.1}$$

Proof. The BVP (1.1), (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Fu$. Thus we set out to verify that the operator F satisfies Avery and Peterson's fixed point theorem which will prove the existence of three fixed points of F which satisfy the conclusion of the theorem.

Firstly, we will show that

$$F : \overline{P(\gamma, r_4)} \to \overline{P(\gamma, r_4)}. \tag{3.2}$$

For any $u \in \overline{P(\gamma, r_4)}$, we have $\gamma(u) = \sup_{t\in[0,T]_{\mathbb{T}}} u^{\Delta}(t) \leq r_4$. From Lemma 2.2, we get $\sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} u(t) \leq Mr_4$. From ($A_1$), we obtain $f(t,u,v) \leq \phi_p\left(\dfrac{r_4}{\lambda_0}\right)$, and so

$$\gamma(Fu) = \sup_{t\in[0,T]_{\mathbb{T}}} (Fu)^{\Delta}(t) = \phi_q\left(\int\limits_0^T g(r) f\big(r, u(r), u^{\Delta}(r)\big)\nabla r\right) \leq$$

$$\leq \phi_q\left(\int\limits_0^T g(r)\nabla r\right)\frac{r_4}{\lambda_0} = r_4.$$

Thus (3.2) holds.

Secondly, we prove that condition (S_1) in Theorem 2.2 holds. Let $u = \dfrac{\lambda_0 r_2}{\mu}t - \dfrac{\lambda_0 r_2}{\mu}\zeta + 2r_2$. Then $\alpha(u) = 2r_2 > r_2$, $\theta(u) = \dfrac{\lambda_0 r_2}{\mu}$ and $\gamma(u) = \dfrac{\lambda_0 r_2}{\mu} < r_4$. So, $\left\{u \in P\left(\gamma, \theta, \alpha, r_2, \dfrac{\lambda_0 r_2}{\mu}, r_4\right) : \alpha(u) > r_2\right\} \neq \varnothing$. On the other hand, for any $\left\{u \in P\left(\gamma, \theta, \alpha, r_2, \dfrac{\lambda_0 r_2}{\mu}, r_4\right) : \alpha(u) > r_2\right\}$, it follows from Lemma 2.2 that $r_2 \leq u(t) \leq Mr_4$, $-r_4 \leq u^{\Delta}(t) \leq r_4$, and for all $t \in [\zeta, T]_{\mathbb{T}}$. From ($A_2$), we get

$$\alpha(Fu) = Fu(\zeta) = \int\limits_0^\zeta \phi_q\left(\int\limits_s^T g(r) f\big(r, u(r), u^\Delta(r)\big)\nabla r\right)\Delta s +$$

$$+B_0\left(\phi_q\left(\int\limits_\nu^T g(r) f\big(r, u(r), u^\Delta(r)\big)\nabla r\right)\right) >$$

$$> (\zeta + B)\phi_q\left(\int\limits_\zeta^T g(r)\phi_p\left(\frac{r_2}{\mu}\right)\nabla r\right) =$$

$$= (\zeta + B)\phi_q\left(\int\limits_\zeta^T g(r)\nabla r\right)\frac{r_2}{\mu} = r_2.$$

Therefore, we have $\alpha(u) > r_2$ for all $u \in P\left(\gamma, \theta, \alpha, r_2, \dfrac{\lambda_0 r_2}{\mu}, r_4\right)$. Consequently, condition (S_1) in Theorem 2.2 is satisfied.

Thirdly, we verify that condition (S_2) of Theorem 2.2 holds. For any $u \in P(\gamma, \alpha, r_2, r_4)$ with $\theta(Fu) > \dfrac{\lambda_0 r_2}{\mu}$ that is

$$\theta(Fu) = (Fu)^\Delta(\tau) = \phi_q\left(\int\limits_\tau^T g(r) f\big(r, u(r), u^\Delta(r)\big)\nabla r\right) > \frac{\lambda_0 r_2}{\mu},$$

we obtain

$$\alpha(Fu) = \inf_{t\in[\zeta,T]_\mathbb{T}} (Fu)(t) \geq \int\limits_0^\zeta \phi_q\left(\int\limits_s^T g(r) f\big(r, u(r), u^\Delta(r)\big)\nabla r\right)\Delta s +$$

$$+A\left(\phi_q\left(\int\limits_\nu^T g(r) f\big(r, u(r), u^\Delta(r)\big)\nabla r\right)\right) >$$

$$> \zeta\phi_q\left(\int\limits_\tau^T g(r) f\big(r, u(r), u^\Delta(r)\big)\nabla r\right) > \zeta\frac{\lambda_0 r_2}{\mu} > r_2.$$

Hence, condition (S_2) in Theorem 2.2 is satisfied.

Finally, we prove that (S_3) in Theorem 2.2 is satisfied. Since $\psi(0) = 0 < r_1$, so, $0 \notin R(\gamma, \psi, r_1, r_4)$. Suppose that $u \in R(\gamma, \psi, r_1, r_4)$ with $\psi(u) = \inf_{t\in[\zeta,T]_\mathbb{T}} u(t) = u(\zeta) = r_1$, Lemma 2.1 implies that

$$\sup_{t\in[0,T]_\mathbb{T}} u(t) = u(T) \leq \frac{T}{\zeta}u(\zeta) = \frac{T}{\zeta}r_1,$$

this yields $0 \leq u(t) \leq \dfrac{T}{\zeta} r_1$ for all $t \in [0,T]_{\mathbb{T}}$. Moreover, $\gamma(u) = \sup_{t\in[0,T]_{\mathbb{T}}} u^{\Delta}(t) \leq r_4$. Hence, by the condition (A$_3$) of this theorem, we have

$$\alpha(Fu) = \inf_{t\in[\zeta,T]_{\mathbb{T}}} (Fu)(t) \leq \int\limits_0^{\zeta} \phi_q\left(\int\limits_0^T g(r) f\big(r, u(r), u^{\Delta}(r)\big) \nabla r \right) \Delta s +$$

$$+A\left(\phi_q\left(\int\limits_{\nu}^T g(r) f\big(r, u(r), u^{\Delta}(r)\big) \nabla r \right)\right) <$$

$$< (\zeta + A)\phi_q\left(\int\limits_0^T g(r) \phi_p\Big(\frac{r_1}{\delta}\Big) \nabla r \right) =$$

$$= (\zeta + A)\phi_q\left(\int\limits_0^T g(r) \nabla r \right) \frac{r_1}{\delta} = r_1.$$

Thus, condition (S$_3$) in Theorem 2.2 holds. As a result, all the conditions of Theorem 2.2 are satisfied.

Theorem 3.1 is proved.

4. Solutions of (1.1) and (1.3) in a cone. Let the Banach space $E = C^1_{ld}\big([0,\sigma(T)]_{\mathbb{T}} \to \mathbb{R}\big)$ with the norm

$$\|u\| = \max\left\{ \sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} |u(t)|,\ \sup_{t\in[0,T]_{\mathbb{T}}} \big|u^{\Delta}(t)\big| \right\},$$

and define the cone $P_1 \subset E$ by

$$P_1 = \big\{ u \in E \colon u(t) \geq 0,\ \ t \in [0,\sigma(T)]_{\mathbb{T}};\ \ u^{\Delta\nabla}(t) \leq 0,\ \ u^{\Delta}(t) \leq 0,\ \ t \in [0,T]_{\mathbb{T}},\ \ u^{\Delta}(0) = 0 \big\}.$$

Fix $\tau \in \mathbb{T}$ such that $0 < \tau < \nu$, and define the nonnegative continuous convex functionals γ and θ, nonnegative continuous concave functional α, and nonnegative continuous functional ψ, respectively, on P_1 by

$$\alpha(u) = \inf_{t\in[\tau,T]_{\mathbb{T}}} u(t) = u(T),$$

$$\psi(u) = \inf_{t\in[\tau,T]_{\mathbb{T}}} u(t) = u(T),$$

$$\gamma(u) = \sup_{t\in[0,T]_{\mathbb{T}}} |u^{\Delta}(t)| = u^{\Delta}(T),$$

$$\theta(u) = \sup_{t\in[\tau,T]_{\mathbb{T}}} |u^{\Delta}(t)| = u^{\Delta}(T).$$

Set

$$\lambda_1 = \phi_q\left(\int\limits_0^T g(r)\nabla r\right),$$

$$\mu_1 = B\phi_q\left(\int\limits_0^\tau g(r)\nabla r\right),$$

$$\delta_1 = A\phi_q\left(\int\limits_0^T g(r)\nabla r\right).$$

We note that $u(t)$ is a solution of (1.1), (1.3) if and only if

$$u(t) = \int\limits_t^T \phi_q\left(\int\limits_0^s g(r)f\big(r, u(r), u^\Delta(r)\big)\nabla r\right)\Delta s +$$

$$+B_1\left(\phi_q\left(\int\limits_0^\nu g(r)f\big(r, u(r), u^\Delta(r)\big)\nabla r\right)\right).$$

Theorem 4.1. *Assume that conditions* $(\mathrm{H}_1)-(\mathrm{H}_5)$ *are satisfied. Let* $0 < r_1 < \dfrac{\zeta}{T}r_2 < \dfrac{\zeta\mu_1}{T\lambda_1}r_4$, $\lambda_1\zeta > \mu_1$, *and suppose that* f *satisfies the following conditions*:

(C_1) $f(t,u,v) \le \phi_p\left(\dfrac{r_4}{\lambda_1}\right)$ *for* $(t,u,v) \in [0,T]_{\mathbb{T}} \times [0, Mr_4] \times [-r_4, r_4]$;

(C_2) $f(t,u,v) > \phi_p\left(\dfrac{r_2}{\mu_1}\right)$ *for* $(t,u,v) \in [\zeta,T]_{\mathbb{T}} \times [r_2, Mr_4] \times [-r_4, r_4]$;

(C_3) $f(t,u,v) < \phi_p\left(\dfrac{r_1}{\delta_1}\right)$ *for* $(t,u,v) \in [0,T]_{\mathbb{T}} \times \left[0, \dfrac{T}{\zeta}r_1\right] \times [-r_4, r_4]$.

Then the BVP (1.1) *and* (1.3) *has at least three positive solutions* u_1, u_2, *and* u_3 *such that*

$$\|u_i\| \le r_4 \quad \textit{for} \quad i = 1,2,3, \quad r_2 < u_1(\zeta), \quad r_1 < u_2(\zeta) \quad \textit{and} \quad u_2(\zeta) < r_2 \quad \textit{with} \quad u_3(\zeta) < r_1. \tag{4.1}$$

5. Example. Let

$$\mathbb{T} = \left\{2 - \left(\frac{1}{3}\right)^{\mathbb{N}_0}\right\} \cup \left\{0, \frac{1}{8}, \frac{1}{4}, \frac{1}{6}, \frac{1}{2}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\right\},$$

where $\mathbb{N}_0$ denotes the set of all nonnegative integers. Take $T = 2$, $p = 7$, $\nu = \dfrac{1}{2}$, $\zeta = 1$, $\tau = \dfrac{3}{2}$, $A = B = \dfrac{1}{100000}$ and choose

$$g(t) = t + \rho(t)$$

and

$$f(t,u,v)=\begin{cases} t+1+|v| & \text{for} \quad (t,u,v)\in[0,2]_{\mathbb{T}}\times[0,4]\times[-6,6], \\ t+|v|+p(u) & \text{for} \quad (t,u,v)\in[0,2]_{\mathbb{T}}\times[4,4.1]\times[-6,6], \\ t+1584+|v| & \text{for} \quad (t,u,v)\in[0,2]_{\mathbb{T}}\times[4.1,20]\times[-6,6]. \end{cases}$$

Here $p(u)$ satisfies $p(4)=1,\ p(4.1)=1584,\ p(u)\colon\mathbb{R}\to\mathbb{R}^+$ is continuous and $p^{\Delta}(u)>0$. Choose $r_1=2,\ r_2=4.1,\ r_4=6$. Then

$$\lambda_0=\left(\int\limits_0^2 g(r)\nabla r\right)^{\frac{1}{6}}\approx 1.260,$$

$$\mu=\left(1+\frac{1}{100000}\right)\left(\int\limits_1^2 g(r)\nabla r\right)^{\frac{1}{6}}\approx 1.201,$$

$$\delta=\left(1+\frac{1}{100000}\right)\left(\int\limits_0^2 g(r)\nabla r\right)^{\frac{1}{6}}\approx 1.25993.$$

It is easy to see that $0<r_1<\dfrac{\zeta}{T}r_2<\dfrac{\zeta\mu}{T\lambda_0}r_4,\ \lambda_0\zeta>\mu$, and $f(t,u,v)$ satisfies that

$$f(t,u,v)<\phi_p\left(\frac{r_4}{\lambda_0}\right)=\left(\frac{6}{1.260}\right)^6\approx 11659.6 \quad \text{for} \quad 0\le t\le 2, \quad 0\le u\le 10.1, \quad |v|\le 6,$$

$$f(t,u,v)>\phi_p\left(\frac{r_2}{\mu}\right)=\left(\frac{4.1}{1.201}\right)^6\approx 1583.26 \quad \text{for} \quad 1\le t\le 2, \quad 4.1\le u\le 10.1, \quad |v|\le 6,$$

$$f(t,u,v)<\phi_p\left(\frac{r_1}{\delta}\right)=\left(\frac{2}{1.25993}\right)^6\approx 15.9993 \quad \text{for} \quad 0\le t\le 2, \quad 0\le u\le 4, \quad |v|\le 6.$$

Then all conditions of Theorem 3.1 hold. Thus by Theorem 3.1, the BVP (1.1), (1.2) has at least three positive solutions $u_1,\ u_2,\ u_3$ such that

$$\|u_i\|\le 6 \quad \text{for} \quad i=1,2,3, \quad 4.1<u_1(1), \quad 2<u_2(1) \qquad \text{and} \quad u_2(1)<4.1 \quad \text{with} \quad u_3(1)<2.$$

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