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V. M. Phung, V. T. Nguyen (Hanoi Nat. Univ. Education, Vietnam),
H. L. Dinh (Hanoi Dept. Education and Training, Vietnam)

COMBINING INTERPOLATION SCHEMES AND LAGRANGE INTERPOLATION ON THE UNIT SPHERE IN $\mathbb{R}^{N+1}$ *

КОМБІНУВАННЯ ІНТЕРПОЛЯЦІЙНИХ СХЕМ ТА ІНТЕРПОЛЯЦІЇ ЛАГРАНЖА НА ОДИНИЧНІЙ СФЕРІ В $\mathbb{R}^{N+1}$

We study Lagrange interpolation in $\mathbb{R}^N$ and on the unit sphere in $\mathbb{R}^{N+1}$. We show that sequences of unisolvent sets can be combined to get other sequences of unisolvent sets such that the existence of the limits is preserved. Moreover, the limiting operators keep the interpolation conditions under the combining process.

Вивчається інтерполяція Лагранжа в $\mathbb{R}^N$ та на одиничній сфері в $\mathbb{R}^{N+1}$. Доведено, що послідовності унірозв'язних множин можна скомбінувати в інші послідовності таким чином, що існування меж збіжності буде збережено. І навіть більше, у такому комбінуванні граничні оператори також зберігають умови інтерполяції.

1. Introduction. Let $\mathcal{P}_d(\mathbb{R}^N)$ be the vector space of all polynomials of degree at most d in $\mathbb{R}^N$. It is known that the dimension $m_d(\mathbb{R}^N)$ of $\mathcal{P}_d(\mathbb{R}^N)$ equals $\binom{N+d}{N}$. Let E be a nonempty subset of $\mathbb{R}^N$. Then the polynomials in $\mathcal{P}_d(\mathbb{R}^N)$, when restricted to E, form a vector space, say $\mathcal{P}_d(E)$. We denote by $m_d(E)$ the dimension of $\mathcal{P}_d(E)$. If $\mathbb{S}^N$ is the unit sphere in $\mathbb{R}^{N+1}$, then

$$m_d(\mathbb{S}^N) = \binom{N+d}{N} + \binom{N+d-1}{N}.$$

More generally, if E is an algebraic variety in $\mathbb{R}^N$, then one can compute $m_d(E)$ precisely (see Subsection 2.2). Since $\mathcal{P}_d(E)$ is a finite dimensional vector space, any two norms on $\mathcal{P}_d(E)$ are equivalent. Hence, the convergence on $\mathcal{P}_d(E)$ can be understood as the convergence under any norm on $\mathcal{P}_d(E)$.

A subset $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_{m_d(E)}\}$ of $m_d(E)$ distinct points of E is said to be unisolvent for $\mathcal{P}_d(E)$ if, for every function f defined on X, there exists a unique $P \in \mathcal{P}_d(E)$ such that $f(\mathbf{x}) = P(\mathbf{x})$ for all $\mathbf{x} \in X$. This function is called the Lagrange interpolation polynomial of f at X on E and is denoted by $\mathbf{L}_E[X; f]$. When $E = \mathbb{R}^N$, we write $\mathbf{L}[X; f]$ for $\mathbf{L}_{\mathbb{R}^N}[X; f]$. Given a basis $\mathcal{B} = \{p_1, \ldots, p_{m_d(E)}\}$ for $\mathcal{P}_d(E)$, the (generalized) Vandermonde determinant with respect to $\mathcal{B}$ and X is defined by

$$\mathrm{VDM}(\mathcal{B}; X) = \det[p_i(\mathbf{x}_j)]_{1 \le i,j \le m_d(E)}.$$

It is known that X is unisolvent for $\mathcal{P}_d(E)$ if and only if $\mathrm{VDM}(\mathcal{B}; X) \neq 0$. The Vandermonde determinant is a polynomial of interpolation points. Hence, it is different from zero for almost all choices of interpolation points. In other words, a subset $A \subset E$ of $m_d(E)$ distinct points is unisolvent for almost all choices of A. On the other hand, given a set of points on E, it is difficult to check whether it is unisolvent.

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Roughly speaking, a Hermite interpolation problem is more general than a Lagrange interpolation problem. More precisely, the problem means to find a polynomial which matches, on a set of distinct points in E, values of a function and its partial derivatives. If the interpolation problem has a unique solution, then we say that the problem is poised. Unlike the univariate Hermite interpolation, the multivariate Hermite interpolation on E is not always poised. Moreover, it is difficult to check whether a particular Hermite problem is poised.

We are concerned with the problem of determining the limits of Lagrange interpolation polynomials, which generalizes the problem considered in [10].

Problem. a) *Construct unisolvent sets for* $\mathcal{P}_d(E)$.

b) *Let* $\{X_n\}$ *be a sequence of unisolvent sets for* $\mathcal{P}_d(E)$. *Find conditions such that the sequence* $\{\mathbf{L}_E[X_n; f]\}$ *converges for every suitably defined function* f *and characterize the limiting operator* $\mathbf{H}(f)$.

The problem has been solved in some special cases. It is expected that $\mathbf{H}(f)$ is a Hermite type projector on E. Let us consider the problem in $\mathbb{R}^N$. If $N = 1$ and X_n coalesces to some points, then Theorem 1.4 in [2] points out that $\{\mathbf{L}[X_n; f]\}$ converges to the univariate Hermite interpolation at the limiting points when f is sufficiently smooth. Bloom and Calvi in [1] gave sufficient conditions to guarantee the convergence of multivariate Lagrange projectors to the Taylor projector. In a recent work, Phung [10] showed that the limit of the bivariate Lagrange interpolation polynomials at Bos configurations distributed on straight lines and circles is a Hermite type interpolation polynomial when the interpolation points coalesce.

The problem is solved in some cases where E is an algebraic hypersurface. When E is a circle in $\mathbb{R}^2$, we showed in [11] (Proposition 4.1) that the Lagrange interpolation on E converges to a Taylor type polynomial when all interpolation points tend to a single point. An extension of this result for irreducible algebraic curves in $\mathbb{C}^2$ was given in [7]. It is worth pointing out that analogous result in [7] also hold when we replace complex curves in $\mathbb{C}^2$ by real curves in $\mathbb{R}^2$ (see Example 2.4 for details). Also in [11], we constructed new Lagrange and Hermite interpolation schemes on 2-sphere $\mathbb{S}^2$. The unisolvent sets for $\mathcal{P}_d(\mathbb{S}^2)$ are located on $d + 1$ circles on $\mathbb{S}^2$ in which the kth circles contains $2k - 1$ points. Fortunately, we can write the interpolation polynomials into Newton forms and use them to prove that Lagrange projectors tend to Hermite type projectors on $\mathbb{S}^2$ (see Example 2.5). In [10], the first author of this paper gave new Lagrange and Hermite interpolation schemes on the unit sphere $\mathbb{S}^2$. More precisely, the unisolvent sets are the images of Bos configurations of points distributed on straight lines and circles in $\mathbb{R}^2$ under the trivial parametrizations of the upper and lower half spheres. Here the special configurations of interpolation points on $\mathbb{S}^2$ enable us to reduce the limiting problem on the sphere to a limiting problem in $\mathbb{R}^2$ which is solvable. As a result, $\mathbf{H}(f)$ is a certain Hermite projector on $\mathbb{S}^2$. For details, we refer the readers to [10].

For convenience, we will say that a sequence of unisolvent sets $\{X_n\} \subset E$ is normal (resp., regular) if E is an algebraic variety in $\mathbb{R}^N$ (resp., $E = \mathbb{R}^N$) and the sequence $\{\mathbf{L}_E[X_n; f]\}$ converges for every suitably defined function f. Precise examples of such sequences are presented in Subsection 2.3. In this paper, we first want to find methods to combine regular and normal sequences to create similar sequences. In this direction, we prove in Theorem 3.1 that a regular sequence in $\mathbb{R}^N$ can combine with a normal sequence on an algebraic variety in $\mathbb{R}^N$ to form a regular sequence. Moreover, the interpolation conditions of the limiting operators preserve under the combining process. In particular, the union of suitable normal sequences on algebraic varieties in $\mathbb{R}^N$ is a regular sequence in which the limiting operator inherits the interpolation conditions from the limiting operators on algebraic varieties.

We also investigate the problem on the unit sphere $\mathbb{S}^N$ in $\mathbb{R}^{N+1}$. We first construct unisolvent sets on $\mathbb{S}^N$. They are of the form

$$X_0 = R^+(A_0) \cup R^-(A_0) \cup R^+(B_0),$$

where $R^{\pm}$ is the trivial parametrizations of the half spheres

$$R^+(\mathbf{x}) = \left(\mathbf{x}, \sqrt{1 - \|\mathbf{x}\|^2}\right), \quad R^-(\mathbf{x}) = \left(\mathbf{x}, -\sqrt{1 - \|\mathbf{x}\|^2}\right), \quad \|\mathbf{x}\| \le 1,$$

and A_0, B_0 are unisolvent for $\mathcal{P}_{d-1}(\mathbb{R}^N)$ and $\mathcal{P}_d(E)$, respectively. Here E is a hyperplane in $\mathbb{R}^N$. We show in Theorem 4.2 that if $\{A_n\}$ is a regular sequence in the unit ball in $\mathbb{R}^N$ and $\{B_n\}$ is a normal sequence on a hyperplane, then

$$X_n = R^+(A_n) \cup R^-(A_n) \cup R^+(B_n)$$

is a normal sequence on $\mathbb{S}^N$. Furthermore, the limiting operators composed with $R^{\pm}$ also preserve the interpolation conditions. Our new theorems generalize results in [10].

Notations and conventions. The points in $\mathbb{R}^N$ are denoted by bold letters. The Euclidean norms of $\mathbf{x} \in \mathbb{R}^N$ is denoted by $\|\mathbf{x}\|$. The symbol $\mathbb{S}^N$ stand for the unit sphere in $\mathbb{R}^{N+1}$. For $\mathbf{a} \in \mathbb{R}^N$ and $r > 0$, we denote by $\mathbb{B}^N(\mathbf{a}, r)$ the Euclidean ball of centre $\mathbf{a}$ and radius r. We write $\mathbb{B}^N = \mathbb{B}^N(0, 1)$, the unit ball. Throughout this paper, we denote by $\mathcal{F}, \mathcal{A}$ the $\mathbb{R}$-algebras of functions defined in $\mathbb{R}^N$ that contain the space of all polynomials $\mathcal{P}(\mathbb{R}^N)$. For a closed subset K of $\mathbb{R}^N$, we write $C^m(K)$ for the space of all continuously differentiable functions in neighborhoods of K. We always assume that d is a positive integer.

2. Regular and normal sequences of unisolvent sets. *2.1. Regular sequences of unisolvent sets in* $\mathbb{R}^N$.

Definition 2.1. *Let $\{A_n\}$ be a sequence of unisolvent sets for $\mathcal{P}_d(\mathbb{R}^N)$ and $\mathcal{F}$ be an algebra of functions defined in $\mathbb{R}^N$. We say the $\{A_n\}$ is $\mathcal{F}$-regular if, for any $f \in \mathcal{F}$, the sequence $\{\mathbf{L}[A_n; f]\}$ is convergent in $\mathcal{P}_d(\mathbb{R}^N)$.*

Let $\{A_n\}$ be a $\mathcal{F}$-regular sequence for $\mathcal{P}_d(\mathbb{R}^N)$. We can define

$$\Lambda(f) := \lim_{n\to\infty} \mathbf{L}[A_n; f], \quad f \in \mathcal{F}.$$

Then $\Lambda\colon \mathcal{F} \to \mathcal{P}_d(\mathbb{R}^N)$ is linear map. It can be regarded as a Hermite type interpolation operator in $\mathbb{R}^N$.

Lemma 2.1. *If $\{A_n\}$ be a $\mathcal{F}$-regular sequence of unisolvent sets for $\mathcal{P}_d(\mathbb{R}^N)$ and Λ is the limit of the sequence of Lagrange projectors $\{\mathbf{L}[A_n; \cdot]\}$, then*

$$\Lambda(f\Lambda(g)) = \Lambda(fg), \quad f, g \in \mathcal{F}.$$

Proof. We first claim that $\mathbf{L}[A_n; f\mathbf{L}[A_n; g]] = \mathbf{L}[A_n; fg]$. Both sides are polynomials of degree at most d and agree at any points of A_n,

$$\mathbf{L}[A_n; f\mathbf{L}[A_n; g]](\mathbf{a}) = f(\mathbf{a})\mathbf{L}[A_n; g](\mathbf{a}) = f(\mathbf{a})g(\mathbf{a}) = \mathbf{L}[A_n; fg](\mathbf{a}), \quad \mathbf{a} \in A_n.$$

The desired relation follows from the uniqueness of Lagrange interpolation. Let $\{p_1, \ldots, p_m\}$ be a basis for $\mathcal{P}_d(\mathbb{R}^N)$ with $m = m_d(\mathbb{R}^N)$. We write $\Lambda(g) = \sum_{i=1}^{m} c_i p_i$ and $\mathbf{L}[A_n; g] = \sum_{i=1}^{m} c_i^n p_i$. By the hypothesis, $\lim_{n\to\infty} c_i^n = c_i,\ i = 1, \ldots, m$. We have

$$\Lambda(f\Lambda(g)) = \lim_{n\to\infty} \mathbf{L}[A_n; f\Lambda(g)] = \lim_{n\to\infty} \sum_{i=1}^{m} c_i \mathbf{L}[A_n; fp_i] =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{m} c_i^n \mathbf{L}[A_n; fp_i] = \lim_{n\to\infty} \mathbf{L}[A_n; f\mathbf{L}[A_n; g]] =$$

$$= \lim_{n\to\infty} \mathbf{L}[A_n; fg] = \Lambda(fg).$$

Here, in the third equation, we use the fact that $\{\mathbf{L}[A_n; fp_i]\}$ is bounded since it converges to $\Lambda(fp_i)$, $i = 1, \ldots, m$.

Lemma 2.2. *Let $\{A_n\}$ be a $\mathcal{F}$-regular sequence for $\mathcal{P}_d(\mathbb{R}^N)$. If $\{P_n\} \subset \mathcal{P}_k(\mathbb{R}^N)$ converges to $P \in \mathcal{P}_k(\mathbb{R}^N)$, then*

$$\lim_{n\to\infty} \mathbf{L}[A_n; fP_n] = \Lambda(fP), \quad f \in \mathcal{F}.$$

Proof. Let $\{p_1, \ldots, p_m\}$ be a basis for $\mathcal{P}_k(\mathbb{R}^N)$ with $m = m_k(\mathbb{R}^N)$. We write $P = \sum_{i=1}^{m} a_i p_i$ and $P_n = \sum_{i=1}^{m} a_i^n p_i$. By the hypothesis, $\lim_{n\to\infty} a_i^n = a_i,\ i = 1, \ldots, m$. It follows that

$$\lim_{n\to\infty} \mathbf{L}[A_n; fP_n] = \lim_{n\to\infty} \sum_{i=1}^{m} a_i^n \mathbf{L}[A_n; fp_i] = \sum_{i=1}^{m} a_i \Lambda(fp_i) = \Lambda(fP),$$

where, in the second equation, we use the fact that $\lim_{n\to\infty} \mathbf{L}[A_n; fp_i] = \Lambda(fp_i),\ i = 1, \ldots, m$.

2.2. Normal sequences of unisolvent sets on algebraic varieties. Throughout this subsection we always assume that E is a real algebraic variety in $\mathbb{R}^N$ such that its ideal

$$\mathcal{I}(E) = \{p \in \mathcal{P}(\mathbb{R}^N) \colon p \text{ is identically zero on } E\}$$

is principal, i.e., generated by a single element $q \in \mathcal{P}(\mathbb{R}^N)$ with $\deg q \geq 1$.

We recall some arguments in [3]. Let $\Phi \colon \mathcal{P}_d(\mathbb{R}^N) \to \mathcal{P}_d(E)$ be the continuous surjective linear map defined by $\Phi(Q) = Q|_E$. Then $\ker\Phi = \mathcal{I}(E) \cap \mathcal{P}_d(\mathbb{R}^N)$. For each $Q \in \ker\Phi$, our assumption implies that q divides Q. This enables us to find $Q_1 \in \mathcal{P}_{d-\deg q}(\mathbb{R}^N)$ such that $Q = qQ_1$. Hence $\ker\Phi \subset q\mathcal{P}_{d-\deg q}(\mathbb{R}^N)$. The converse inclusion is trivial. It follows that $\ker\Phi = q\mathcal{P}_{d-\deg q}(\mathbb{R}^N)$. Consequently,

$$m_d(E) = \dim\mathcal{P}_d(\mathbb{R}^N) - \dim\ker\Phi = m_d(\mathbb{R}^N) - m_{d-\deg q}(\mathbb{R}^N).$$

Here we make the convention that $m_{d-\deg q}(\mathbb{R}^N) = 0$ when $\deg q > d$.

Next we give an analog of the notion of regular sequences presented in the above subsection.

Definition 2.2. *Let E be an algebraic variety in $\mathbb{R}^N$ such that $\mathcal{I}(E)$ is generated by a non-constant polynomial q. Let $\{B_n\} \subset E$ be a sequence of unisolvent sets for $\mathcal{P}_d(E)$ and $\mathcal{A}$ be an algebra of functions defined on E. The sequence $\{B_n\}$ is said to be $\mathcal{A}$-normal if $\{\mathbf{L}_E[B_n; f]\}$ is convergent in $\mathcal{P}_d(E)$ for every $f \in \mathcal{A}$.*

Let $\Phi \colon \mathcal{P}_d(\mathbb{R}^N) \to \mathcal{P}_d(E)$ be the continuous surjective linear map defined above. Note that $\ker \Phi$ is a subspace of the finite-dimensional vector space $\mathcal{P}_d(\mathbb{R}^N)$. We now denote by $\mathcal{Q}$ the supplementary space of $\ker \Phi$ in $\mathcal{P}_d(\mathbb{R}^N)$, i.e.,

$$\mathcal{P}_d(\mathbb{R}^N) = \mathcal{Q} \oplus \ker \Phi.$$

Then Φ restricted on $\mathcal{Q}$ is a bijective from $\mathcal{Q}$ onto $\mathcal{P}_d(E)$. Furthermore, the two maps

$$\Phi|_{\mathcal{Q}} \colon \mathcal{Q} \to \mathcal{P}_d(E) \quad \text{and} \quad (\Phi|_{\mathcal{Q}})^{-1} \colon \mathcal{P}_d(E) \to \mathcal{Q}$$

are continuous maps between two normed spaces. If $P \in \mathcal{P}_d(E)$, then $(\Phi|_{\mathcal{Q}})^{-1}(P) \in \mathcal{P}_d(\mathbb{R}^N)$ and $(\Phi|_{\mathcal{Q}})^{-1}(P)|_E = P$.

If B_0 be a unisolvent set for $\mathcal{P}_d(E)$, then we define

$$\mathbf{I}[B_0; f] := (\Phi|_{\mathcal{Q}})^{-1}(\mathbf{L}_E[B_0; f]), \quad f \in \mathcal{A}.$$

It is easy to see that $\mathbf{I}[B_0; \cdot] \colon \mathcal{A} \to \mathcal{Q}$ is a linear map.

Lemma 2.3. *The operator $\mathbf{I}[B_0; \cdot]$ has the following properties:*

a) *for every $f \in \mathcal{A}$, $\mathbf{I}[B_0; f]$ interpolates f at B_0, i.e., $\mathbf{I}[B_0; f](\mathbf{b}) = f(\mathbf{b})$, $\mathbf{b} \in B_0$;*

b) *if $f \in \mathcal{A}$, $f = 0$ on B_0, then $\mathbf{I}[B_0; f] = 0$.*

Proof. By definition we have

$$\mathbf{I}[B_0; f](\mathbf{b}) = \mathbf{L}_E[B_0; f](\mathbf{b}) = f(\mathbf{b}) \quad \forall \mathbf{b} \in B_0.$$

If $f = 0$ on B_0, then $\mathbf{L}_E[B_0; f] = 0$. It follows that $\mathbf{I}[B_0; f] = (\Phi|_{\mathcal{Q}})^{-1}(\mathbf{L}_E[B_0; f]) = 0$.

The following result gives a connection between the above operator and a normal sequence.

Lemma 2.4. *Let E be an algebraic variety in $\mathbb{R}^N$ such that $\mathcal{I}(E)$ is generated by a non-constant polynomial q. Let $B_n \subset E$ be a unisolvent set for $\mathcal{P}_d(E)$ and $\mathcal{A}$ be an algebra of functions. Then the sequence $\{B_n\}$ is $\mathcal{A}$-normal for $\mathcal{P}_d(E)$ if and only if the sequence $\{\mathbf{I}[B_n; f]\}$ is convergent in $\mathcal{P}_d(\mathbb{R}^N)$ for every $f \in \mathcal{A}$, where $\mathbf{I}[B_n; \cdot] \colon \mathcal{A} \to \mathcal{Q}$ defined by*

$$\mathbf{I}[B_n; f] := \Phi^{-1}(\mathbf{L}_E[B_n; f]), \quad f \in \mathcal{A}.$$

Proof. We first assume that $\{B_n\}$ is $\mathcal{A}$-normal for $\mathcal{P}_d(E)$. Since Φ^{-1} is continuous and $\{\mathbf{L}_E[B_n; f]\}$ is convergent, $\mathbf{I}[B_n; f]$ is also convergent.

Conversely, the sequence $\{\mathbf{L}_E[B_n; f]\}$ is convergent, because $\{\mathbf{I}[B_n; f]\}$ is convergent and Φ is continuous.

We set

$$\Pi(f) := \lim_{n\to\infty} \mathbf{I}[B_n; f], \quad f \in \mathcal{A}.$$

Then $\Pi \colon \mathcal{A} \to \mathcal{P}_d(\mathbb{R}^N)$ is a linear map which can be viewed as a Hermite type interpolation operator on E. Next we investigate some properties of Π.

Lemma 2.5. *Let E be an algebraic variety in $\mathbb{R}^N$ such that $\mathcal{I}(E)$ is generated by a non-constant polynomial q. Let $\{B_n\} \subset E$ be an $\mathcal{A}$-normal sequence for $\mathcal{P}_d(E)$. Then, for any $f, g \in \mathcal{A}$, we have*:

a) $\Pi(qf) = 0$;

b) $\Pi(f\Pi(g)) = \Pi(fg)$; *in particular,* $\Pi(\Pi(g)) = \Pi(g)$.

Proof. a) Since $B_n \subset E$, the function qf vanishes on B_n. Hence $\mathbf{I}[B_n; qf] = 0$ for all $n \geq 1$. It follows that $\Pi(qf) = \lim_{n\to\infty} \mathbf{I}[B_n; qf](qf) = 0$.

b) We have $f\mathbf{I}[B_n; g] - fg = 0$ on B_n. Hence, by Lemma 2.3, $\mathbf{I}[B_n; f\mathbf{I}[B_n; g] - fg] = 0$. It follows that

$$\mathbf{I}[B_n; f\mathbf{I}[B_n; g]] = \mathbf{I}[B_n; fg].$$

We now apply the arguments given in the proof of Lemma 2.1, with $\mathbf{L}[A_n; \cdot]$ and Λ replaced by $\mathbf{I}[B_n; \cdot]$ and Π, respectively, to obtain the desired relation.

Lemma 2.6. *Let E be an algebraic variety in $\mathbb{R}^N$ such that $\mathcal{I}(E)$ is generated by a non-constant polynomial q. Let $\{B_n\}$ be an $\mathcal{A}$-normal sequence for $\mathcal{P}_d(E)$. If $\{P_n\} \subset \mathcal{P}_k(\mathbb{R}^N)$ converges to $P \in \mathcal{P}_k(\mathbb{R}^N)$, then*

$$\lim_{n\to\infty} \mathbf{I}[B_n; fP_n] = \Pi(fP), \quad f \in \mathcal{A}.$$

Proof. The proof is similar to that of Lemma 2.2. Let $\{p_1, \ldots, p_m\}$ be a basis for $\mathcal{P}_k(\mathbb{R}^N)$ with $m = m_k(\mathbb{R}^N)$. We write $P = \sum_{i=1}^m a_i p_i$ and $P_n = \sum_{i=1}^m a_i^n p_i$. By the hypothesis, $\lim_{n\to\infty} a_i^n = a_i$, $i = 1, \ldots, m$. It follows that

$$\lim_{n\to\infty} \mathbf{I}[B_n; fP_n] = \lim_{n\to\infty} \sum_{i=1}^m a_i^n \mathbf{I}[B_n; fp_i] = \sum_{i=1}^m a_i \Pi(fp_i) = \Pi(fP),$$

where, in the second equation, we use the fact that $\lim_{n\to\infty} \mathbf{I}[B_n; fp_i] = \Pi(fp_i)$, $i = 1, \ldots, m$.

2.3. Examples.

***Example* 2.1.** Let $\{A_n\}$ be a sequence of unisolvent sets for $\mathcal{P}_d(\mathbb{R}^N)$ such that, for every multi-index α with $|\alpha| = d + 1$,

$$\lim_{n\to\infty} \mathbf{L}[A_n; \mathbf{x}^\alpha] = 0.$$

Bloom and Calvi proved in [1] that

$$\lim_{n\to\infty} \sup\{\|\mathbf{x}\| : \mathbf{x} \in A_n\} = 0$$

and

$$\lim_{n\to\infty} \mathbf{L}[A_n; f] = \mathbf{T}_0^d(f) \quad \forall f \in C^{m_d(\mathbb{R}^N)-1}(\{0\}),$$

where $\mathbf{T}_0^d(f)$ stands for the Taylor expansion of f at 0 to the order d. In other words, $\{A_n\}$ is $C^{m_d(\mathbb{R}^N)-1}(\{0\})$-regular. In [1], the authors also gave some examples of $\{A_n\}$ satisfying the above assumption.

***Example* 2.2.** A set of N hyperplanes $H = \{h_1, \ldots, h_N\}$ in $\mathbb{R}^N$ is said to be in general position if the intersection of the N hyperplanes is a singleton, that is,

$$\bigcap_{j=1}^N h_j = \{\mathbf{a}_H\}.$$

More generally, a collection $\mathcal{H}$ of $d \geq N$ hyperplanes in $\mathbb{R}^N$ is said to be in general position if

(1) every $H \in \binom{\mathcal{H}}{N}$, a subset of N hyperplanes of $\mathcal{H}$, is in general position;

(2) the map

$$H \in \binom{\mathcal{H}}{N} \longmapsto \mathbf{a}_H = \bigcap_{j=1}^{N} h_j$$

is one-to-one.

Let us set

$$\Theta_{\mathcal{H}} = \left\{ \mathbf{a}_H : H \in \binom{\mathcal{H}}{N} \right\},$$

which is called a natural lattice of degree $d - N$. It was proved in [8] that $\Theta_{\mathcal{H}}$ is unisolvent for $\mathcal{P}_{d-N}(\mathbb{R}^N)$. Moreover, the corresponding Lagrange interpolation polynomial has a simple formula.

Let $d \geq N$ and let $\Theta^{(s)}$ be a sequence of natural lattices of degree $d - N$ in $\mathbb{R}^N$. We assume that $\Theta^{(s)}$ is the lattice generated by the family of hyperplanes

$$\mathcal{H}^{(s)} = \{h_1^{(s)}, \ldots, h_d^{(s)}\} \quad \text{with} \quad h_j^{(s)}(\mathbf{x}) = \langle \mathbf{n}_j^{(s)}, \mathbf{x} \rangle - c_j^{(s)}, \ \|\mathbf{n}_j^{(s)}\| = 1, \ j = 1, \ldots, d,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^N$. Consider the following two conditions:

(C_1) all points of the lattices tend to the origin as $s \to \infty$, that is, $\max\{\|\mathbf{a}\| : \mathbf{a} \in \Theta^{(s)}\} \to 0$ as $s \to \infty$;

(C_2) the volumes

$$\operatorname{vol}\left(\mathbf{n}_{j_1}^{(s)}, \ldots, \mathbf{n}_{j_N}^{(s)}\right), \quad 1 \leq j_1 < j_2 < \ldots < j_N \leq d,$$

of the parallelotope spanned by the vectors $\mathbf{n}_{j_1}^{(s)}, \ldots, \mathbf{n}_{j_N}^{(s)}$ are bounded from below, always from 0, uniformly in s.

We proved in [6] (Theorem 3.1) that if the above two conditions hold, then

$$\lim_{s \to \infty} \mathbf{L}[\Theta^{(s)}; f] = \mathbf{T}_0^{d-N}(f), \quad f \in C^{d-N+1}(\{0\}).$$

Hence, we can say that $\{\Theta^{(s)}\}$ is $C^{d-N+1}(\{0\})$-regular.

***Example* 2.3.** Let $\delta > 0$ and $d \geq 2$ be a positive integer. We define $m = \left[\dfrac{d}{2}\right] + 1$ and $S = \{s_1, \ldots, s_m\}$ with $s_k = d - 2k + 2$ for $k = 1, \ldots, m$. Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be m distinct points in $\mathbb{R}^2$. Each point $\mathbf{a}_k$ is associated to a sequence of circles $\{C_k^n\}$ with $C_k^n = \{\mathbf{x} : \|\mathbf{x} - \mathbf{a}_k\| = r_{k,n}\}$ such that $\lim_{n\to\infty} r_{k,n} = 0$. For $1 \leq k \leq m$ and $n \geq 1$, let X_k^n be a δ-separate set of $2s_k + 1$ points on C_k^n, i.e.,

$$\|\mathbf{b} - \mathbf{c}\| \geq \delta r_{k,n}, \quad \mathbf{b}, \mathbf{c} \in X_k^n, \quad \mathbf{b} \neq \mathbf{c}.$$

Then $X^n := \cup_{k=1}^m X_k^n$ is a unisolvent set for $\mathcal{P}_d(\mathbb{R}^2)$ and is called a Bos configuration on circles. Let $\mathcal{F}$ be the class of all bivariate functions of class C^{s_k} in neighborhoods of the $\mathbf{a}_k$'s. It was proved in [9] that, for any $f \in \mathcal{F}$,

$$\lim_{n \to \infty} \mathbf{L}[X^n; f] = \mathbf{H}[(A, S); f],$$

where $\mathbf{H}[(A, S); f]$ is a Hermite type projector satisfying the relations

$$\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)^k \mathbf{H}[(A,S);f](\mathbf{a}_k) = \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)^k f(\mathbf{a}_k), \quad 1 \le k \le m, \quad 0 \le j \le s_k,$$

where i is the imaginary unit and $\mathbf{x} = (x_1, x_2)$. It follows that $\{X^n\}$ is $\mathcal{F}$-regular. A generalization of the above result can be found in [10].

***Example* 2.4.** Normal sets can be constructed on irreducible algebraic curves in $\mathbb{R}^2$. Indeed, let $A^{(n)} = \{\mathbf{a}_0^{(n)}, \ldots, \mathbf{a}_{2d}^{(n)}\}$ be distinct points on the circle $C(0,\rho)$ and $\mathbf{b} \in C(0,\rho)$ such that $\mathbf{a}_j^{(n)} \to \mathbf{b}$ as $n \to \infty$. Let g be a real-valued functions in $C^{2d}(\Omega)$, where Ω is a neighborhood of $\mathbf{b}$ in $C(0,\rho)$. Proposition 4.1 in [11] asserts that the following limit exists:

$$\lim_{n\to\infty} \mathbf{L}_{C(0,\rho)}[A^{(n)}, g]. \tag{2.1}$$

Moreover, the limit depends only on $\mathbf{b}$ and g and is denoted by $\mathbf{T}^d_{\mathbf{b}}(g)$ which satisfies the relations

$$\left(\mathbf{T}^d_{\mathbf{b}}(g)(\rho\cos\alpha, \rho\sin\alpha)\right)^{(k)}\Big|_{\alpha=\alpha^*} = (g(\rho\cos\alpha, \rho\sin\alpha))^{(k)}\Big|_{\alpha=\alpha^*}, \quad k = 0, \ldots, 2d,$$

where $\mathbf{b} = (\rho\cos\alpha^*, \rho\sin\alpha^*)$. In other words, $\{A^{(n)}\}$ is $\mathcal{A}$-normal on $C(0,\rho)$, where $\mathcal{A}$ is the algebra of all functions of class C^{2d} in neighborhoods of $\mathbf{b}$ on $C(0,\rho)$.

More general result also holds when we replace the circle by irreducible algebraic curves in $\mathbb{R}^2$. In [7], we studied the polynomial interpolation on irreducible algebraic curves in $\mathbb{C}^2$. However, with simple adaptions, every result remains true in the real settings. The passage to the real case can be found in [4].

Let q be an irreducible polynomial in $\mathbb{R}^2$ such that $V := \{\mathbf{x} \in \mathbb{R}^2 : q(\mathbf{x}) = 0\}$ contains at least one regular point. In [5], the authors defined the notion of d-Taylorian points on V. Note that all but finitely many points on V are d-Taylorian (see [5], Theorem 4.10). Bos and Calvi proved in [4] that, for every function f of class $C^{m_d(V)-1}$ on a neighbourhood of a d-Taylorian point $\mathbf{a} \in V$, there exists a unique polynomial $P \in \mathcal{P}_d(V)$ such that, for every local parametrization $\mathcal{L} = (0, U, R)$ of V at $\mathbf{a}$ with $R(0) = \mathbf{a}$, we have

$$(P \circ R)^{(i)}(0) = (f \circ R)^{(i)}(0), \quad i = 0, \ldots, m_d(V) - 1.$$

The above interpolation polynomial is called the d-Taylor polynomial of f at $\mathbf{a}$ and is denoted by $\mathbf{T}^d_{\mathbf{a}}(f)$. We proved in [7] that it is the limit of Lagrange interpolation on V. More precisely, we proved in Theorems 4.1 and 4.2 that if $A_n \subset V$, $n \in \mathbb{N}$ is a sequence of unisolvent sets for $\mathcal{P}_d(V)$ whose points tend to a d-Taylorian point $\mathbf{a}$, i.e., $\max\{\|\mathbf{x} - \mathbf{a}\| : \mathbf{x} \in A_n\} \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} \mathbf{L}_V[A_n\,;\, f] = \mathbf{T}^d_{\mathbf{a}}(f)$$

for every function f of class $C^{m_d(V)-1}$ on a neighborhood of $\mathbf{a}$ in V. Hence every sequence of unisolvent sets $\{A_n\} \subset V$ tending to a d-Taylorian point $\mathbf{a}$ is normal.

***Example* 2.5.** In [11], we give some normal sequences on the unit sphere in $\mathbb{R}^3$. Associated with each point $\mathbf{a} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in \mathbb{S}^2$, we denote $R_{\mathbf{a}}$ by the local parametrization of $\mathbb{S}$ at $\mathbf{a}$,

$$R_{\mathbf{a}} = (R^1_{\mathbf{a}}, R^2_{\mathbf{a}}, R^3_{\mathbf{a}}),$$

where

$$R^1_{\mathbf{a}}(u,v) = (\cos\theta\cos\varphi)u - (\sin\varphi)v + (\sin\theta\cos\varphi)\sqrt{1-u^2-v^2},$$

$$R^2_{\mathbf{a}}(u,v) = (\cos\theta\sin\varphi)u + (\cos\varphi)v + (\sin\theta\sin\varphi)\sqrt{1-u^2-v^2},$$

$$R^3_{\mathbf{a}}(u,v) = -(\sin\theta)u + (\cos\theta)\sqrt{1-u^2-v^2}.$$

For $\rho \in (0,1]$, we consider the linear polynomial

$$q_{\mathbf{a},\rho}(\mathbf{x}) = (\sin\theta\cos\varphi)x_1 + (\sin\theta\sin\varphi)x_2 + (\cos\theta)x_3 - \sqrt{1-\rho^2}.$$

The plane $\{q_{\mathbf{a},\rho} = 0\}$ cuts a small spherical cap off the sphere in which the circle of the base denoted by $\mathcal{C}(\mathbf{a},\rho)$ is of radius ρ and the peak point is $\mathbf{a}$.

Let $\delta > 0$. Let $\mathbf{a}_0,\ldots,\mathbf{a}_d$ be $d+1$ distinct points on $\mathbb{S}^2$. Each point $\mathbf{a}_j$ is associated with a sequence of circles $\mathcal{C}(\mathbf{a}_j,\rho_j^{(n)})$ where $\rho_j^{(n)} \in (0,1)$ and $\lim_{n\to\infty}\rho_j^{(n)} = 0$. For each $n \geq 1$, let $A_j^{(n)}$ be a set of $2(d-j)+1$ distinct points on $\mathcal{C}(\mathbf{a}_j,\rho_j^{(n)})$ such that $A_j^{(n)}$ is δ-separate. Let us set $A^{(n)} = \bigcup_{j=0}^d A_j^{(n)}$. Let $\mathcal{A}$ be the algebra of functions of class $C^{(d-j)}$ in a neighborhood of $\mathbf{a}_j$ in $\mathbb{S}^2$, $j = 0,\ldots,d$. For each $f \in \mathcal{A}$, we have

$$\lim_{n\to\infty}\mathbf{L}_{\mathbb{S}^2}[A^{(n)};f] = \Pi^{(1)}(f).$$

Here the right-hand side is a Hermite type interpolation defined in [11] (Theorem 3.4) which satisfies the relation

$$\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)^k (\Pi^{(1)}(f)\circ R_{\mathbf{a}_j})(0,0) = \left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)^k (f\circ R_{\mathbf{a}_j})(0,0),$$

$$j = 0,\ldots,d, \quad k = 0,1\ldots,d-j.$$

The above assertion is proved in [11] (Theorem 3.7) and gives an $\mathcal{A}$-normal sequence.

Another type of normal sequence is constructed in [11] (Theorem 4.4).

Let $\mathbf{a}_0,\ldots,\mathbf{a}_d$ be $d+1$ not necessarily distinct points on $\mathbb{S}^2$ and $\rho_0,\ldots,\rho_d \in (0,1]$. On each circle $\mathcal{C}(\mathbf{a}_j,\rho_j)$, we take a point $\mathbf{b}_j$ which does not lie on $\mathcal{C}(\mathbf{a}_k,\rho_k)$ for $j > k$. For $j = 0,\ldots,d$ and $n \geq 1$, let $A_j^{(n)}$ be a set of $2(d-j)+1$ distinct points on $\mathcal{C}(\mathbf{a}_j,\rho_j)$ such that $A_j^{(n)} \to \mathbf{b}_j$. Set $A^{(n)} = \bigcup_{j=0}^d A_j^{(n)}$. Then, for each function f of class $C^{2(d-k)}$ in a neighborhood of $\mathbf{b}_k$ on $\mathbb{S}^2$, $k = 0,\ldots,d$, we have

$$\lim_{n\to\infty}\mathbf{L}_{\mathbb{S}^2}[A^{(n)};f] = \Pi^{(2)}(f), \tag{2.2}$$

where $A_j^{(n)} \to \mathbf{b}_j$ means that all points in $A_j^{(n)}$ tend to $\mathbf{b}_j$ as $n \to \infty$. Here the right-hand side is a Hermite type interpolation defined in [11] (Theorem 4.3) which satisfies the relation

$$\Big((\Pi^{(2)}(f)\circ R_{\mathbf{a}_j})(\rho_j\cos\alpha,\rho_j\sin\alpha)\Big)^{(k)}\Big|_{\alpha=\alpha_j} = \big((f\circ R_{\mathbf{a}_j})(\rho_j\cos\alpha,\rho_j\sin\alpha)\big)^{(k)}\Big|_{\alpha=\alpha_j}$$

for all $j = 0,\ldots,d$ and $k = 0,\ldots,2(d-j)$.

***Example* 2.6.** Let q be a linear polynomial in $\mathbb{R}^N$ and $V = \{\mathbf{x} \in \mathbb{R}^N : q(\mathbf{x}) = 0\}$. Then V can be viewed as a $(N-1)$-dimensional space. Then any $\mathcal{F}$-regular sequence in V is a $\mathcal{F}$-normal sequence on the subset V of $\mathbb{R}^N$. Hence Examples 2.1 – 2.3 give normal sequences on V. This type of normal sequence is used in Theorem 4.2 below.

3. Combining interpolation schemes. The aim of this section is to study the behavior of Lagrange projectors in $\mathbb{R}^N$ when the interpolation points are collected from regular sequences and normal sequences. The method is inspired from [3].

Theorem 3.1. *Let d, k be non-negative integer with $k < d$. Let E be an algebraic variety in $\mathbb{R}^N$ such that $\mathcal{I}(E)$ is generated by a non-constant polynomial q with $\deg q = d - k$. Let $\{B_n\} \subset E$ be an $\mathcal{A}$-normal sequence for $\mathcal{P}_d(\mathbb{R}^N)$. Let $\{A_n\}$ be a $\mathcal{F}$-regular sequence for $\mathcal{P}_k(\mathbb{R}^N)$ such that $A \cap \{q = 0\} = \varnothing$ with $A = \bigcup_{n=1}^{\infty} A_n$. Assume that $1/q \in \mathcal{F}$. Then $\{A_n \cup B_n\}$ is $(\mathcal{A} \cap \mathcal{F})$-regular for $\mathcal{P}_d(\mathbb{R}^N)$. Moreover, if Λ and Π are define by*

$$\Lambda(f) := \lim_{n\to\infty} \mathbf{L}[A_n; f], \quad \Pi(f) := \lim_{n\to\infty} \mathbf{I}[B_n; f], \quad f \in \mathcal{A} \cap \mathcal{F},$$

then the operator $\mathbf{H}$ defined by

$$\mathbf{H}(f) := \lim_{n\to\infty} \mathbf{L}[A_n \cup B_n; f]$$

satisfies the relations

$$\Lambda(\mathbf{H}(f)) = \Lambda(f), \quad \Pi(\mathbf{H}(f)) = \Pi(f).$$

Proof. We first prove that $X_n := A_n \cup B_n$ is unisolvent for $\mathcal{P}_d(\mathbb{R}^N)$. Since $A_n \cap B_n = \varnothing$, we have

$$\sharp X_n = \sharp A_n + \sharp B_n = \dim \mathcal{P}_k(\mathbb{R}^N) + \dim \mathcal{P}_d(E) = m_k(\mathbb{R}^N) + m_d(\mathbb{R}^N) - m_{d-\deg q}(\mathbb{R}^N) = m_d(\mathbb{R}^N).$$

Hence, it suffices to show that $P \in \mathcal{P}_d(\mathbb{R}^N)$ that vanishes on X_n is identically zero. Since $P = 0$ on B_n and B_n is unisolvent for $\mathcal{P}_d(E)$, $P = 0$ on E. This enables us to find $P_1 \in \mathcal{P}_{d-\deg q}(\mathbb{R}^N)$ such that $P = qP_1$. Since $\{q = 0\} \cap A_n = \varnothing$, P_1 must vanish on A_n. It follows that $P_1 = 0$ since it belongs to $\mathcal{P}_k(\mathbb{R}^N)$. Hence, $P = 0$.

We next prove a Newton form formula for interpolation polynomials

$$\mathbf{L}[X_n; f] = \mathbf{I}[B_n; f] + q\mathbf{L}\left[A_n; \frac{f - \mathbf{I}[B_n; f]}{q}\right]. \tag{3.1}$$

Indeed, the right-hand side of (3.1) denoted by Q is a polynomial of degree at most d in $\mathbb{R}^N$. For each $\mathbf{b} \in B_n$ we have $q(\mathbf{b}) = 0$. Hence $Q(\mathbf{b}) = \mathbf{I}[B_n; f](\mathbf{b}) = f(\mathbf{b})$. On the other hand, for each $\mathbf{a} \in A_n$, we have

$$Q(\mathbf{a}) = \mathbf{I}[B_n; f](\mathbf{a}) + q(\mathbf{a})\mathbf{L}\left[A_n; \frac{f - \mathbf{I}[B_n; f]}{q}\right](\mathbf{a}) =$$

$$= \mathbf{I}[B_n; f](\mathbf{a}) + q(\mathbf{a})\frac{f(\mathbf{a}) - \mathbf{I}[B_n; f](\mathbf{a})}{q(\mathbf{a})} = f(\mathbf{a}).$$

From what has already been proved, we conclude that Q interpolates f at X_n. Therefore, $Q = \mathbf{L}[X_n; f]$.

Next we find the limit of polynomials in (3.1). Note that $\{\mathbf{I}[B_n; f]\} \subset \mathcal{P}_d(\mathbb{R}^N)$ converges to $\Pi(f)$. Hence, we can use Lemma 2.2, Lemma 2.6 and (3.1) to get

$$\lim_{n\to\infty} \mathbf{L}[X_n; f] = \lim_{n\to\infty} \left(\mathbf{I}[B_n; f] + q\mathbf{L}\left[A_n; \frac{f}{q}\right] - q\mathbf{L}\left[A_n; \frac{\mathbf{I}[B_n; f]}{q}\right]\right) =$$

$$= \Pi(f) + q\left(\Lambda\left(\frac{f}{q}\right) - \Lambda\left(\frac{\Pi(f)}{q}\right)\right) = \Pi(f) + q\Lambda\left(\frac{f-\Pi(f)}{q}\right).$$

It follows that $\{X_n\}$ is $(\mathcal{F} \cup \mathcal{A})$-regular and

$$\mathbf{H}(f) = \Pi(f) + q\Lambda\left(\frac{f-\Pi(f)}{q}\right), \quad f \in \mathcal{A} \cap \mathcal{F}.$$

By using Lemma 2.5, we obtain

$$\Pi(\mathbf{H}(f)) = \Pi(\Pi(f)) + \Pi\left(q\Lambda\left(\frac{f-\Pi(f)}{q}\right)\right) = \Pi(f).$$

To prove the last relation, we use Lemma 2.1 to get

$$\Lambda(\mathbf{H}(f)) = \Lambda(\Pi(f)) + \Lambda\left(q\Lambda\left(\frac{f-\Pi(f)}{q}\right)\right) = \Lambda(\Pi(f)) + \Lambda\left(q\frac{f-\Pi(f)}{q}\right) = \Lambda(f).$$

Theorem is proved.

Theorem 3.2. *Let $m \geq 2$ be a positive integer. For each $1 \leq j \leq m$, let E_j be an algebraic variety in $\mathbb{R}^N$ such that $\mathcal{I}(E_j)$ is generated by a non-constant polynomial q_j with $\deg q_j = r_j$. Let $d \in \mathbb{N}$ be such that*

$$r_1 + \ldots + r_{m-1} < d \leq r_1 + \ldots + r_m.$$

We define $s_1, s_2, \ldots, s_m$ by the relation

$$s_1 = d, \quad s_j = d - r_1 - \ldots - r_{j-1}, \quad j = 2, \ldots, m.$$

Let $\{B_{j,n}\} \subset E_j$ be a $\mathcal{A}_j$-normal sequence for $\mathcal{P}_{s_j}(E_j)$ such that $\{q_k = 0\} \cap (\bigcup_{n=1}^{\infty} B_{j,n}) = \varnothing$ for $j > k$ and $1/q_i \in \mathcal{A}_{i+1}$ for $1 \leq i \leq m-1$. Set $X_n = \bigcup_{j=1}^{m} B_{j,n}$ for $n \geq 1$.

a) *In the case $d < r_1 + \ldots + r_m$, the sequence $\{X_n\}$ is $\left(\bigcap_{j=1}^{m} \mathcal{A}_j\right)$-regular. Moreover, the limiting operator defined*

$$\lim_{n\to\infty} \mathbf{L}[X_n; f] = \mathbf{H}(f), \quad f \in \bigcap_{j=1}^{m} \mathcal{A}_j$$

satisfies the relation

$$\Pi_j(\mathbf{H}(f)) = \Pi_j(f), \quad 1 \leq j \leq m, \tag{3.2}$$

where

$$\Pi_j(f) = \lim_{n\to\infty} \mathbf{I}[B_{j,n}; f].$$

b) *In the case $d = r_1 + \ldots + r_m$, if $\{\mathbf{a}_{m+1,n}\}$ is a sequence in $\mathbb{R}^N$ lying outside $\bigcup_{j=1}^{m}\{q_j = 0\}$ and converging to $\mathbf{a} \notin \bigcup_{j=1}^{m}\{q_j = 0\}$, then the sequence $\{X_n \cup \{\mathbf{a}_{m+1,n}\}\}$ is $(C^0(\{\mathbf{a}\}) \cap \cap_{j=1}^{m} \mathcal{A}_j)$-regular. Moreover, the limit operator defined*

$$\lim_{n\to\infty} \mathbf{L}[X_n; f] = \mathbf{H}(f), \quad f \in C^0(\{\mathbf{a}\}) \cap \bigcap_{j=1}^{m} \mathcal{A}_j$$

satisfies (3.2) *along with the additional relation $\mathbf{H}(f)(\mathbf{a}) = f(\mathbf{a})$.*

Proof. We only prove the statement corresponding to the case $d < r_1 + \ldots + r_m$. Since $B_{j,n}$ is unisolvent for $\mathcal{P}_{s_j}(E_j)$, Theorem 3.3 in [3] asserts that $\bigcup_{j=k}^m B_{j,n}$ is unisolvent for $\mathcal{P}_{s_k}(\mathbb{R}^N)$ for $1 \le k \le m$ and $n \ge 1$. In particular, X_n is unisolvent for $\mathcal{P}_d(\mathbb{R}^N)$. The proof is now by induction in m.

We first assume that $m = 2$. Then $X_n = B_{1,n} \cup B_{2,n}$. Since $B_{2,n}$ is unisolvent for $\mathcal{P}_{d-r_1}(\mathbb{R}^N)$, $\mathbf{L}[B_{2,n}; f] = \mathbf{I}[B_{2,n}; f]$, and hence $\{B_{2,n}\}$ is $\mathcal{A}_2$-regular,

$$\lim_{n\to\infty} \mathbf{L}[B_{2,n}; f] = \Pi_2(f), \quad f \in \mathcal{A}_2.$$

By using Theorem 3.1, we get the regularity of $\{X_n\}$. Moreover,

$$\Pi_j(\mathbf{H}(f)) = \Pi_j(f), \quad j = 1, 2.$$

Assume that the assertion holds up to $m - 1 \ge 2$; we will prove it for m. We set $\widetilde{X}_n = \bigcup_{j=2}^m B_{j,n}$. Then $\widetilde{X}_n$ is unisolvent for $\mathcal{P}_{s_2}(\mathbb{R}^N)$ and, by the induction hypothesis, $\{\widetilde{X}_n\}$ is $\left(\bigcap_{j=2}^m \mathcal{A}_j\right)$-regular. Furthermore, the limiting operator

$$\widetilde{\mathbf{H}}(f) := \lim_{n\to\infty} \mathbf{L}[\widetilde{X}_n; f], \quad f \in \bigcap_{j=2}^m \mathcal{A}_j,$$

satisfies the relation

$$\Pi_j(\widetilde{\mathbf{H}}(f)) = \Pi_j(f), \quad 2 \le j \le m. \tag{3.3}$$

Applying Theorem 3.1 for $\{B_{1,n}\}$ and $\{\widetilde{X}_n\}$, we conclude that $\{X_n\}$ is $\left(\bigcap_{j=1}^m \mathcal{A}_j\right)$-regular and

$$\widetilde{\mathbf{H}}(\mathbf{H}(f)) = \widetilde{\mathbf{H}}(f), \quad \Pi_1(\mathbf{H}(f)) = \Pi_1(f),$$

where

$$\mathbf{H}(f) := \lim_{n\to\infty} \mathbf{L}[X_n; f].$$

Combining the above relation with (3.3), we obtain, for $2 \le j \le m$,

$$\Pi_j(f) = \Pi_j(\widetilde{\mathbf{H}}(f)) = \Pi_j\left(\widetilde{\mathbf{H}}(\mathbf{H}(f))\right) = \Pi_j(\mathbf{H}(f)).$$

Here, in the third relation we use (3.3) for $\mathbf{H}(f)$ in the place of f. The assertion holds for m.

Theorem is proved.

Remark **3.1.** From Theorems 3.1 and 3.2, we can use examples in Subsection 2.3 to build new regular sequences. The details are left to the readers.

4. Polynomial interpolation on the unit sphere. In this section, we construct unisolvent sets for $\mathcal{P}_d(\mathbb{S}^N)$ from unisolvent sets lying in unit ball $\mathbb{B}^N$. We also investigate the limit of the Lagrange interpolation on $\mathbb{S}^N$ corresponding a regular sequence and a normal sequence in $\mathbb{B}^N$. We recall the trivial parametrizations of the half spheres

$$R^+(\mathbf{x}) = \left(\mathbf{x}, \sqrt{1 - \|\mathbf{x}\|^2}\right), \quad R^-(\mathbf{x}) = \left(\mathbf{x}, -\sqrt{1 - \|\mathbf{x}\|^2}\right), \quad \mathbf{x} \in \mathbb{B}^N.$$

Theorem 4.1. *Let $A_0 \subset \mathbb{B}^N$ be unisolvent for $\mathcal{P}_{d-1}(\mathbb{R}^N)$. Let E be a hyperplane in $\mathbb{R}^N$ such that $A_0 \cap E = \varnothing$. Let $B_0 \subset E \cap \mathbb{B}^N$ be unisolvent for $\mathcal{P}_d(E)$. Then the set*

$$X_0 = R^+(A_0) \cup R^-(A_0) \cup R^+(B_0)$$

is unisolvent for $\mathcal{P}_d(\mathbb{S}^N)$. Moreover, if f is a function defined on $\mathbb{S}^N$, then

$$\mathbf{L}_{\mathbb{S}^N}[X_0; f] \circ R^{\pm}(\mathbf{x}) = \frac{\pm\sqrt{1-\|\mathbf{x}\|^2}\mathbf{L}[A_0; f_1](\mathbf{x}) + \mathbf{L}[(A_0; f_2), (B_0; f_3)](\mathbf{x})}{2}, \quad \mathbf{x} \in \overline{\mathbb{B}^N}, \tag{4.1}$$

where

$$f_1(\mathbf{x}) = \frac{f \circ R^+(\mathbf{x}) - f \circ R^-(\mathbf{x})}{\sqrt{1-\|\mathbf{x}\|^2}}, \quad f_2(\mathbf{x}) = f \circ R^+(\mathbf{x}) + f \circ R^-(\mathbf{x}), \quad \mathbf{x} \in \mathbb{B}^N,$$

and

$$f_3(\mathbf{x}) = 2f \circ R^+(\mathbf{x}) - \sqrt{1-\|\mathbf{x}\|^2}\mathbf{L}[A_0; f_1](\mathbf{x}), \quad \mathbf{x} \in \mathbb{B}^N.$$

Proof. We first prove that X_0 is unisolvent for $\mathcal{P}_d(\mathbb{S}^N)$. The proof is motivated by [12] (Theorem 2.1) and is similar to [10] (Theorem 3.1). We take a non-zero affine polynomial q in $\mathbb{R}^N$ such that $E = \{\mathbf{x} \in \mathbb{R}^N : q(\mathbf{x}) = 0\}$. We have $\sharp B_0 = \dim \mathcal{P}_d(E) = m_d(\mathbb{R}^N) - m_{d-1}(\mathbb{R}^N)$. Hence

$$\sharp X_0 = 2m_{d-1}(\mathbb{R}^N) + m_d(\mathbb{R}^N) - m_{d-1}(\mathbb{R}^N) = m_d(\mathbb{R}^N) + m_{d-1}(\mathbb{R}^N) = \dim \mathcal{P}_d(\mathbb{S}^N).$$

To prove the theorem, it suffices to verify that if $P \in \mathcal{P}_d(\mathbb{S}^N)$ that vanishes on X_0, i.e.,

$$P \circ R^+(\mathbf{a}) = P \circ R^-(\mathbf{a}) = P \circ R^+(\mathbf{b}) = 0, \quad \forall \mathbf{a} \in A_0 \quad \mathbf{b} \in B_0, \tag{4.2}$$

then P is identically zero. Let us set

$$P_1(\mathbf{x}) := \frac{P \circ R^+(\mathbf{x}) - P \circ R^-(\mathbf{x})}{\sqrt{1-\|\mathbf{x}\|^2}}. \tag{4.3}$$

Then P_1 belongs to $\mathcal{P}_{d-1}(\mathbb{R}^N)$. Relation (4.2) implies that $P_1(\mathbf{a}) = 0$ for all $\mathbf{a} \in A_0$. Since A_0 is unisolvent for $\mathcal{P}_{d-1}(\mathbb{R}^N)$, P_1 must be identically zero. This enables us to write $P \circ R^+(\mathbf{x}) =$ $= P \circ R^-(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{B}^N$. Likewise, consider the polynomial $P_2 \in \mathcal{P}_d(\mathbb{R}^N)$ defined by

$$P_2(\mathbf{x}) := P \circ R^+(\mathbf{x}) + P \circ R^-(\mathbf{x}). \tag{4.4}$$

Then $P_2(\mathbf{x}) = 2P \circ R^+(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{B}$. From (4.2) we see that

$$P_2(\mathbf{a}) = P_2(\mathbf{b}) = 0 \quad \forall \mathbf{a} \in A_0 \quad \forall \mathbf{b} \in B_0.$$

This forces $P_2 = 0$, because $A_0 \cup B_0$ is unisolvent for $\mathcal{P}_d(\mathbb{R}^N)$ due to Theorem 3.1. From what has already been proved, we conclude that

$$P \circ R^+(\mathbf{x}) = P \circ R^-(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \mathbb{B}^N.$$

It follows that $P = 0$ on $\mathbb{S}^N$, and the proof of the first assertion is complete.

It remains to prove the formulas. For convenience, we set $P = \mathbf{L}_{\mathbb{S}^N}[X; f]$. Let P_1, P_2 be defined as in (4.3) and (4.4). By the interpolation condition, we have

$$P_1(\mathbf{a}) = \frac{P \circ R^+(\mathbf{a}) - P \circ R^-(\mathbf{a})}{\sqrt{1 - \|\mathbf{a}\|^2}} = \frac{f \circ R^+(\mathbf{a}) - f \circ R^-(\mathbf{a})}{\sqrt{1 - \|\mathbf{a}\|^2}} = f_1(\mathbf{a}), \quad \mathbf{a} \in A_0.$$

It follows that $P_1 = \mathbf{L}[A_0, f_1]$. Combining this with the setting in (4.3), we get

$$P \circ R^-(\mathbf{x}) = P \circ R^+(\mathbf{x}) - \sqrt{1 - \|\mathbf{x}\|^2}\mathbf{L}[A_0, f_1](\mathbf{x}), \quad \mathbf{x} \in \mathbb{B}^N. \tag{4.5}$$

From (4.4), it follows that

$$P_2(\mathbf{a}) = f \circ R^+(\mathbf{a}) + f \circ R^-(\mathbf{a}) = f_2(\mathbf{a}), \quad \mathbf{a} \in A_0. \tag{4.6}$$

On the other hand, for all $\mathbf{b} \in B_0$, by using (4.5), we obtain

$$P_2(\mathbf{b}) = P \circ R^+(\mathbf{b}) + P \circ R^-(\mathbf{b}) = 2f \circ R^+(\mathbf{b}) - \sqrt{1 - \|\mathbf{b}\|^2}\mathbf{L}[A_0, f_1](\mathbf{b}) = f_3(\mathbf{b}). \tag{4.7}$$

We conclude from (4.6) and (4.7) that P_2 interpolates f_2 at A_0 and f_3 at B_0. Hence

$$P_2 = \mathbf{L}[(A_0; f_2), (B_0; f_3)].$$

Consequently,

$$\begin{aligned} P \circ R^+(\mathbf{x}) - P \circ R^-(\mathbf{x}) &= \sqrt{1 - \|\mathbf{x}\|^2}\mathbf{L}[A_0, f_1](\mathbf{x}), \\ P \circ R^+(\mathbf{x}) + P \circ R^-(\mathbf{x}) &= \mathbf{L}[(A_0; f_2), (B_0; f_3)](\mathbf{x}), \end{aligned} \quad \mathbf{x} \in \mathbb{B}^N.$$

Combining the last two relations, we obtain the desired equations in $\mathbb{B}^N$. By continuity, we get the equations in $\overline{\mathbb{B}^N}$.

Theorem is proved.

Theorem 4.2. *Let $\rho \in (0, 1)$. Let E be a hyperplane in $\mathbb{R}^N$ generated by an affine polynomial q. Let $\{B_n\} \subset E \cap \mathbb{B}^N(0, \rho)$ be an $\mathcal{A}$-normal sequence for $\mathcal{P}_d(E)$ corresponding to the sequence of linear maps $\{\mathbf{I}[B_n; \cdot]\}$. Let $\{A_n\} \subset \mathbb{B}^N(0, \rho)$ be a $\mathcal{F}$-regular sequence for $\mathcal{P}_{d-1}(\mathbb{R}^N)$ with $A_n \cap E = \varnothing$, $n \geq 1$. We define*

$$X_n = R^+(A_n) \cup R^-(A_n) \cup R^+(B_n).$$

Assume that $(1 - \|\mathbf{x}\|^2)^{\pm 1/2}, 1/q \in \mathcal{F}$ and $(1 - \|\mathbf{x}\|^2)^{1/2} \in \mathcal{A}$. We set

$$\mathcal{C} = \Big\{ f : \mathbb{S}^N \to \mathbb{R} \big| f \circ R^\pm \in \mathcal{F}, f \circ R^+ \in \mathcal{A} \Big\}.$$

Then, for any function $f \in \mathcal{C}$, the following limit exists:

$$\lim_{n \to \infty} \mathbf{L}_{\mathbb{S}^N}[X_n; f].$$

Moreover, the limit denoted by $\mathbf{H}_{\mathbb{S}^N}$ satisfies the relations

$$\Lambda(\mathbf{H}_{\mathbb{S}^N} \circ R^+) = \Lambda(f \circ R^+), \quad \Lambda(\mathbf{H}_{\mathbb{S}^N} \circ R^-) = \Lambda(f \circ R^-), \quad \Pi(\mathbf{H}_{\mathbb{S}^N} \circ R^+) = \Pi(f \circ R^+),$$

where Λ and Π are defined by

$$\Lambda(g) = \lim_{n \to \infty} \mathbf{L}[A_n; g], \quad g \in \mathcal{F}, \qquad \Pi(h) = \lim_{n \to \infty} \mathbf{I}[B_n; h], \quad h \in \mathcal{A}.$$

Proof. We first find the limit of the sequence of Lagrange interpolation polynomials. By Theorem 4.1, X_n is unisolvent for $\mathcal{P}_d(\mathbb{S}^N)$. For convenience, we set $P_n := \mathbf{L}_{\mathbb{S}^N}[X_n; f]$. Then Theorem 4.1 gives

$$P_n \circ R^{\pm}(\mathbf{x}) = \frac{\pm\sqrt{1-\|\mathbf{x}\|^2}\mathbf{L}[A_n; f_1](\mathbf{x}) + \mathbf{L}[(A_n; f_2), (B_n; f_{3,n})](\mathbf{x})}{2}, \quad \mathbf{x} \in \overline{\mathbb{B}^N}, \tag{4.8}$$

where

$$f_1(\mathbf{x}) = \frac{f \circ R^+(\mathbf{x}) - f \circ R^-(\mathbf{x})}{\sqrt{1-\|\mathbf{x}\|^2}}, \quad f_2(\mathbf{x}) = f \circ R^+(\mathbf{x}) + f \circ R^-(\mathbf{x}), \quad \mathbf{x} \in \mathbb{B}^N,$$

and

$$f_{3,n}(\mathbf{x}) = 2f \circ R^+(\mathbf{x}) - \sqrt{1-\|\mathbf{x}\|^2}\mathbf{L}[A_n; f_1](\mathbf{x}), \quad \mathbf{x} \in \mathbb{B}^N.$$

Analysis similar to that in the proof of Theorem 3.1 shows that

$$\mathbf{L}[(A_n; f_2), (B_n; f_{3,n})] = \mathbf{I}[B_n; f_{3,n}] + q\mathbf{L}\left[A_n; \frac{f_2 - \mathbf{I}[B_n; f_{3,n}]}{q}\right] =$$

$$= \mathbf{I}[B_n; f_{3,n}] + q\left(\mathbf{L}\left[A_n; \frac{f_2}{q}\right] - \mathbf{L}\left[A_n; \frac{\mathbf{I}[B_n; f_{3,n}]}{q}\right]\right). \tag{4.9}$$

We will find the limit of each term in (4.9). By hypothesis we have

$$\lim_{n\to\infty} \mathbf{L}[A_n; f_1] = \Lambda(f_1), \quad \lim_{n\to\infty} \mathbf{L}\left[A_n; \frac{f_2}{q}\right] = \Lambda\left(\frac{f_2}{q}\right). \tag{4.10}$$

Hence Lemma 2.6 shows that

$$\lim_{n\to\infty} \mathbf{I}[B_n; f_{3,n}] = \lim_{n\to\infty}\left(\mathbf{I}[B_n; 2f \circ R^+] - \mathbf{I}\left[B_n; \sqrt{1-\|\mathbf{x}\|^2}\mathbf{L}[A_n; f_1](\mathbf{x})\right]\right) =$$

$$= \Pi(2f \circ R^+) - \Pi\left(\sqrt{1-\|\mathbf{x}\|^2}\Lambda(f_1)(\mathbf{x})\right) =$$

$$= \Pi\left(2f \circ R^+(\mathbf{x}) - \sqrt{1-\|\mathbf{x}\|^2}\Lambda(f_1)(\mathbf{x})\right) =: \Phi(f).$$

By using Lemma 2.2, we obtain

$$\lim_{n\to\infty} \mathbf{L}\left[A_n; \frac{\mathbf{I}[B_n; f_{3,n}]}{q}\right] = \Lambda\left(\frac{\Phi(f)}{q}\right).$$

From (4.9), we see that

$$\lim_{n\to\infty} \mathbf{L}[(A_n; f_2), (B_n; f_{3,n})] = \Phi(f) + q\left(\Lambda\left(\frac{f_2}{q}\right) - \Lambda\left(\frac{\Phi(f)}{q}\right)\right) =: \Psi(f). \tag{4.11}$$

Since $\mathcal{P}_d(\mathbb{R}^N)$ is a finite dimensional vector space, the convergence on $\mathcal{P}_d(\mathbb{R}^N)$ can be understood as the convergence under any norm on $\mathcal{P}_d(\mathbb{R}^N)$. Hence the convergence in (4.10) and (4.11) can be regarded as the uniform convergence on $\overline{\mathbb{B}^N}$, because the relation

$$p \longmapsto \sup_{\mathbf{x}\in\overline{\mathbb{B}^N}} |p(\mathbf{x})|, \quad p \in \mathcal{P}_d(\mathbb{R}^N)$$

define a norm on $\mathcal{P}_d(\mathbb{R}^N)$. It follows from (4.8) that

$$P_n \circ R^{\pm}(\mathbf{x}) \longrightarrow \frac{\pm\sqrt{1-\|\mathbf{x}\|^2}\Lambda(f_1)(\mathbf{x}) + \Psi(f)(\mathbf{x})}{2} \quad \text{uniformly on} \quad \overline{\mathbb{B}^N}. \tag{4.12}$$

Note that $\Lambda(f_1) \in \mathcal{P}_{d-1}(\mathbb{R}^N)$ and $\Psi(f) \in \mathcal{P}_d(\mathbb{R}^N)$. We define

$$\mathbf{H}_{\mathbb{S}^N}(f)(\mathbf{x}, x_{N+1}) = \frac{x_{N+1}\Lambda(f_1)(\mathbf{x}) + \Psi(f)(\mathbf{x})}{2}. \tag{4.13}$$

Evidently, $\mathbf{H}_{\mathbb{S}^N}(f) \in \mathcal{P}_d(\mathbb{R}^{N+1})$ and

$$\mathbf{H}_{\mathbb{S}^N}(f) \circ R^{\pm}(\mathbf{x}) = \frac{\pm\sqrt{1-\|\mathbf{x}\|^2}\Lambda(f_1)(\mathbf{x}) + \Psi(f)(\mathbf{x})}{2}, \quad \mathbf{x} \in \overline{\mathbb{B}^N}. \tag{4.14}$$

Combining (4.12) and (4.14), we deduce that $\{P_n\}$ converges to $\mathbf{H}_{\mathbb{S}^N}$ uniformly on $\mathbb{S}^N$.

It remains to prove the desired properties of $\mathbf{H}_{\mathbb{S}^N}(f)$. From (4.14) we have

$$\frac{\mathbf{H}_{\mathbb{S}^N}(f) \circ R^{+}(\mathbf{x}) - \mathbf{H}_{\mathbb{S}^N}(f) \circ R^{-}(\mathbf{x})}{\sqrt{1-\|\mathbf{x}\|^2}} = \Lambda(f_1)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{B}^N, \tag{4.15}$$

and

$$\mathbf{H}_{\mathbb{S}^N}(f) \circ R^{+}(\mathbf{x}) + \mathbf{H}_{\mathbb{S}^N}(f) \circ R^{-}(\mathbf{x}) = \Psi(f)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{B}^N. \tag{4.16}$$

It follows from (4.15) that

$$\Lambda\left(\mathbf{H}_{\mathbb{S}^N}(f) \circ R^{+}(\mathbf{x}) - \mathbf{H}_{\mathbb{S}^N}(f) \circ R^{-}(\mathbf{x})\right) = \Lambda\left(\sqrt{1-\|\mathbf{x}\|^2}\Lambda(f_1)(\mathbf{x})\right) =$$
$$= \Lambda\left(\sqrt{1-\|\mathbf{x}\|^2} f_1(\mathbf{x})\right) = \Lambda\left(f \circ R^{+}(\mathbf{x}) - f \circ R^{-}(\mathbf{x})\right), \tag{4.17}$$

where we use Lemma 2.1 in the second relation. From (4.16) we see that

$$\Lambda\left(\mathbf{H}_{\mathbb{S}^N}(f) \circ R^{+}(\mathbf{x}) + \mathbf{H}_{\mathbb{S}^N}(f) \circ R^{-}(\mathbf{x})\right) = \Lambda\left(\Psi(f)(\mathbf{x})\right).$$

The next goal is to determine $\Lambda\left(\Psi(f)\right)$. Using Lemma 2.1 again, we conclude from the definition of $\Psi(f)$ that

$$\Lambda\left(\Psi(f)\right) = \Lambda\left(\Phi(f)\right) + \Lambda\left(q\Lambda\left(\frac{f_2}{q}\right)\right) - \Lambda\left(q\Lambda\left(\frac{\Phi(f)}{q}\right)\right) =$$
$$= \Lambda\left(\Phi(f)\right) + \Lambda\left(q\frac{f_2}{q}\right) - \Lambda\left(q\frac{\Phi(f)}{q}\right) = \Lambda(f_2).$$

Hence

$$\Lambda\left(\mathbf{H}_{\mathbb{S}^N}(f) \circ R^{+}(\mathbf{x}) + \mathbf{H}_{\mathbb{S}^N}(f) \circ R^{-}(\mathbf{x})\right) = \Lambda\left(f \circ R^{+}(\mathbf{x}) + f \circ R^{-}(\mathbf{x})\right). \tag{4.18}$$

Combining (4.17) and (4.18), we obtain

$$\Lambda(\mathbf{H}_{\mathbb{S}^N} \circ R^+) = \Lambda(f \circ R^+), \quad \Lambda(\mathbf{H}_{\mathbb{S}^N} \circ R^-) = \Lambda(f \circ R^-).$$

From (4.16) we conclude that

$$\Pi\left(\mathbf{H}_{\mathbb{S}^N}(f) \circ R^+(\mathbf{x}) + \mathbf{H}_{\mathbb{S}^N}(f) \circ R^-(\mathbf{x})\right) = \Pi\left(\Psi(f)(\mathbf{x})\right).$$

On the other hand,

$$\Pi\left(\Psi(f)\right) = \Pi\left(\Phi(f)\right) + \Pi\left(q\Lambda\left(\frac{f_2}{q}\right)\right) - \Pi\left(q\Lambda\left(\frac{\Phi(f)}{q}\right)\right) = \Pi\left(\Phi(f)\right) =$$

$$= \Pi\left(\Pi\left(2f \circ R^+(\mathbf{x}) - \sqrt{1 - \|\mathbf{x}\|^2}\Lambda(f_1)(\mathbf{x})\right)\right) =$$

$$= \Pi\left(2f \circ R^+(\mathbf{x}) - \sqrt{1 - \|\mathbf{x}\|^2}\Lambda(f_1)(\mathbf{x})\right) =$$

$$= \Pi\left(2f \circ R^+(\mathbf{x}) - \mathbf{H}_{\mathbb{S}^N}(f) \circ R^+(\mathbf{x}) + \mathbf{H}_{\mathbb{S}^N}(f) \circ R^-(\mathbf{x})\right),$$

where we use Lemma 2.5 in the second and forth equations. It follows that

$$\Pi\left(\mathbf{H}_{\mathbb{S}^N}(f) \circ R^+(\mathbf{x}) + \mathbf{H}_{\mathbb{S}^N}(f) \circ R^-(\mathbf{x})\right) =$$

$$= \Pi\left(2f \circ R^+(\mathbf{x}) - \mathbf{H}_{\mathbb{S}^N}(f) \circ R^+(\mathbf{x}) + \mathbf{H}_{\mathbb{S}^N}(f) \circ R^-(\mathbf{x})\right).$$

The last relation finally gives

$$\Pi(\mathbf{H}_{\mathbb{S}^N} \circ R^+) = \Pi(f \circ R^+).$$

Theorem is proved.

***Corollary* 4.1.** *Under the assumptions of Theorem* 4.2, *we have*

$$\mathbf{H}_{\mathbb{S}^N}(f)(\mathbf{x}, x_{N+1}) = \frac{x_{N+1}\Lambda(f_1)(\mathbf{x}) + \Phi(f)(\mathbf{x})}{2},$$

where

$$\Psi(f) = \Phi(f) + q\left(\Lambda\left(\frac{f_2}{q}\right) - \Lambda\left(\frac{\Phi(f)}{q}\right)\right)$$

with

$$\Phi(f) = \Pi\left(2f \circ R^+(\mathbf{x}) - \sqrt{1 - \|\mathbf{x}\|^2}\Lambda(f_1)(\mathbf{x})\right).$$

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