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GENERALIZED VECTOR-VALUED PARANORMED SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

УЗАГАЛЬНЕНІ ВЕКТОРНОЗНАЧНІ ПАРАНОРМОВНІ ПРОСТОРОВІ ПОСЛІДОВНОСТІ, ЩО ВИЗНАЧАЮТЬСЯ ПОСЛІДОВНІСТЮ ФУНКЦІЙ ОРЛІЧА

We introduced a class of generalized vector-valued paranormed sequence space $X[E, A, \Delta_v^m, M, p]$ by using a sequence of Orlicz functions $M = (M_k)$, a non-negative infinite matrix $A = [a_{nk}]$, generalized difference operator Δ_v^m and bounded sequence of positive real numbers p_k with $\inf_k p_k > 0$. Properties related to this space are studied under certain conditions. Some inclusion relations are obtained and a result related to subspace with Orlicz functions satisfying Δ_2-condition has also been proved.

Введено клас узагальнених векторнозначних паранормовних послідовностей простору $X[E, A, \Delta_v^m, M, p]$ на базі послідовності функцій Орліча $M = (M_k)$, невід'ємної нескінченної матриці $A = [a_{nk}]$, узагальненого різницевого оператора Δ_v^m та обмеженої послідовності додатних дійсних чисел p_k з $\inf_k p_k > 0$. Властивості, пов'язані з цим простором, вивчаються за наявності деяких умов. Отримано деякі співвідношення включення та доведено результати, які відносяться до підпростору з функціями Орліча, що задовольняють Δ_2-умову.

1. Introduction. The theory of sequence spaces has been one of the most active area of research in functional analysis. Generalization of $\ell_p,\ p \geq 1,\ c_0$ and c has been studied by many authors with the help of difference operator, modulus function and Orlicz functions in the last five decades.

Kizmaz [9] introduced the notion of difference operator Δ and studied difference sequence spaces $\ell_\infty(\Delta),\ c(\Delta)$ and $c_0(\Delta)$. Et and Çolak [4] generalized the operator by introducing the spaces $\ell_\infty(\Delta^m),\ c(\Delta^m)$ and $c_0(\Delta^m)$ for non-negative integer m. Further, Et and Esi [5] generalized these spaces by taking the sequence $v = (v_k)$ of non-zero complex numbers which are defined as follows:

$$X(\Delta_v^m) = \big\{x = (x_k) \in w : \Delta_v^m x \in X\big\} \quad \text{for} \quad X = \ell_\infty, c \text{ and } c_0,$$

where w is the space of all complex sequences, $\Delta_v^0 x = (v_k x_k)$ and

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i} \quad \text{for} \quad m \in \mathbb{N}.$$

In 1971, Lindenstrauss and Tzafriri [11] used the idea of an Orlicz function M to construct the sequence space l_M as follows:

$$l_M = \left\{x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \quad \text{for some} \quad \rho > 0\right\}.$$

They proved that l_M is Banach space under the following norm:

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}.$$

In 1994, Parashar and Choudhary [16] generalized the space l_M to $l_M(p)$ by using bounded sequence of real numbers (p_k) as follows:

$$l_M(p) = \left\{ x \in \omega : \sum_{k=1}^{\infty} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} < \infty \quad \text{for some} \quad \rho > 0 \right\},$$

$$W_0(M,p) = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{x_k}{\rho}\right)\right]^{p_k} = 0 \quad \text{for some} \quad \rho > 0 \right\}$$

and

$$W_\infty(M,p) = \left\{ x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{x_k}{\rho}\right)\right]^{p_k} < \infty \quad \text{for some} \quad \rho > 0 \right\}.$$

For $M(x) = x$, above sequence spaces become $\ell(p)$, $[C,1,p]_0$ and $[C,1,p]_\infty$, respectively, studied by Maddox [13].

Mursaleen et al. [15] introduced sequence spaces $c_0(M,\Delta,p)$ and $\ell_\infty(M,\Delta,p)$ as follows:

$$c_0(M,\Delta,p) = \left\{ x \in \omega : \lim_{k\to\infty} \left[M\left(\frac{|\Delta x_k|}{\rho}\right)\right]^{p_k} = 0 \quad \text{for some} \quad \rho > 0 \right\},$$

$$\ell_\infty(M,\Delta,p) = \left\{ x \in \omega : \sup_k \left[M\left(\frac{|\Delta x_k|}{\rho}\right)\right]^{p_k} < \infty \quad \text{for some} \quad \rho > 0 \right\}.$$

In 2005, Tripathy and Sarma [22] introduced spaces $c_0(M,\Delta,p,q)$ and $\ell_\infty(M,\Delta,p,q)$ in semi-normed space (E,q) as follows:

$$c_0(M,\Delta,p,q) = \left\{ x \in \omega(E) : \lim_{k\to\infty} \left(\frac{1}{p_k}\right) \left[M\left(\frac{q(\Delta x_k)}{\rho}\right)\right]^{p_k} = 0 \quad \text{for some} \quad \rho > 0 \right\},$$

$$\ell_\infty(M,\Delta,p,q) = \left\{ x \in \omega(E) : \sup_k \left(\frac{1}{p_k}\right) \left[M\left(\frac{q(\Delta x_k)}{\rho}\right)\right]^{p_k} < \infty \quad \text{for some} \quad \rho > 0 \right\}.$$

By using sequence of Orlicz functions, Bektaş [3] constructed space $l_M(\Delta_v^m, p, q, s)$ as follows:

$$l_M(\Delta_v^m,p,q,s) = \left\{ x \in \omega(E) : \sum_{k=1}^{\infty} k^{-s} \left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k} < \infty \quad \text{for some} \quad \rho > 0, s \geq 0 \right\}.$$

If $M_k = M$ and $v_k = 1$ for all k, then above space becomes $l_M(\Delta^m,p,q,s)$ which is discussed by Tripathy et al. [17]. Further, for $m = 0$, the space $l_M(\Delta^m,p,q,s)$ reduces to $l_M(p,q,s)$, which is studied by Bektaş and Altin [2].

Esi [7] used non-negative regular matrix to introduce spaces $W_0(A,M,p)$ and $W_\infty(A,M,p)$ as follows:

$$W_0(A,M,p) = \left\{ x \in \omega : \lim_{n\to\infty} \sum_k a_{nk} \left[M\left(\frac{x_k}{\rho}\right)\right]^{p_k} = 0 \quad \text{for some} \quad \rho > 0 \right\},$$

$$W_\infty(A,M,p) = \left\{ x \in \omega : \sup_n \sum_k a_{nk} \left[M\left(\frac{x_k}{\rho}\right)\right]^{p_k} < \infty \quad \text{for some} \quad \rho > 0 \right\}.$$

If $M(x) = x$, then above spaces reduce to $[A,p]_0$ and $[A,p]_\infty$, studied by Maddox [14]. Orlicz function and generalized difference operator were frequently used to introduce scalar and vector-valued sequence spaces by the researchers in [1, 6, 18 – 21] and many others.

Above development motivated us to introduce a class of vector-valued sequence spaces $X[E, A, \Delta_v^m, M, p]$ by using non-negative matrix $A = [a_{nk}]$, generalized difference operator Δ_v^m and a sequence of Orlicz functions $M = (M_k)$ which generalizes many known scalar and vector-valued sequence spaces.

1.1. A new sequence space $X[E, A, \Delta_v^m, M, p]$. Let $M = (M_k)$ be a sequence of Orlicz functions, $v = (v_k)$ be any fixed sequence of non-zero complex numbers, $A = [a_{nk}]$ be a non-negative infinite matrix, i.e., $a_{nk} \geq 0$ for all $n, k \in \mathbb{N}$ and (p_k) be a bounded sequence of positive real numbers such that $\inf_k p_k > 0$. Further, let (E, q) be a seminormed space and X be a normal (or solid) sequence space. We define

$$X\big[E, A, \Delta_v^m, M, p\big] =$$

$$= \left\{ x = (x_k) \in W(E) \colon \left(\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^m x_k)}{\rho} \right) \right]^{p_k} \right) \in X \quad \text{for some} \quad \rho > 0 \right\},$$

where $\Delta_v^0 x_k = v_k x_k$ and

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i} \quad \text{for} \quad m \in \mathbb{N}.$$

Class of vector-valued sequences $\big[X[E, A, \Delta_v^m, M, p]\big]$ is also defined by

$$\big[X[E, A, \Delta_v^m, M, p]\big] =$$

$$= \left\{ x = (x_k) \in W(E) \colon \left(\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^m x_k)}{\rho} \right) \right]^{p_k} \right) \in X \quad \text{for every} \quad \rho > 0 \right\}.$$

Clearly, $\big[X[E, A, \Delta_v^m, M, p]\big]$ is a subspace of $X[E, A, \Delta_v^m, M, p]$.

1.2. Particular cases:

(i) If we choose $X = \ell_\infty, E = \mathbb{C}, m = 0$, $a_{nk} = 1$ for $n \geq k$ and 0 otherwise, $p_k = 1$ for all k, $M_k = M$ for all k and $v_k = 1$ for all k, the space $X[E, A, \Delta_v^m, M, p]$ reduces to l_M [11].

(ii) If we choose $X = c_0$ and ℓ_∞, $E = \mathbb{C}, m = 0$, $a_{nk} = \dfrac{1}{k}$ for $n \geq k$ and 0 otherwise, $M_k = M$ for all k and $v_k = 1$ for all k, the space $X[E, A, \Delta_v^m, M, p]$ becomes $W_0(M, p)$ and $W_\infty(M, p)$, respectively [16].

(iii) If we choose $X = c_0$ and ℓ_∞, $E = \mathbb{C}$, $m = 0$, $M_k = M$ for all k and $v_k = 1$ for all k, the space $X[E, A, \Delta_v^m, M, p]$ reduces to $W_0(A, M, p)$ and $W_\infty(A, M, p)$, respectively [7].

(iv) If we choose $X = c_0$ and ℓ_∞, $E = \mathbb{C}, m = 1$, $A = [a_{nk}]$ such that $a_{nk} = 1$ for $n = k$ and 0 otherwise, $M_k = M$ for all k and $v_k = 1$ for all k, the space reduces to $c_0(M, \Delta, p)$ and $\ell_\infty(M, \Delta, p)$, respectively [15].

(v) If we choose $X = c_0$ and ℓ_∞, $m = 1$, $a_{nk} = \dfrac{1}{p_k}$ for $n = k$ and 0 otherwise, $M_k = M$ for all k and $v_k = 1$ for all k, the space $X[E, A, \Delta_v^m, M, p]$ reduces to $c_0(M, \Delta, p, q)$ and $\ell_\infty(M, \Delta, p, q)$, respectively [22].

(vi) If we choose $X = \ell_\infty$, $m = 0$, $a_{nk} = k^{-s}$ for all n, $M_k = M$ for all k and $v_k = 1$ for all k, the space $X[E, A, \Delta_v^m, M, p]$ reduces to $l_M(p, q, s)$ [2].

(vii) If we choose $X = \ell_\infty$ and $a_{nk} = k^{-s}$ for all n, the space $X[E, A, \Delta_v^m, M, p]$ reduces to $l_M(\Delta_v^m, p, q, s)$ [3].

2. Some definitions and known results.

***Result* 1** [12]**.** For a_k and b_k in $\mathbb{C}$, the following inequalities hold:

$$|a_k + b_k|^{p_k} \le T\{|a_k|^{p_k} + |b_k|^{p_k}\}, \tag{2.1}$$

$$|\lambda|^{p_k} \le \max\left(1, |\lambda|^H\right), \tag{2.2}$$

where (p_k) is a bounded sequence of real numbers with $0 < p_k \le \sup_k p_k = H$, $T = \max(1, 2^{H-1})$ and λ in $\mathbb{C}$.

Definition 1 [8]**.** *A sequence space X is called normal* (*or solid*) *space if*

$$x = (x_k) \in X \quad \textit{and} \quad |\lambda_k| \le 1 \quad \textit{for each} \quad k \in \mathbb{N} \Rightarrow \lambda x = (\lambda_k x_k) \in X,$$

where $\lambda = (\lambda_k)$ is a scalar sequence of real or complex numbers.

Definition 2 [11]**.** *An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.*

***Remark* 1** [10]**.** An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \le KM(u)$, $u \ge 0$.

The Δ_2-condition is equivalent to the inequality $M(Lu) \le KLM(u)$ which holds for all values of u and for $L > 1$.

3. Results on sequence space $X[E, A, \Delta_v^m, M, p]$.

Theorem 1. *$X[E, A, \Delta_v^m, M, p]$ is a linear space over $\mathbb{C}$.*

Proof. Let $x, y \in X[E, A, \Delta_v^m, M, p]$ and $\alpha, \beta \in \mathbb{C}$. Then there exist some positive numbers ρ_1 and ρ_2 such that

$$\left(\sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho_1}\right)\right]^{p_k}\right) \in X \quad \text{and} \quad \left(\sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q(\Delta_v^m y_k)}{\rho_2}\right)\right]^{p_k}\right) \in X.$$

Let $\rho_3 = \max\left(2|\alpha|\rho_1, 2|\beta|\rho_2\right)$. By using the subadditive property of seminorm q, non-decreasing and convexity of Orlicz functions, for each n, we have

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q(\Delta_v^m(\alpha x_k + \beta y_k))}{\rho_3}\right)\right]^{p_k} \le T\sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho_1}\right)\right]^{p_k} +$$

$$+T\sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q(\Delta_v^m y_k)}{\rho_2}\right)\right]^{p_k}, \quad \text{by using} \quad (2.1).$$

Since X is a normal space, so $\left(\sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\dfrac{q(\Delta_v^m(\alpha x_k + \beta y_k))}{\rho_3}\right)\right]^{p_k}\right) \in X$. Thus, $X[E, A, \Delta_v^m, M, p]$ is a linear space.

Theorem 2. *The sequence space $X[E, A, \Delta_v^m, M, p]$ is a paranormed space under paranorm g defined by*

$$g(x) = \sum_{k=1}^{m} q(x_k) + \inf\left\{\rho^{\frac{p_n}{H}} : \sup_n \left[\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k}\right]^{\frac{1}{H}} \le 1, n \in \mathbb{N}\right\},$$

where $H = \max(1, \sup_k p_k)$.

Proof. As $q(\theta) = 0$ and $M_k(0) = 0$ for all $k \in \mathbb{N}$, so $\inf\{\rho^{\frac{p_n}{H}}\} = 0$ which implies that $g(\theta) = 0$ for $x = \theta$. Clearly, $g(x) \ge 0$ and $g(-x) = g(x)$ for any $x \in X[E, A, \Delta_v^m, M, p]$. To show that $g(x+y) \le g(x) + g(y)$, let $x, y \in X[E, A, \Delta_v^m, M, p]$. Then there exist $\rho_1 > 0, \rho_2 > 0$ such that

$$\sup_n \left[\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho_1}\right)\right]^{p_k}\right]^{\frac{1}{H}} \le 1 \quad \text{and} \quad \sup_n \left[\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho_2}\right)\right]^{p_k}\right]^{\frac{1}{H}} \le 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by using convexity of Orlicz function and Minkowski's inequality, we have

$$\sup_n \left[\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m (x_k + y_k))}{\rho}\right)\right]^{p_k}\right]^{\frac{1}{H}} \le \frac{\rho_1}{\rho} \sup_n \left[\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho_1}\right)\right]^{p_k}\right]^{\frac{1}{H}} +$$

$$+\frac{\rho_2}{\rho} \sup_n \left[\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho_2}\right)\right]^{p_k}\right]^{\frac{1}{H}} \le 1.$$

Now,

$$g(x+y) =$$

$$= \sum_{k=1}^{m} q(x_k + y_k) + \inf\left\{\rho^{\frac{p_n}{H}} : \sup_n \left[\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q\big(\Delta_v^m (x_k + y_k)\big)}{\rho}\right)\right]^{p_k}\right]^{\frac{1}{H}} \le 1, n \in \mathbb{N}\right\} \le$$

$$\le \sum_{k=1}^{m} q(x_k) + \sum_{k=1}^{m} q(y_k) + \inf\Bigg\{(\rho_1 + \rho_2)^{\frac{p_n}{H}} : \sup_n \left[\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho_1}\right)\right]^{p_k}\right]^{\frac{1}{H}} \le 1,$$

$$\sup_n \left[\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m y_k)}{\rho_2}\right)\right]^{p_k}\right]^{\frac{1}{H}} \le 1\Bigg\} \le$$

$$\le g(x) + g(y).$$

To prove continuity of scalar multiplication, let λ is fixed number in $\mathbb{C}$. Then

$$g(\lambda x) = \sum_{k=1}^{m} q(\lambda x_k) + \inf\left\{\rho^{\frac{p_n}{H}} : \sup_n \left[\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q\big(\Delta_v^m (\lambda x_k)\big)}{\rho}\right)\right]^{p_k}\right]^{\frac{1}{H}} \le 1, n \in \mathbb{N}\right\} =$$

$$= |\lambda| \sum_{k=1}^{m} q(x_k) + \inf \left\{ (r|\lambda|)^{\frac{p_n}{H}} : \sup_n \left[\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^m x_k)}{r} \right) \right]^{p_k} \right]^{\frac{1}{H}} \leq 1, n \in \mathbb{N} \right\} \leq$$

$$\leq \max(1, |\lambda|) g(x),$$

where $r = \dfrac{\rho}{|\lambda|}$. Thus $g(\lambda x) \to 0$ as $x \to 0$.

Now, we will prove that $g(\lambda_i x) \to 0$ as $\lambda_i \to 0$ for a fixed x. As $\lambda_i \to 0$, there exists a positive integer m_0 such that $|\lambda_i| < 1$ for all $i \geq m_0$. By non-decreasing property of Orlicz function, for all $i \geq m_0$, we have

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\big(\Delta_v^m (\lambda_i x_k)\big)}{\rho} \right) \right]^{p_k} \leq \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^m x_k)}{\rho} \right) \right]^{p_k} < \infty,$$

which implies that for every $\varepsilon > 0$, there exists a positive integer k_0 such that

$$\sum_{k=k_0}^{\infty} a_{nk} \left[M_k \left(\frac{q\big(\Delta_v^m (\lambda_i x_k)\big)}{\rho} \right) \right]^{p_k} < \frac{\varepsilon}{2}. \tag{3.1}$$

Now, we define a function f by

$$f(t) = \sum_{k=1}^{k_0} a_{nk} \left[M_k \left(\frac{q\big(\Delta_v^m (t x_k)\big)}{\rho} \right) \right]^{p_k}.$$

Clearly, $f(t)$ is continuous at 0 and $f(0) = 0$. This implies that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\big|f(t)\big| < \dfrac{\varepsilon}{2}$ whenever $|t| < \delta$. Since $\lambda_i \to 0$, so there exists positive integer m_1 such that $|\lambda_i| < \delta$ for all $i \geq m_1$. Which gives us $\big|f(\lambda_i)\big| < \dfrac{\varepsilon}{2}$ for $i \geq m_1$, i.e.,

$$\sum_{k=1}^{k_0} a_{nk} \left[M_k \left(\frac{q\big(\Delta_v^m (\lambda_i x_k)\big)}{\rho} \right) \right]^{p_k} < \frac{\varepsilon}{2}. \tag{3.2}$$

By inequalities (3.1) and (3.2), for $i \geq m_1$, we have

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\big(\Delta_v^m (\lambda_i x_k)\big)}{\rho} \right) \right]^{p_k} < \varepsilon.$$

Using above inequality, we can obtain $g(\lambda_i x) \to 0$ as $\lambda_i \to 0$.

Theorem 2 is proved.

***Remark* 2.** Sequence space $X[E, A, \Delta_v^m, M, p]$ is not a total paranormed space because $g(x) = 0$ need not imply $x = \theta$ due to seminorm q.

Theorem 3. *Let $M = (M_k)$ and $T = (T_k)$ be any two sequences of Orlicz functions. If each T_k satisfies Δ_2-condition, then $X[E, A, \Delta_v^m, M, p] \subseteq X[E, A, \Delta_v^m, T \circ M, p]$, where $T \circ M = (T_k \circ M_k)$.*

Proof. Let $x \in X[E, A, \Delta_v^m, M, p]$, i.e., $\left(\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k}\right) \in X$.

Case (i) : If $M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right) \leq 1$, then by convexity of Orlicz functions, for each $n \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} a_{nk}\left[(T_k \circ M_k)\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k} \leq \sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right) T_k(1)\right]^{p_k} \leq$$

$$\leq \max\left(1, [T(1)]^H\right) \sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k},$$

where $T(1) = \sup_k T_k(1)$.

Case (ii): If $M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right) > 1$. Then by Δ_2-condition of Orlicz function, for each $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{\infty} a_{nk}\left[(T_k \circ M_k)\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k} \leq \sum_{k=1}^{\infty} a_{nk}\left[K M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right) T_k(1)\right]^{p_k} \leq$$

$$\leq \max(1, [KT(1)]^H) \sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k}, \quad \text{where} \quad K > 0.$$

As X is a normal space, so $x \in X[E, A, \Delta_v^m, T \circ M, p]$ in both cases. Hence, required inclusion follows.

Theorem 4. *Let $M = (M_k)$, $T = (T_k)$ be any two sequences of Orlicz functions. Then*

(i) $X[E, A, \Delta_v^m, M, p] \cap X[E, A, \Delta_v^m, T, p] \subseteq X[E, A, \Delta_v^m, M + T, p]$

and

(ii) $X[E, A, \Delta_v^m, T, p] \subseteq X[E, A, \Delta_v^m, M, p]$, *if* $\sup_u \left[\frac{M_k(u)}{T_k(u)}\right] < \infty$ *for each* $k \in \mathbb{N}$.

Proof. (i) Let $x \in X[E, A, \Delta_v^m, M, p] \cap X[E, A, \Delta_v^m, T, p]$. By using inequality (2.1), for each n, we obtain

$$\sum_{k=1}^{\infty} a_{nk}\left[(M_k + T_k)\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k} \leq$$

$$\leq T \sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k} + T \sum_{k=1}^{\infty} a_{nk}\left[T_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k}.$$

Since X is a normal space, so $x \in X[E, A, \Delta_v^m, M + T, p]$. Thus, we get the required result.

(ii) Let $x \in X[E, A, \Delta_v^m, T, p]$. Then $\left(\sum_{k=1}^{\infty} a_{nk}\left[T_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k}\right) \in X$. Since $\sup_u \left[\frac{M_k(u)}{T_k(u)}\right] < \infty$ for each $k \in \mathbb{N}$, so there exists $\eta > 0$ such that $M_k(u) \leq \eta T_k(u)$ for each $k \in \mathbb{N}$ and for all $u > 0$. Now,

$$\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k} \leq \sum_{k=1}^{\infty} a_{nk}\left[\eta T_k\left(\frac{q(\Delta_v^m x_k)}{\rho}\right)\right]^{p_k} \leq$$

$$\leq \max(1, \eta^H) \sum_{k=1}^{\infty} a_{nk} \left[T_k \left(\frac{q(\Delta_v^m x_k)}{\rho} \right) \right]^{p_k}, \quad \text{by using} \quad (2.2).$$

Since X is a normal space, so $x \in X[E, A, \Delta_v^m, M, p]$ and thus inclusion follows.

Theorem 5. *Let* X_1 *and* X_2 *be two normal sequence spaces with* $X_1 \subseteq X_2$. *Then* $X_1[E, A, \Delta_v^m, M, p] \subseteq X_2[E, A, \Delta_v^m, M, p]$.

Proof. Inclusion follows by the definition of $X[E, A, \Delta_v^m, M, p]$.

Theorem 6. *Let* $A = [a_{nk}]$ *be non-negative infinite matrix such that* $a_{nk} \leq a_{n(k+1)}$ *for all* $n, k \in \mathbb{N}$ *and* $m \geq 1$. *Suppose* (M_k) *is non-decreasing sequence of Orlicz functions, i.e.,* $M_k(x) \leq$ $\leq M_{k+1}(x)$ *for all* $x \geq 0$. *Then*

$$X[E, A, \Delta_v^l, M, p] \subset X[E, A, \Delta_v^{l+1}, M, p] \quad \textit{for any} \quad l \in \{1, 2, \ldots, m-1\}.$$

Proof. Let $x \in X[E, A, \Delta_v^l, M, p]$. Then

$$\left(\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^l x_k)}{\rho} \right) \right]^{p_k} \right) \in X \quad \text{for some} \quad \rho > 0.$$

Since seminorm q is subadditive and each M_k is non-decreasing convex function, so we have

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^{l+1} x_k)}{2\rho} \right) \right]^{p_k} = \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^l x_k - \Delta_v^l x_{k+1})}{2\rho} \right) \right]^{p_k} \leq$$

$$\leq T \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^l x_k)}{\rho} \right) \right]^{p_k} + T \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^l x_{k+1})}{\rho} \right) \right]^{p_k} \leq$$

(by using inequality $\quad (2.1)$)

$$\leq T \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^l x_k)}{\rho} \right) \right]^{p_k} + T \sum_{k=1}^{\infty} a_{n(k+1)} \left[M_{k+1} \left(\frac{q(\Delta_v^l x_{k+1})}{\rho} \right) \right]^{p_k}.$$

As X is a normal space, so $\left(\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^{l+1} x_k)}{\rho} \right) \right]^{p_k} \right) \in X$, i.e., $x \in X[E, A, \Delta_v^{l+1}, M, p]$. Consequently, $X[E, A, \Delta_v^l, M, p] \subseteq X[E, A, \Delta_v^{l+1}, M, p]$.

Now, for strictness of inclusion, let us consider the following example.

Let $E = \mathbb{C}, A = [a_{nk}]$ such that $a_{nk} = 1$ for $n = k$, and 0 otherwise, $p_k = 1$ for all k, $M_k(x) = x$ for all k, $v_k = \dfrac{1}{k}$ for any k and $x_k = k^{l+1}$ for any k. Then $\Delta_v^{l+1} x_k = (0, 0, \ldots)$, which means $x \in c_0[E, A, \Delta_v^{l+1}, M, p]$. But $\Delta_v^l x_k = (-1)^l l!$, which implies that $x \notin c_0[E, A, \Delta_v^l, M, p]$.

Theorem 7. *The sequence space* $X[E, A, \Delta_v^m, M, p]$ *is a normal space if* $m = 0$.

Proof. Let $x \in X[E, A, \Delta_v^0, M, p]$, i.e., $\left(\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^0 x_k)}{\rho} \right) \right]^{p_k} \right) \in X$. Again, let (λ_k) be a sequence of scalars such that $|\lambda_k| \leq 1$ for all $k \in \mathbb{N}$. Then by non-decreasing property of Orlicz function, we have

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^0 (\lambda_k x_k))}{\rho} \right) \right]^{p_k} \leq \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(\Delta_v^0 x_k)}{\rho} \right) \right]^{p_k} \quad \text{for all} \quad n \in \mathbb{N}.$$

As X is a normal space, so $\left(\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^0(\lambda_k x_k))}{\rho}\right)\right]^{p_k}\right) \in X$ and result follows.

Theorem 8. *Let X_1 and X_2 be two normal sequence spaces with $X_1 \subseteq X_2$. Then $X_1[E, A, \Delta_v^m, M, p] \subseteq X_2[E, A, \Delta_v^m, M, p]$.*

Proof. Inclusion follows by the definition of $X[E, A, \Delta_v^m, M, p]$.

Theorem 9. *Let $A = [a_{nk}]$ be non-negative infinite matrix such that $a_{nk} \leq a_{n(k+1)}$ for all $n, k \in \mathbb{N}$ and $m \geq 1$. Suppose (M_k) is non-decreasing sequence of Orlicz functions, i.e., $M_k(x) \leq$ $\leq M_{k+1}(x)$ for all $x \geq 0$. Then*

$$X[E, A, \Delta_v^l, M, p] \subset X[E, A, \Delta_v^{l+1}, M, p] \quad \textit{for any} \quad l \in \{1, 2, \ldots, m-1\}.$$

Proof. Let $x \in X[E, A, \Delta_v^l, M, p]$. Then

$$\left(\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^l x_k)}{\rho}\right)\right]^{p_k}\right) \in X \quad \text{for some} \quad \rho > 0.$$

Since seminorm q is subadditive and each M_k is non-decreasing convex function, so we have

$$\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^{l+1} x_k)}{2\rho}\right)\right]^{p_k} = \sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^l x_k - \Delta_v^l x_{k+1})}{2\rho}\right)\right]^{p_k} \leq$$

$$\leq T\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^l x_k)}{\rho}\right)\right]^{p_k} + T\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^l x_{k+1})}{\rho}\right)\right]^{p_k} \Rightarrow$$

$$\text{(by using inequality} \quad (2.1))$$

$$\Rightarrow \sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^{l+1} x_k)}{2\rho}\right)\right]^{p_k} \leq T\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^l x_k)}{\rho}\right)\right]^{p_k} +$$

$$+T\sum_{k=1}^{\infty} a_{n(k+1)}\left[M_{k+1}\left(\frac{q(\Delta_v^l x_{k+1})}{\rho}\right)\right]^{p_k}.$$

As X is a normal space, so $\left(\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^{l+1} x_k)}{\rho}\right)\right]^{p_k}\right) \in X$, i.e., $x \in X[E, A, \Delta_v^{l+1}, M, p]$. Consequently, $X[E, A, \Delta_v^l, M, p] \subseteq X[E, A, \Delta_v^{l+1}, M, p]$.

Now, for strictness of inclusion, let us consider the following example.

Let $E = \mathbb{C}, A = [a_{nk}]$ such that $a_{nk} = 1$ for $n = k$, and 0 otherwise, $p_k = 1$ for all k, $M_k(x) = x$ for all k, $v_k = \dfrac{1}{k}$ for any k and $x_k = k^{l+1}$ for any k. Then $\Delta_v^{l+1} x_k = (0, 0, \ldots)$, which means $x \in c_0[E, A, \Delta_v^{l+1}, M, p]$. But $\Delta_v^l x_k = (-1)^l l!$, which implies that $x \notin c_0[E, A, \Delta_v^l, M, p]$.

Theorem 10. *The sequence space $X[E, A, \Delta_v^m, M, p]$ is a normal space if $m = 0$.*

Proof. Let $x \in X[E, A, \Delta_v^0, M, p]$, i.e., $\left(\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^0 x_k)}{\rho}\right)\right]^{p_k}\right) \in X$. Again, let (λ_k) be a sequence of scalars such that $|\lambda_k| \leq 1$ for all $k \in \mathbb{N}$. Then by non-decreasing property of Orlicz function, we have

$$\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^0(\lambda_k x_k))}{\rho}\right)\right]^{p_k} \leq \sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q(\Delta_v^0 x_k)}{\rho}\right)\right]^{p_k} \quad \text{for all} \quad n \in \mathbb{N}.$$

As X is a normal space, so $\left(\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\dfrac{q(\Delta_v^0(\lambda_k x_k))}{\rho}\right)\right]^{p_k}\right) \in X$ and result follows.

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