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SEVERAL JENSEN – GRÜSS INEQUALITIES WITH APPLICATIONS IN INFORMATION THEORY

КІЛЬКА НЕРІВНОСТЕЙ ЙЄНСЕНА – ГРЮССА ТА ЇХ ЗАСТОСУВАННЯ В ТЕОРІЇ ІНФОРМАЦІЇ

Several integral Jensen – Grüss inequalities are proved together with their refinements. Some new bounds for integral Jensen – Chebyshev inequality are obtained. The multidimensional integral variants are also presented. In addition, some integral Jensen – Grüss inequalities for monotone and completely monotone functions are established. Finally, as an application, we present the refinements for Shannon's entropy.

Доведено кілька інтегральних нерівностей Йєнсена – Грюсса та їх уточнення. Отримано деякі нові оцінки для інтегральної нерівності Йєнсена – Чебишова. Також наведено багатовимірні інтегральні варіанти. Крім того, встановлено деякі інтегральні нерівності Йєнсена – Грюсса для монотонних і цілком монотонних функцій. Насамкінець в якості додатка наведено уточнення, отримані для ентропії Шеннона.

1. Introduction. Jensen inequality is the most notable inequality and many other inequalities can be deduced from it as its consequences. This inequality has huge impact in solving many optimization problems, e.g., information theory, probability theory, applied statistics, control theory and computer sciences. Taking into consideration the tremendous applications of Jensen's inequality in various fields of mathematics and other applied sciences, the generalizations and improvements of Jensen's inequality has been a topic of supreme interest for the researchers during the last few decades as evident from a large number of publications on the topic see [4, 7, 10, 11, 19, 21].

Theorem A (classical Jensen's inequality, see [18])**.** *Let h be an integrable function on a probability space $(\Omega,\mathcal{A},\mu)$ taking values in an interval $l\subset\mathbb{R}$. Then $\int_\Omega hd\mu$ lies in I. If φ is a convex function on I such that $\varphi\circ h$ is integrable, then*

$$\varphi\left(\int\limits_\Omega hd\mu\right)\le\int\limits_\Omega\varphi\circ hd\mu.$$

There are two other important inequalities in mathematical analysis namely Chebyshev inequality [18, p. 197] or [12, p. 240] and *Grüss inequality* [8]. To start with, we let $\varphi,h\in L[u,v]$ and $\rho\colon [u,v]\to\mathbb{R}^+$ be Lebesgue integrable functions. Then we consider the following weighted Chebyshev functional:

$$\mathfrak{C}(\varphi,h;\rho)=\frac{1}{P}\int\limits_u^v\rho(\zeta)\varphi(\zeta)h(\zeta)d\zeta-\frac{1}{P}\int\limits_u^v\rho(\zeta)\varphi(\zeta)d\zeta\frac{1}{P}\int\limits_u^v\rho(\zeta)h(\zeta)d\zeta, \tag{1}$$

where $P=\int_u^v\rho(\zeta)d\zeta$.

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If $\rho(\zeta) = 1$ for all $\zeta \in [u, v]$, then we define Chebyshev functional $\mathfrak{C}(\varphi, h) = \mathfrak{C}(\varphi, h; 1)$.

The renowned Grüss inequality states that

$$\mathfrak{C}(\varphi, h; \rho) \leq \frac{1}{4}(\Lambda - \lambda)(\Psi - \psi), \tag{2}$$

where $\Lambda, \lambda, \Psi, \psi$ are real numbers with the property

$$-\infty < \lambda \leq \varphi \leq \Lambda < \infty, \quad -\infty < \psi \leq h \leq \Psi < \infty \quad \text{a.e. on} \quad [u, v]. \tag{3}$$

In 1934, G. Grüss [8] gives proof without weights however the same proof hold for weighted version also.

Moreover, we need to mention here the weighted version of Korkine's identity [12, p. 242]

$$\mathfrak{C}(\varphi, h; \rho) = \frac{1}{2P^2} \int\limits_u^v \int\limits_u^v \rho(\tau)\rho(\zeta)(\varphi(\tau) - \varphi(\zeta))(h(\tau) - h(\zeta))d\tau d\zeta. \tag{4}$$

We give variety of upper bounds for Jensen's difference in terms of the Grüss and Chebyshev inequalities. We also present multidimensional case of *Jensen – Grüss inequality* and formulate its bounds in case for monotonic functions. We also point out some applications of such results in information theory, namely we provide some new upper bounds for the Shannon entropy.

2. Jensen – Grüss inequality.

Theorem 2.1. *Let $\varphi\colon I = [u, v] \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping with continuous first derivative. Let $h\colon I \to I$ such that h, $\varphi \circ h$, $\varphi' \circ h \in L[u, v]$, and suppose that there exist $\lambda, \Lambda, \psi, \Psi \in \mathbb{R}$ such that*

$$\lambda \leq h(\zeta) \leq \Lambda, \quad \psi \leq \varphi'(\zeta) \leq \Psi \quad \textit{for all} \quad \zeta \in I.$$

Then, for all $\rho(\zeta) \geq 0$ such that $P = \int_u^v \rho(\zeta)d\zeta > 0$ exists, we have the following refinements:

$$\left| \frac{1}{P} \int\limits_u^v \rho(\zeta)(\varphi \circ h)(\zeta)d\zeta - \varphi\left(\frac{1}{P} \int\limits_u^v \rho(\zeta)h(\zeta)d\zeta \right) \right| \leq$$

$$\leq \frac{\Psi - \psi}{2} \left(\frac{1}{P} \int\limits_u^v \rho(\zeta)h^2(\zeta)d\zeta - \left(\frac{1}{P} \int\limits_u^v \rho(\zeta)h(\zeta)d\zeta \right)^2 \right)^{\frac{1}{2}} \leq$$

$$\leq \frac{(\Lambda - \lambda)(\Psi - \psi)}{4}. \tag{5}$$

Proof. Employing the mean-value theorem for points $c, d \in I$, we can write that there exists ξ, $c \leq \xi \leq d$, such that

$$\varphi(c) - \varphi(d) = \varphi'(\xi)(c - d). \tag{6}$$

Using (6) for

$$c = \bar{h} = \frac{1}{P}\int\limits_u^v \rho(\zeta)h(\zeta)d\zeta$$

and $d = h$, we conclude that there exists g, $\bar{h} \le g \le h$, such that

$$\varphi\big(\bar{h}\big) - \varphi(h) = \varphi'(g)\big(\bar{h} - h\big). \tag{7}$$

Now multiplying (7) by $\rho(\zeta)$ and integrating over $[u, v]$ yields

$$P\varphi(\bar{h}) - \int\limits_u^v \rho(\zeta)\varphi(h(\zeta))d\zeta = \bar{h}\int\limits_u^v \rho(\zeta)\varphi'(g(\zeta))d\zeta - \int\limits_u^v \rho(\zeta)\varphi'(g(\zeta))h(\zeta)d\zeta.$$

Dividing by P, we get

$$\frac{1}{P}\int\limits_u^v \rho(\zeta)(\varphi\circ h)(\zeta)d\zeta - \varphi\left(\frac{1}{P}\int\limits_u^v \rho(\zeta)h(\zeta)d\zeta\right) =$$

$$= \frac{1}{P}\int\limits_u^v \rho(\zeta)\varphi'(g(\zeta))h(\zeta)d\zeta - \frac{1}{P}\int\limits_u^v \rho(\zeta)h(\zeta)d\zeta\frac{1}{P}\int\limits_u^v \rho(\zeta)\varphi'(g(\zeta))d\zeta.$$

Now taking modulus on both sides and using weighted Korkine's identity (4), gives

$$\left|\frac{1}{P}\int\limits_u^v \rho(\zeta)(\varphi\circ h)(\zeta)d\zeta - \varphi\left(\frac{1}{P}\int\limits_u^v \rho(\zeta)h(\zeta)d\zeta\right)\right| =$$

$$= \left|\frac{1}{P}\int\limits_u^v \rho(\zeta)\varphi'(g(\zeta))h(\zeta)d\zeta - \frac{1}{P}\int\limits_u^v \rho(\zeta)h(\zeta)d\zeta\frac{1}{P}\int\limits_u^v \rho(\zeta)\varphi'(g(\zeta))d\zeta\right| =$$

$$= \big|\mathfrak{C}(h, \varphi'(g); \rho)\big| \le \frac{1}{2P^2}\int\limits_u^v\int\limits_u^v \rho(\zeta)\rho(\tau)(|h(\zeta) - h(\tau)|)\times$$

$$\times\big(\big|\varphi'(g(\zeta)) - \varphi'(g(\tau))\big|\big)d\zeta d\tau.$$

Now applying Cauchy – Buniakowsky – Schwartz inequality for double integrals, we can state that the last expression is less than

$$\left|\frac{1}{P}\int\limits_u^v \rho(\zeta)(\varphi\circ h)(\zeta)d\zeta - \varphi\left(\frac{1}{P}\int\limits_u^v \rho(\zeta)h(\zeta)d\zeta\right)\right| \le$$

$$\le \mathfrak{C}^{\frac{1}{2}}(h, h; \rho)\mathfrak{C}^{\frac{1}{2}}\big(\varphi'(g), \varphi'(g); \rho\big). \tag{8}$$

Now utilizing weighted Grüss inequality (2) on second term, we obtain

$$\mathfrak{C}^{\frac{1}{2}}(h, h; \rho)\frac{1}{2}(\Psi - \psi) =$$

$$= \left(\frac{1}{P} \int_u^v \rho(\zeta) h^2(\zeta) d\zeta - \left(\frac{1}{P} \int_u^v \rho(\zeta) h(\zeta) d\zeta \right)^2 \right)^{\frac{1}{2}} \frac{\Psi - \psi}{2}.$$

Now utilizing weighted Grüss inequality (2) on first term, we get

$$\leq \frac{(\Lambda - \lambda)(\Psi - \psi)}{4}.$$

Theorem 2.1 is proved.

***Remark* 2.1.** It is important to note that the first inequality in (5) is valid without the bounds of function h.

We give the following interesting corollaries.

***Corollary* 2.1.** *Under the assumptions of Theorem* 2.1, *suppose that* φ' *is Lipschitzian with the constant* $L > 0$, *i.e.,*

$$\left| \varphi'(x) - \varphi'(y) \right| \leq L|x - y|$$

for all $x, y \in \mathrm{Range}(h)$, *where* $\lambda \leq \mathrm{Range}(h) \leq \Lambda$. *Then we have the following refinements:*

$$\left| \frac{1}{P} \int_u^v \rho(\zeta)(\varphi \circ h)(\zeta) d\zeta - \varphi\left(\frac{1}{P} \int_u^v \rho(\zeta) h(\zeta) d\zeta \right) \right| \leq$$

$$\leq L \frac{\Lambda - \lambda}{2} \left(\frac{1}{P} \int_u^v \rho(\zeta) h^2(\zeta) d\zeta - \left(\frac{1}{P} \int_u^v \rho(\zeta) h(\zeta) d\zeta \right)^2 \right)^{\frac{1}{2}} \leq$$

$$\leq L \frac{(\Lambda - \lambda)^2}{4}.$$

Proof. By using (8) and Lipschitzian condition on φ', we get

$$\left| \frac{1}{P} \int_u^v \rho(\zeta)(\varphi \circ h)(\zeta) d\zeta - \varphi\left(\frac{1}{P} \int_u^v \rho(\zeta) h(\zeta) d\zeta \right) \right| \leq$$

$$\leq L \mathfrak{C}^{\frac{1}{2}}(h, h; \rho) \mathfrak{C}^{\frac{1}{2}}(id(\zeta), id(\zeta); \rho),$$

where $id(\zeta) = \zeta$. Now applying weighted Grüss inequality (2) successively on right-hand side, we obtain

$$\leq L \frac{\Lambda - \lambda}{2} \mathfrak{C}^{\frac{1}{2}}(h, h; \rho) \leq$$

$$\leq L \frac{(\Lambda - \lambda)^2}{4}.$$

***Corollary* 2.2.** *With the assumptions of Corollary* 2.1, *further suppose that* φ'' *is bounded, that is,* $L = \|\varphi''\|$ *and* $\|\cdot\|$ *is defined as the sup-norm. Then we get the following refinements:*

$$\left|\frac{1}{P}\int_u^v \rho(\zeta)(\varphi\circ h)(\zeta)d\zeta - \varphi\left(\frac{1}{P}\int_u^v \rho(\zeta)h(\zeta)d\zeta\right)\right| \leq$$

$$\leq \|\varphi''\|\frac{\Lambda-\lambda}{2}\left(\frac{1}{P}\int_u^v \rho(\zeta)h^2(\zeta)d\zeta - \left(\frac{1}{P}\int_u^v \rho(\zeta)h(\zeta)d\zeta\right)^2\right)^{\frac{1}{2}} \leq$$

$$\leq \|\varphi''\|\frac{(\Lambda-\lambda)^2}{4}.$$

3. Jensen – Chebyshev inequality. We need the following lemma of our interest.

Lemma 3.1 [3]. *Let $h\colon [u,v] \to \mathbb{R}$ be an absolutely continuous function such that $(h')^2 \in \in L[u,v]$ and weight ρ be a positive integrable function such that*

$$P(z) = \int_u^z \rho(\zeta)d\zeta \quad and \quad \breve{P}(z) = P(z)\int_u^v \zeta\rho(\zeta)d\zeta - P\int_u^z \zeta\rho(\zeta)d\zeta.$$

Then we have the following inequality:

$$\mathfrak{C}(h,h;\rho) \leq \frac{1}{P^2}\int_u^v \breve{P}(z)[h'(z)]^2 dz \tag{9}$$

provided that integral on the right-hand side of above inequality exists. Also the inequality in (9) *is sharp.*

Theorem 3.1. *Let $\varphi\colon I = [u,v] \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping with continuous first derivative. Let $h\colon I \to I$ be absolutely continuous such that h, $\varphi\circ h$, $\varphi'\circ h$, $(h')^2 \in L[u,v]$, and suppose that there exist ψ, $\Psi \in \mathbb{R}$ such that*

$$\psi \leq \varphi'(\zeta) \leq \Psi \quad for\ all \quad \zeta \in I.$$

Then, for all $\rho \geq 0$ such that $P = \int_u^v \rho(\zeta)d\zeta > 0$ exists and $\breve{P}(\cdot)$ be as given in Lemma 3.1, *we have the following refinements:*

$$\left|\frac{1}{P}\int_u^v p(\zeta)(\varphi\circ h)(\zeta)d\zeta - \varphi\left(\frac{1}{P}\int_u^v p(\zeta)h(\zeta)d\zeta\right)\right| \leq$$

$$\leq \frac{\Psi-\psi}{2}\left(\frac{1}{P}\int_u^v p(\zeta)h^2(\zeta)d\zeta - \left(\frac{1}{P}\int_u^v p(\zeta)h(\zeta)d\zeta\right)^2\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{\Psi-\psi}{2P}\left(\int_u^v \breve{P}(\zeta)\left[h'(\zeta)\right]^2 d\zeta\right)^{\frac{1}{2}}. \tag{10}$$

Proof. We have already established in Theorem 2.1 that

$$\left|\frac{1}{P}\int_u^v \rho(\zeta)(\varphi\circ h)(\zeta)d\zeta - \varphi\left(\frac{1}{P}\int_u^v \rho(\zeta)h(\zeta)d\zeta\right)\right| =$$

$$= \left|\frac{1}{P}\int_u^v p(\zeta)\varphi'(g(\zeta))h(\zeta)d\zeta - \frac{1}{P}\int_u^v \rho(\zeta)h(\zeta)d\zeta\frac{1}{P}\int_u^v \rho(\zeta)\varphi'(g(\zeta))d\zeta\right| =$$

$$= \left|\mathfrak{C}(h,\varphi'(g);\rho)\right| \leq \mathfrak{C}^{\frac{1}{2}}(h,h;\rho)\mathfrak{C}^{\frac{1}{2}}\left(\varphi'(g),\varphi'(g);\rho\right).$$

Now utilizing weighted Grüss inequality (2) on second term, we get

$$\leq \mathfrak{C}^{\frac{1}{2}}(h,h;\rho)\frac{1}{2}(\Psi-\psi) =$$

$$= \left(\frac{1}{P}\int_u^v \rho(\zeta)h^2(\zeta)d\zeta - \left(\frac{1}{P}\int_u^v \rho(\zeta)h(\zeta)d\zeta\right)^2\right)^{\frac{1}{2}}\frac{\Psi-\psi}{2}.$$

Now utilizing Lemma 3.1 on first term, we obtain

$$\leq \left(\frac{1}{P^2}\int_u^v \breve{P}(\zeta)\left[h'(\zeta)\right]^2 d\zeta\right)^{\frac{1}{2}}\frac{\Psi-\psi}{2}.$$

Theorem 3.1 is proved.

4. Jensen – Chebyshev norm estimates. We start this section by notating the following classes that we used:

(M_1) $C(u,v)$ denote the space of all functions $\rho>0$ continuous on (u,v) such that

$$\int_u^v \rho(\zeta)d\zeta = P < \infty.$$

(M_2) $W_r^2(u,v)$ denote the space of all functions h which are locally absolutely continuous on (u,v), with

$$\int_u^v rh'^2(\zeta)d\zeta < \infty.$$

Define

$$\|h\|_r = \left(\int_u^v r(\zeta)h^2(\zeta)d\zeta\right)^{\frac{1}{2}}.$$

In [13], G. V. Milovanović and I. Z. Milovanović gave weighted norm estimates of Chebyshev functional.

Theorem 4.1. *Let* $\rho \in C(u,v),\ r(\zeta) = \dfrac{1}{\rho(\zeta)}$ *and* $h, g \in W_r^2(u,v)$. *Then the following inequality holds:*

$$|\mathfrak{C}(h,g;\rho)| \leq \frac{P}{\pi^2}\left\|h'\right\|_r \left\|g'\right\|_r. \tag{11}$$

If $h(\zeta) = A + B\sin\Delta(\zeta),\ g(\zeta) = C + D\sin\Delta(\zeta)$, *where*

$$\Delta(\zeta) = \left(\frac{\pi}{P}\int\limits_{\zeta}^{b}\rho(t)dt - \int\limits_{a}^{\zeta}\rho(t)dt\right),$$

the equality appears in (11).

Now we are in position to state our results of this section.

Theorem 4.2. *Let* $\varphi\colon I = (u,v) \subset \mathbb{R} \to \mathbb{R}$ *be differentiable mapping with continuous first derivative. Let* $h\colon I \to I$ *be such that* $h \in W_r^2(I)$ *and* $h,\ \varphi \circ h,\ \varphi' \circ h \in L[u,v]$ *and suppose that there exist* $\psi,\ \Psi \in \mathbb{R}$ *such that*

$$\psi \leq \varphi'(\zeta) \leq \Psi \quad \textit{for all} \quad \zeta \in I.$$

Then, for all $\rho \in C(u,v),\ r(\zeta) = \dfrac{1}{\rho(\zeta)}$, *we have the following refinements:*

$$\left|\frac{1}{P}\int\limits_{u}^{v} p(\zeta)(\varphi\circ h)(\zeta)d\zeta - \varphi\left(\frac{1}{P}\int\limits_{u}^{v} p(\zeta)h(\zeta)d\zeta\right)\right| \leq$$

$$\leq \frac{\Psi-\psi}{2}\left(\frac{1}{P}\int\limits_{u}^{v} p(\zeta)h^2(\zeta)d\zeta - \left(\frac{1}{P}\int\limits_{u}^{v} p(\zeta)h(\zeta)d\zeta\right)^2\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{\Psi-\psi}{2}\frac{\sqrt{P}}{\pi}\left|h'\right|_r. \tag{12}$$

Proof. We have already established in Theorem 2.1 that

$$\left|\mathfrak{C}\big(h,\varphi'(g);\rho\big)\right| \leq \mathfrak{C}^{\frac{1}{2}}(h,h;\rho)\mathfrak{C}^{\frac{1}{2}}\big(\varphi'(g),\varphi'(g);\rho\big) \leq \mathfrak{C}^{\frac{1}{2}}(h,h;\rho)\frac{1}{2}(\Psi-\psi) =$$

$$= \left(\frac{1}{P}\int\limits_{u}^{v}\rho(\zeta)h^2(\zeta)d\zeta - \left(\frac{1}{P}\int\limits_{u}^{v}\rho(\zeta)h(\zeta)d\zeta\right)^2\right)^{\frac{1}{2}}\frac{\Psi-\psi}{2}.$$

Now utilizing Theorem 4.1 on first term, we get

$$\leq \left(\frac{P}{\pi^2}\|h'\|_r^2\right)^{\frac{1}{2}}\frac{\Psi-\psi}{2}.$$

Theorem 4.2 is proved.

5. Multidimensional Jensen – Grüss inequality. Let $(\Omega, \mathcal{A}, \mu)$ be a space with positive finite measure. Let $L = L_1(\Omega, \mathcal{A}, \mu)$ and for $h \in L$ define

$$\bar{h} = \frac{1}{\mu(\Omega)} \int\limits_{\Omega} h(\zeta) d\mu(\zeta).$$

Let $U \subset \mathbb{R}^n$ be a convex set, and φ be an arbitrary convex function on U. If $h_1, h_2, \ldots, h_n$ are functions in class L_1 (i.e., μ-measurable functions), then the following multidimensional version of Jensen's integral inequality [18, p. 51] is valid:

$$\frac{1}{\mu(\Omega)} \int\limits_{\Omega} \varphi(h_1(\zeta), h_2(\zeta), \ldots, h_n(\zeta)) d\mu(\zeta) -$$

$$-\varphi\left(\frac{1}{\mu(\Omega)} \int\limits_{\Omega} h_1(\zeta) d\mu(\zeta), \frac{1}{\mu(\Omega)} \int\limits_{\Omega} h_2(\zeta) d\mu(\zeta), \ldots, \frac{1}{\mu(\Omega)} \int\limits_{\Omega} h_n(\zeta) d\mu(\zeta)\right) \geq 0.$$

Inorder to give multidimensional Jensen – Grüss integral version, we denote

$$\mathbf{h}(\zeta) = (h_1(\zeta), h_2(\zeta), \ldots, h_n(\zeta))$$

be n-tuple of functions of class L_1 and we denote

$$\frac{1}{\mu(\Omega)} \int\limits_{\Omega} \mathbf{h}(\zeta) d\mu(\zeta)$$

be the n-tuple

$$\left(\frac{1}{\mu(\Omega)} \int\limits_{\Omega} h_1(\zeta) d\mu(\zeta), \frac{1}{\mu(\Omega)} \int\limits_{\Omega} h_2(\zeta) d\mu(\zeta), \ldots, \frac{1}{\mu(\Omega)} \int\limits_{\Omega} h_n(\zeta) d\mu(\zeta)\right).$$

Theorem 5.1. *Let $\varphi \colon U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable mapping with continuous partial derivatives, where U is a convex point set in $\mathbb{R}^n$. Let $(\Omega, \mathcal{A}, \mu)$ be a space with positive finite measure and $L = L_1(\Omega, \mathcal{A}, \mu)$. Also, let $\mathbf{h} \colon \Omega \to U \subset \mathbb{R}^n$ such that $h_i(\zeta), \varphi \circ \mathbf{h}(\zeta)$, $\varphi' \circ \mathbf{h}(\zeta) \in L_1$ (i.e., μ-measurable functions) for all $\zeta \in \Omega$ and $i = 1, 2, \ldots, n$. Suppose that there exist $\boldsymbol{\lambda}, \boldsymbol{\Lambda}, \boldsymbol{\psi}, \boldsymbol{\Psi} \in \mathbb{R}^n$ such that*

$$\boldsymbol{\lambda} \leq \mathbf{h} \leq \boldsymbol{\Lambda} \quad \textit{(the order is onsiderd cordinatewise)}$$

and

$$\boldsymbol{\psi} \leq \nabla\varphi(\boldsymbol{\zeta}) \leq \boldsymbol{\Psi} \quad \textit{for all} \quad \boldsymbol{\zeta} \in \operatorname{dom}(\varphi).$$

Then we have the inequalities

$$\left| \frac{1}{\mu(\Omega)} \int\limits_{\Omega} \varphi(h_1(\zeta), h_2(\zeta), \ldots, h_n(\zeta)) d\mu(\zeta) - \right.$$

$$\left. -\varphi\left(\frac{1}{\mu(\Omega)} \int\limits_{\Omega} h_1(\zeta) d\mu(\zeta), \frac{1}{\mu(\Omega)} \int\limits_{\Omega} h_2(\zeta) d\mu(\zeta), \ldots, \frac{1}{\mu(\Omega)} \int\limits_{\Omega} h_n(\zeta) d\mu(\zeta)\right) \right| \leq$$

$$\leq \frac{1}{2}\|\boldsymbol{\Psi}-\boldsymbol{\psi}\|\left(\frac{1}{\mu(\Omega)}\int\limits_{\Omega}\|\mathbf{h}(\zeta)\|^2 d\mu(\zeta)-\left\|\frac{1}{\mu(\Omega)}\int\limits_{\Omega}\mathbf{h}(\zeta)d\mu(\zeta)\right\|^2\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{4}\|\boldsymbol{\Lambda}-\boldsymbol{\lambda}\|\,\|\boldsymbol{\Psi}-\boldsymbol{\psi}\|.$$

Proof. Employing the mean-value theorem in multidimensional case for points $\mathbf{c},\mathbf{d}\in\operatorname{dom}(\varphi)$, we conclude that there exist $\alpha\in(0,1)$ such that

$$\varphi(\mathbf{c})-\varphi(\mathbf{d})=\langle\nabla\varphi(\boldsymbol{\xi}),\mathbf{c}-\mathbf{d}\rangle, \tag{13}$$

where $\boldsymbol{\xi}=\mathbf{d}+\alpha(\mathbf{c}-\mathbf{d})$. Using (13) for

$$\mathbf{c}=\bar{\mathbf{h}}=\left(\bar{h}_1,\bar{h}_2,\ldots,\bar{h}_n\right)=\frac{1}{\mu(\Omega)}\int\limits_{\Omega}\mathbf{h}(\zeta)d\mu(\zeta)$$

be the n-tuple

$$\left(\frac{1}{\mu(\Omega)}\int\limits_{\Omega}h_1(\zeta)d\mu(\zeta),\frac{1}{\mu(\Omega)}\int\limits_{\Omega}h_2(\zeta)d\mu(\zeta),\ldots,\frac{1}{\mu(\Omega)}\int\limits_{\Omega}h_n(\zeta)d\mu(\zeta)\right),$$

$\mathbf{d}=\mathbf{h}=(h_1,h_2,\ldots,h_n)$ and $\boldsymbol{\xi}=\mathbf{g}=(g_1,g_2,\ldots,g_n)$, where $g_i(\zeta)\in L_1$ (i.e., μ-measurable functions) for all $\zeta\in\Omega$ and $i=1,2,\ldots,n$, we have

$$\varphi\big(\bar{\mathbf{h}}\big)-\varphi(\mathbf{h})=\big\langle\nabla\varphi(\mathbf{g}),\bar{\mathbf{h}}-\mathbf{h}\big\rangle.$$

Integrating over Ω w.r.t. μ yields

$$\mu(\Omega)\varphi\big(\bar{\mathbf{h}}\big)-\int\limits_{\Omega}\varphi(\mathbf{h}(\zeta))d\mu(\zeta)=\int\limits_{\Omega}\big\langle\nabla\varphi(\mathbf{g}),\bar{\mathbf{h}}\big\rangle d\mu(\zeta)-\int\limits_{\Omega}\langle\nabla\varphi(\mathbf{g}),\mathbf{h}\rangle d\mu(\zeta).$$

Dividing by $\mu(\Omega)$, we get

$$\frac{1}{\mu(\Omega)}\int\limits_{\Omega}\varphi(h_1(\zeta),h_2(\zeta),\ldots,h_n(\zeta))d\mu(\zeta)-$$

$$-\varphi\left(\frac{1}{\mu(\Omega)}\int\limits_{\Omega}h_1(\zeta)d\mu(\zeta),\ldots,\frac{1}{\mu(\Omega)}\int\limits_{\Omega}h_n(\zeta)d\mu(\zeta)\right)=$$

$$=\frac{1}{\mu(\Omega)}\int\limits_{\Omega}\langle\nabla\varphi(\mathbf{g}),\mathbf{h}\rangle d\mu(\zeta)-\left\langle\frac{1}{\mu(\Omega)}\int\limits_{\Omega}\nabla\varphi(\mathbf{g}(\zeta))d\zeta,\bar{\mathbf{h}}\right\rangle.$$

Rest of the proof can be completed by method used to prove multidimensional discrete version, given in the proof of Theorem 1 from [6].

A multidimensional generalization of Lupas – Ostrowski inequality was given in [20]. For instance, we give the following theorem for two variables.

Theorem 5.2 [20]. *Let*

$$\rho \in C(u_1, v_1), \quad \mathfrak{q} \in C(u_2, v_2), \quad \rho > 0, \quad \mathfrak{q} > 0,$$

$$\int\limits_{u_1}^{v_1} \rho(t)dt = P < \infty, \quad \int\limits_{u_2}^{v_2} \mathfrak{q}(t)dt = Q < \infty.$$

Let $h\colon (u_1, v_1) \times (u_2, v_2) \to \mathbb{R}$ *be a function such that* $h(\cdot, x_2)$ *is locally absolutely continuous on* (u_1, v_1) *for almost every* $x_2 \in (u_2, v_2)$ *and* $h(x_1, \cdot)$ *is locally absolutely continuous on* (u_2, v_2) *for almost every* $x_1 \in (u_1, v_1)$. *Suppose that*

$$\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(x_1)\mathfrak{q}(x_2)h^2(x_1, x_2)dx_1dx_2 < \infty$$

and

$$\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \left[\frac{\mathfrak{q}(x_2)}{\rho(x_1)}\left(\frac{\partial h}{\partial x_1}\right)^2 + \frac{\rho(x_1)}{\mathfrak{q}(x_2)}\left(\frac{\partial h}{\partial x_1}\right)^2\right] dx_1dx_2 < \infty.$$

Also let g *satisfy the same condition as* h, *then we have*

$$|\mathfrak{C}(h, g; \rho, \mathfrak{q})| \le \frac{1}{\pi^2}\|\nabla h; \rho, \mathfrak{q}\|_2 \|\nabla g; \rho, \mathfrak{q}\|_2,$$

where

$$\mathfrak{C}(h, g; \rho, \mathfrak{q}) = A(h, g; \rho, \mathfrak{q}) - A(h; \rho, \mathfrak{q})A(g; \rho, \mathfrak{q}), \tag{14}$$

$$A(h; \rho, \mathfrak{q}) = \frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_1}^{v_1} \rho(x_1)\mathfrak{q}(x_2)h(x_1, x_2)dx_1dx_2 \tag{15}$$

and

$$\|\nabla h; \rho, \mathfrak{q}\|_2 = \left(\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \left[\frac{P}{Q}\frac{\mathfrak{q}(x_2)}{\rho(x_1)}\left(\frac{\partial h}{\partial x_1}\right)^2 + \frac{Q}{P}\frac{\rho(x_1)}{\rho(x_2)}\left(\frac{\partial h}{\partial x_2}\right)^2\right] dx_1dx_2\right)^{\frac{1}{2}}. \tag{16}$$

Theorem 5.3. *Let* $\varphi\colon I \subset \mathbb{R} \to \mathbb{R}$ *be differentiable mapping with continuous first derivative. Let* $h\colon (u_1, v_1) \times (u_2, v_2) \to I$ *be as defined in Theorem* 5.2 *such that* h, $\varphi \circ h$, $\varphi' \circ h \in L((u_1, v_1) \times \times (u_2, v_2))$, *and suppose that there exist* ψ, $\Psi \in \mathbb{R}$ *such that*

$$\psi \le \varphi'(\zeta) \le \Psi \quad \textit{for all} \quad \zeta \in I.$$

Then, for

$$\rho \in C(u_1, v_1), \quad \mathfrak{q} \in C(u_2, v_2), \quad \rho > 0, \quad \mathfrak{q} > 0,$$

$$\int\limits_{u_1}^{v_1} \rho(t)dt = P < \infty, \quad \int\limits_{u_2}^{v_2} \mathfrak{q}(t)dt = Q < \infty,$$

we have the following refinements:

$$\left|\frac{1}{PQ}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)q(\zeta_2)(\varphi\circ h)(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2-\varphi\left(\frac{1}{PQ}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)\mathfrak{q}(\zeta_2)h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2\right)\right|\leq$$

$$\leq\frac{\Psi-\psi}{2}\left[\frac{1}{PQ}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)\mathfrak{q}(\zeta_2)h^2(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2-\right.$$

$$\left.-\left(\frac{1}{PQ}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)\mathfrak{q}(\zeta_2)h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2\right)^2\right]^{\frac{1}{2}}\leq$$

$$\leq\frac{\Psi-\psi}{2\pi}\|\nabla h;\rho,\mathfrak{q}\|_2,$$

where $\|\nabla h;\rho,\mathfrak{q}\|_2$ *is given in* (16).

Proof. Employing the mean-value theorem for points $c,d\in I$, we can write that there exists ξ, $c\leq\xi\leq d$, such that

$$\varphi(c)-\varphi(d)=\varphi'(\xi)(c-d). \tag{17}$$

Using (17) for

$$c=\bar{h}=\frac{1}{PQ}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)\mathfrak{q}(\zeta_2)\ h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2$$

and $d=h(\zeta_1,\zeta_2)$, we conclude that there exists g, $\bar{h}\leq g\leq h$, such that

$$\varphi\big(\bar{h}\big)-\varphi(h)=\varphi'(g)\big(\bar{h}-h\big). \tag{18}$$

Now multiplying (18) by $\rho(\zeta_1)$ and $\mathfrak{q}(\zeta_2)$ and integrating over (u_1,v_1) and (u_2,v_2) yields

$$PQ\,\varphi\left(\frac{1}{PQ}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)\mathfrak{q}(\zeta_2)h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2\right)-\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)\mathfrak{q}(\zeta_2)\varphi(h(\zeta_1,\zeta_2))d\zeta_1 d\zeta_2=$$

$$=\bar{h}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)\mathfrak{q}(\zeta_2)\varphi'(g(\zeta_1,\zeta_2))d\zeta_1 d\zeta_2-\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)\mathfrak{q}(\zeta_2)\varphi'(g(\zeta_1,\zeta_2))h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2.$$

Dividing by PQ, we get

$$\frac{1}{PQ}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)\mathfrak{q}(\zeta_2)\varphi(h(\zeta_1,\zeta_2))d\zeta_1 d\zeta_2-$$

$$-\varphi\left(\frac{1}{PQ}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\rho(\zeta_1)\mathfrak{q}(\zeta_2)h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2\right)=$$

$$= \frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(\zeta_1)\mathfrak{q}(\zeta_2)\varphi'(g(\zeta_1,\zeta_2))h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2 -$$

$$-\frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(\zeta_1)\mathfrak{q}(\zeta_2)h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2 \frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(\zeta_1)\mathfrak{q}(\zeta_2)\varphi'(g(\zeta_1,\zeta_2))d\zeta_1 d\zeta_2.$$

Now taking modulus on both sides and using representation $\mathfrak{C}(h,g;\rho,\mathfrak{q})$ given in (14), we obtain

$$\left|\frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(\zeta_1)\mathfrak{q}(\zeta_2)\varphi(h(\zeta_1,\zeta_2))d\zeta_1 d\zeta_2 -\right.$$

$$\left. -\varphi\left(\frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(\zeta_1)\mathfrak{q}(\zeta_2)h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2\right)\right| =$$

$$= \left|\frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(\zeta_1)\mathfrak{q}(\zeta_2)\varphi'(g(\zeta_1,\zeta_2))h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2 -\right.$$

$$\left. -\frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(\zeta_1)\mathfrak{q}(\zeta_2)h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2 \; \frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(\zeta_1)\mathfrak{q}(\zeta_2)\varphi'(g(\zeta_1,\zeta_2))d\zeta_1 d\zeta_2\right| =$$

$$= \left|\mathfrak{C}\big(h,\varphi'(g);\rho,\mathfrak{q}\big)\right|.$$

Now applying Cauchy – Schwartz inequality, we can state that the last expression is less than

$$\left|\frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(\zeta_1)\mathfrak{q}(\zeta_2)\varphi(h(\zeta_1,\zeta_2))d\zeta_1 d\zeta_2 - \varphi\left(\frac{1}{PQ}\int\limits_{u_1}^{v_1}\int\limits_{u_2}^{v_2} \rho(\zeta_1)\mathfrak{q}(\zeta_2)h(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2\right)\right| \le$$

$$\le \mathfrak{C}^{\frac{1}{2}}(h,h;\rho,\mathfrak{q})\mathfrak{C}^{\frac{1}{2}}\big(\varphi'(g),\varphi'(g);\rho,\mathfrak{q}\big) \le \frac{\Psi-\psi}{2}\,\mathfrak{C}^{\frac{1}{2}}(h,h;\rho,\mathfrak{q}). \tag{19}$$

Finally, employing Theorem 5.2 on the second expression of (19), we complete the proof of Theorem 5.3.

***Remark* 5.1.** It is of worth mentioning that using result given in Remark 9 in [20], we can also give extension for function h with more variables.

6. Jensen – Grüss inequality and monotonic functions. S. Bernstein in [5] introduce the term absolutely monotonic function on interval $[u,v]$, if $h \in C^k[u,v]$ and satisfies

$$h^{(k)}(\zeta) \ge 0, \quad k = 0,1,\ldots, \quad \zeta \in (u,v),$$

and completely monotonic function if

$$(-1)^k h^{(k)}(\zeta) \ge 0, \quad k = 0,1,\ldots, \quad \zeta \in (u,v).$$

G. Grüss in [8] also gave results for monotone functions given as:

If φ and h are absolutely monotone functions on $(0,1)$ satisfying (3), then

$$\mathfrak{C}(\varphi,\varphi) \leq \frac{4}{45}\,(\Lambda-\lambda)^2 \tag{20}$$

and

$$|\mathfrak{C}(\varphi,h)| \leq \frac{4}{45}(\Lambda-\lambda)(\Psi-\psi). \tag{21}$$

The constant $4/45$ is best possible both for (20) and (21) and this can been seen by putting $\varphi(u) = u^2$. G. Grüss used Bernstein's polynomial to prove (20) and (21). However, E. Landau [14] gave an easy proof by using his proposition (see [12, p. 297]). In [15], he also proved that inequalities (20) and (21) still hold provided that the functions φ and h are monotonic of order 4. He also proved bounds for Chebyshev functional for monotone functions of order $k = 1, 2, 3$ respectively as:

$$|\mathfrak{C}(\varphi,h)| \leq \frac{1}{4}\,(\Lambda-\lambda)(\Psi-\psi) \quad \text{for} \quad k=1, \tag{22}$$

$$|\mathfrak{C}(\varphi,h)| \leq \frac{1}{9}\,(\Lambda-\lambda)(\Psi-\psi) \quad \text{for} \quad k=2, \tag{23}$$

and

$$|\mathfrak{C}(\varphi,h)| \leq \frac{9}{100}\,(\Lambda-\lambda)(\Psi-\psi) \quad \text{for} \quad k=3. \tag{24}$$

G. Hardy [16] obtained the following result:

Let φ and h be totally monotonic function on $(0,\infty)$ and $\varphi, h \in L(0,v)$ satisfying (3). Then the following inequality is valid:

$$|\mathfrak{C}(\varphi,h)| \leq \frac{1}{12}\,(\Lambda-\lambda)(\Psi-\psi). \tag{25}$$

Now we give several Jensen–Grüss inequalities for monotone functions.

Theorem 6.1. *Let $\varphi\colon I = (0,1) \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping with continuous first derivative. Let $h\colon (0,1) \to I$ be absolutely monotone function on $(0,1)$ such that $h,\ \varphi\circ h,\ \varphi'\circ h \in L(0,1)$, and suppose that there exist $\lambda,\ \Lambda,\ \psi,\ \Psi \in \mathbb{R}$ such that*

$$\lambda \leq h(\zeta) \leq \Lambda, \quad \psi \leq \varphi'(\zeta) \leq \Psi \quad \textit{for all} \quad \zeta \in I.$$

Then we have the following refinements:

$$\left|\int\limits_0^1 (\varphi\circ h)(\zeta)d\zeta - \varphi\left(\int\limits_0^1 h(\zeta)d\zeta\right)\right| \leq$$

$$\leq \frac{\Psi-\psi}{2}\left(\int\limits_0^1 h^2(\zeta)d\zeta - \left(\int\limits_0^1 h(\zeta)d\zeta\right)^2\right)^{\frac{1}{2}} \leq \frac{(\Lambda-\lambda)(\Psi-\psi)}{3\sqrt{5}}.$$

Proof. Now from the proof of Theorem 2.1 without weights, putting $(u,v)=(0,1)$ and $\rho(\zeta)=1$ for all $\zeta\in(0,1)$, we get

$$\left|\int_0^1(\varphi\circ h)(\zeta)d\zeta-\varphi\left(\int_0^1 h(\zeta)d\zeta\right)\right|=$$

$$=\left|\int_0^1\varphi'(g(\zeta))h(\zeta)d\zeta-\int_0^1 h(\zeta)d\zeta\int_0^1\varphi'(g(\zeta))d\zeta\right|=$$

$$=\left|\mathfrak{C}(h,\varphi'(g))\right|\leq\frac{1}{2}\int_0^1\int_0^1(\|h(\zeta)-h(\tau)|)\big(\left|\varphi'(g(\zeta))-\varphi'(g(\tau))\right|\big)d\zeta d\tau.$$

Now applying Cauchy – Buniakowsky – Schwartz inequality for double integrals, we can state that the last expression is less than

$$\left|\int_0^1(\varphi\circ h)(\zeta)d\zeta-\varphi\left(\int_0^1 h(\zeta)d\zeta\right)\right|\leq\mathfrak{C}^{\frac{1}{2}}(h,h)\mathfrak{C}^{\frac{1}{2}}\big(\varphi'(g),\varphi'(g)\big).$$

Now utilizing Grüss inequality (2) without weights on second term, we get

$$\leq\mathfrak{C}^{\frac{1}{2}}(h,h)\frac{1}{2}(\Psi-\psi)=\left(\int_0^1 h^2(\zeta)d\zeta-\left(\int_0^1 h(\zeta)d\zeta\right)^2\right)^{\frac{1}{2}}\frac{\Psi-\psi}{2}.$$

Now utilizing Grüss type inequality (20) for h to be absolutely monotone functions on $(0,1)$ on first term, we have

$$\leq\left(\frac{2}{3\sqrt{5}}\left(\Lambda-\lambda\right)\right)\frac{\Psi-\psi}{2}.$$

Theorem 6.1 is proved.

The next results entails the bounds of Jensen – Grüss type inequalities for monotonic function of different orders.

***Corollary* 6.1.** *Under the assumptions of Theorem* 6.1, *let* $h\colon(0,1)\to I$ *be monotonic function on* $(0,1)$ *of order* $k=1,2,3$ *such that* $h,\ \varphi\circ h,\ \varphi'\circ h\in L(0,1)$, *and suppose that there exist* λ, $\Lambda,\ \psi,\ \Psi\in\mathbb{R}$ *such that*

$$\lambda\leq h(\zeta)\leq\Lambda,\quad \psi\leq\varphi'(\zeta)\leq\Psi\quad \textit{for all}\quad \zeta\in I.$$

Then we have the following several refinements:

$$\left|\int_0^1(\varphi\circ h)(\zeta)d\zeta-\varphi\left(\int_0^1 h(\zeta)d\zeta\right)\right|\leq$$

$$\leq \frac{\Psi - \psi}{2} \left(\int_0^1 h^2(\zeta) d\zeta - \left(\int_0^1 h(\zeta) d\zeta \right)^2 \right)^{\frac{1}{2}} \leq$$

$$\leq \frac{(\Lambda - \lambda)(\Psi - \psi)}{4} \quad \textit{for} \quad k = 1,$$

$$\left| \int_0^1 (\varphi \circ h)(\zeta) d\zeta - \varphi \left(\int_0^1 h(\zeta) d\zeta \right) \right| \leq$$

$$\leq \frac{\Psi - \psi}{2} \left(\int_0^1 h^2(\zeta) d\zeta - \left(\int_0^1 h(\zeta) d\zeta \right)^2 \right)^{\frac{1}{2}} \leq$$

$$\leq \frac{(\Lambda - \lambda)(\Psi - \psi)}{6} \quad \textit{for} \quad k = 2,$$

and

$$\left| \int_0^1 (\varphi \circ h)(\zeta) d\zeta - \varphi \left(\int_0^1 h(\zeta) d\zeta \right) \right| \leq$$

$$\leq \frac{\Psi - \psi}{2} \left(\int_0^1 h^2(\zeta) d\zeta - \left(\int_0^1 h(\zeta) d\zeta \right)^2 \right)^{\frac{1}{2}} \leq$$

$$\leq \frac{3}{20} (\Lambda - \lambda)(\Psi - \psi) \quad \textit{for} \quad k = 3. \tag{26}$$

Proof. We establish the proof, when h is monotonic of order $k = 3$. We have already established in the proof of Theorem 6.1 that

$$\left| \int_0^1 (\varphi \circ h)(\zeta) d\zeta - \varphi \left(\int_0^1 h(\zeta) d\zeta \right) \right| \leq \mathfrak{C}^{\frac{1}{2}}(h, h) \mathfrak{C}^{\frac{1}{2}} \big(\varphi'(g), \varphi'(g) \big) \leq$$

$$\leq \mathfrak{C}^{\frac{1}{2}}(h, h) \frac{1}{2} (\Psi - \psi) = \left(\int_0^1 h^2(\zeta) d\zeta - \left(\int_0^1 h(\zeta) d\zeta \right)^2 \right)^{\frac{1}{2}} \frac{\Psi - \psi}{2}.$$

Now utilizing Grüss type inequality (24) for $h = \varphi$ on first term, we get

$$\leq \left(\frac{3}{10} (\Lambda - \lambda) \right) \frac{\Psi - \psi}{2}$$

and (26) is established.

Other cases when h is monotonic of order $k = 1, 2$ can be obtained analogously by applying inequalities (22) and (23), respectively.

Corollary 6.1 is proved.

Now, we give refinements by using Grüss bounds obtained by G. Hardy for completely monotone functions.

Theorem 6.2. *Let $\varphi\colon I = (0, v) \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping with continuous first derivative. Let $h\colon (0,\infty) \to I$ be completely monotone function on $(0,\infty)$ such that $h,\ \varphi \circ h,$ $\varphi' \circ h \in L(0, v)$, and suppose that there exist $\lambda,\ \Lambda,\ \psi,\ \Psi \in \mathbb{R}$ such that*

$$\lambda \le h(\cdot) \le \Lambda \quad \textit{and} \quad \psi \le \varphi'(\cdot) \le \Psi.$$

Then we have the following refinements:

$$\left| \int\limits_0^v (\varphi \circ h)(\zeta) d\zeta - \varphi\left(\int\limits_0^v h(\zeta) d\zeta \right) \right| \le$$

$$\le \frac{\Psi - \psi}{2} \left(\int\limits_0^v h^2(\zeta) d\zeta - \left(\int\limits_0^v h(\zeta) d\zeta \right)^2 \right)^{\frac{1}{2}} \le \frac{(\Lambda - \lambda)(\Psi - \psi)}{4\sqrt{3}}.$$

Proof. From the proof of Theorem 2.1 without weights, putting $(u, v) = (0, v)$ and $\rho(\zeta) = 1$ for all $\zeta \in (0, v)$, we obtain

$$\left| \frac{1}{v} \int\limits_0^v (\varphi \circ h)(\zeta) d\zeta - \varphi\left(\frac{1}{v} \int\limits_0^v h(\zeta) d\zeta \right) \right| =$$

$$= \left| \frac{1}{v} \int\limits_0^v \varphi'(g(\zeta)) h(\zeta) d\zeta - \frac{1}{v^2} \int\limits_0^v h(\zeta) d\zeta \int\limits_0^v \varphi'(g(\zeta)) d\zeta \right| =$$

$$= \left| \mathfrak{C}(h, \varphi'(g)) \right| \le \frac{1}{2v^2} \int\limits_0^v \int\limits_0^v (\|h(\zeta) - h(\tau)\|) \left(\left| \varphi'(g(\zeta)) - \varphi'(g(\tau)) \right| \right) d\zeta d\tau.$$

Now applying Cauchy – Buniakowsky – Schwartz inequality for double integrals, we can state that the last expression is less than

$$\left| \frac{1}{v} \int\limits_0^v (\varphi \circ h)(\zeta) d\zeta - \varphi\left(\frac{1}{v} \int\limits_0^v h(\zeta) d\zeta \right) \right| \le \mathfrak{C}^{\frac{1}{2}}(h, h) \mathfrak{C}^{\frac{1}{2}}\left(\varphi'(g), \varphi'(g) \right).$$

Now utilizing Grüss inequality (2) without weights on second term, we get

$$\le \mathfrak{C}^{\frac{1}{2}}(h, h) \frac{1}{2} (\Psi - \psi) =$$

$$= \left(\frac{1}{v} \int\limits_0^v h^2(\zeta) d\zeta - \left(\frac{1}{v} \int\limits_0^v h(\zeta) d\zeta \right)^2 \right)^{\frac{1}{2}} \frac{\Psi - \psi}{2}.$$

Now utilizing Grüss type inequality (25) for $h = \varphi$ to be completely monotone function on first term, we have

$$\leq \left(\frac{1}{\sqrt{12}}(\Lambda - \lambda)\right)\frac{\Psi - \psi}{2}.$$

Theorem 6.2 is proved.

7. Applications in information theory. Information theory is the study of data which manages the capacity, measurement and correspondence of data. The subject studies all the theoretical problems related to information transformation over the communication channels. Being a theoretical substance, data can't be measured without any problem. Claude Shannon, who is these days regarded as the "Father of Information Theory", presented a theory in order to quantify the communication of information [22]. Shannon's theory deals with the problem of how to transmit information most efficiently through a given channel. It also tackles the issues of communication security. Shannon's formula states that we will gain the largest amount of Shannon's information when dealing with systems whose individual possible outcomes are equally likely to occur. Shannon's entropy is a measure of the potential reduction in uncertainty in the receiver's knowledge. Shannon's entropy and related measures are increasingly used in molecular ecology and population genetics, information theory, dynamical systems and statistical physics.

Jensen's integral inequality is importantly used to construct many information inequalities [1, 2, 9]. In this section, we present some important applications in information theory of our main results.

Consider the set of probability density functions

$$\mathcal{P} = \left\{\rho | \rho \colon I \to \mathbb{R},\ \rho(\zeta) \geq 0 \ \text{ and } \int_u^v p(\zeta)d\zeta = 1\right\}.$$

For positive probability density function $\rho \in \mathcal{P}$, the **Shannon entropy** is defined as [17]

$$H(\rho) = -\int_u^v \rho(\zeta)\ln\rho(\zeta)d\zeta.$$

Theorem 7.1. *Under the assumptions of Theorem* 2.1 *with* $I = [u, v] \subset (0, \infty)$, *suppose that* $\rho \in \mathcal{P}$ *be positive probability distribution function, defined on* I, *such that there exist constants* $0 < \lambda,\ \Lambda \leq 1$ *such that*

$$\lambda \leq \rho(\zeta) \leq \Lambda \quad \textit{for all} \quad \zeta \in I.$$

Then we have the following refinements:

$$\left|H(\rho) + \ln\left(\int_u^v \rho^2(\zeta)d\zeta\right)\right| \leq$$

$$\leq \frac{\Lambda - \lambda}{2\Lambda\lambda}\left(\int_u^v \rho^3(\zeta)d\zeta - \left(\int_u^v \rho^2(\zeta)d\zeta\right)^2\right)^{\frac{1}{2}} \leq \frac{(\Lambda - \lambda)^2}{4\Lambda\lambda}.$$

Proof. Substituting $\varphi(\zeta) := -\ln(\zeta)$ and $h(\zeta) = \rho(\zeta)$ in (5), we will obtain our results.

Theorem 7.2. *Under the assumptions of Theorem* 3.1 *with* $I = [u, v] \subset (0, \infty)$, *suppose that* $\rho \in \mathcal{P}$ *be positive probability distribution function, defined on* I, *such that* $(\rho')^2 \in L[u, v]$ *and there exist constants* $0 < \lambda,\ \Lambda \leq 1$ *such that*

$$\lambda \leq \rho(\zeta) \leq \Lambda \quad \textit{for all} \quad \zeta \in I.$$

Then we have the following refinements:

$$\left| H(\rho) + \ln\left(\int_u^v \rho^2(\zeta)d\zeta\right)\right| \leq$$

$$\leq \frac{\Lambda - \lambda}{2\Lambda\lambda}\left(\int_u^v \rho^3(\zeta)d\zeta - \left(\int_u^v \rho^2(\zeta)d\zeta\right)^2\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{\Lambda - \lambda}{2\Lambda\lambda}\left(\int_u^v \breve{P}(\zeta)\left[\rho'(\zeta)\right]^2 d\zeta\right)^{\frac{1}{2}}.$$

Proof. Substituting $\phi(\zeta) := -\ln(\zeta)$ and $h(\zeta) = \rho(\zeta)$ in (10), we will obtain our results.

Theorem 7.3. *Under the assumptions of Theorem* 4.2 *with* $I = [u, v] \subset (0, \infty)$, *suppose that* $\rho \in \mathcal{P}$ *be positive probability distribution function, defined on* I, *such that* $\rho \in W_r^2(I)$ *and there exist constants* $0 < \lambda,\ \Lambda \leq 1$ *such that*

$$\lambda \leq \rho(\zeta) \leq \Lambda \quad \textit{for all} \quad \zeta \in I.$$

Then we have the following refinements:

$$\left| H(\rho) + \ln\left(\int_u^v \rho^2(\zeta)d\zeta\right)\right| \leq$$

$$\leq \frac{\Lambda - \lambda}{2\Lambda\lambda}\left(\int_u^v \rho^3(\zeta)d\zeta - \left(\int_u^v \rho^2(\zeta)d\zeta\right)^2\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{\Lambda - \lambda}{2\Lambda\lambda}\,\frac{1}{\pi}\left(\int_u^v \frac{1}{\rho(\zeta)}\left[\rho'(\zeta)\right]^2 d\zeta\right)^{\frac{1}{2}}.$$

Proof. Substituting $\varphi(\zeta) := -\ln(\zeta)$ and $h(\zeta) = \rho(\zeta)$ in (12), we will obtain our results.

8. Conclusion. In this paper, we use Jensen'n integral difference and give bounds by using Grüss inequality. We vary bounds by imposing conditions on the un known function h by employing Chebyshev bounds and Chebyshev norm estimations. We also present multidimensional version of our results. Some bounds for absolutely monotone and completely monotone functions are also obtained. Finally, as an application we conclude our paper by giving new bounds for Shannon's entropy. Our results will be of general interest for many researchers.

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