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ZARISKI TOPOLOGY OVER MULTIPLICATION KRASNER HYPERMODULES

ТОПОЛОГІЯ ЗАРИСЬКОГО НАД МУЛЬТИПЛІКАТИВНИМИ ГІПЕРМОДУЛЯМИ КРАСНЕРА

In this paper, we introduce the notion of multiplication Krasner hypermodules over commutative hyperrings and topologize the collection of all multiplication Krasner hypermodules. In addition, we investigate some properties of this topological space.

Мета цієї роботи — визначення поняття мультиплікативних гипермодулів Краснера над комутативними гіперкільцями й топологізація колекції всіх мультиплікативних гипермодулів Краснера, а також вивчення деяких властивостей цього топологічного простору.

1. Introduction. Notions of hypergroups, hypermodules and hyperrings have many important roles in hyperstructures. Some authors have gotten many conclusions about these theories (see [2, 7, 8, 10, 13]). It may also be eligible for reference [20] certain information about the theory of rings and modules.

We recall some definitions and propositions from above references which we need to develop our paper.

In this paper, we use $\circ\colon M\times M\to P^*(M)$ instead of $\cdot\colon M\times M\to M$, where M is a non-empty set and $P^*(M)$ the set of all non-empty subsets of M. The map $\circ$ is called a *hyperoperation* on M. Thus, we use $X\circ Y=\bigcup_{x\in X,\, y\in Y} x\circ y$, $m\circ X=\{m\}\circ X$ and $X\circ m=X\circ\{m\}$ for all $m\in M$ and $X,Y\in P^*(M)$. The hyperstructure (M,o) is called a *semihypergroup* if, for all x,y,z of M, we have $(x\circ y)\circ z=x\circ(y\circ z)$. A semihypergroup $(M,\circ)$ is called a *hypergroup* if, for all $m\in M$, $m\circ M=M\circ m=M$ [6].

A non-empty subset N of a hypergroup $(M,\circ)$ is called *subhypergroup* if, for all $n\in N$, we have $n\circ N=N\circ n=N$. A hypergroup $(M,\circ)$ is called *commutative* if $x\circ y=y\circ x$ for all $x,y\in M$. A commutative hypergroup $(M,\circ)$ is said to be *canonical*, if there exists a unique $0\in M$ such that for all $m\in M$, $m\circ 0=\{m\}$, for all $m\in M$, there exists a unique $m^{-1}\in M$ such that $0\in m\circ m^{-1}$; if $x\in y\circ z$, then $y\in xoz^{-1}$ and $z\in y^{-1}\circ x$ for all $x,y,z\in M$ [6].

The triple $(R,\uplus,\circ)$ is called a *hyperring*, if $(R,\uplus)$ is a semihypergroup, (R,o) is a semihypergroup and $\circ$ is a distributive over $\uplus$ [8]. A *Krasner hyperring* is an algebraic structure $(R,\uplus,\circ)$ which satisfies the following axioms:

(1) $(R,\uplus)$ is a canonical hypergroup;

(2) $(R,\circ)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x\circ 0=0\circ x=0$;

(3) the multiplication is distributive with respect to the hyperoperation $\uplus$.

A Krasner hyperring $(R,+,.)$ is called *commutative (with unit element)* if $(R,.)$ is a commutative semigroup (with unit element) [8]. A Krasner hyperring R is called a *Krasner hyperfield*, if $(R\setminus\{0\},.)$ is a group [8]. Let A and B be non-empty subsets of a hyperring R. The sum $A+B$ is defined by $A+B=\{\,x\,|\,x\in a+b\ \text{ for some } a\in A,\, b\in B\,\}$. The product AB is defined by $AB = \left\{ x \mid x \in \sum_{i=1}^{n} a_i b_i,\, a_i \in A, b_i \in B,\, n \in \mathbb{Z}^+ \right\}$. If A and B are hyperideals of R, then $A + B$ and AB are also hyperideals of R. Let $(R, +, .)$ be a ring and I be a subset of R. G is called a *multiplicative subgroup* of R if and only if $(I, .)$ is a group. Moreover, if I is such that $R = RI$ and $rG = Gr$ for every $r \in R$, then I is called a *normal subgroup* of R. As a normal subgroup I of R induces an equivalence relation P in R and a partition of R in equivalence classes which inherits a hyperring structure from R. Hyperrings obtained via this construction are called *quotient hyperrings* and are denoted by R/I [8]. A non-empty subset I of a hyperring R is called a *hyperideal* if $(I, \uplus)$ is a subhypergroup of $(R, \uplus)$ and $r \circ a \cup a \circ r \subseteq I$ for all $a \in I,\ r \in R$ [8]. A non-empty subset I of a Krasner hyperring $(R, \uplus, \circ)$ is called a *left hyperideal* of R if $(I, \uplus)$ is a canonical subhypergroup of $(R, \uplus)$ and for all $a \in I$ and $r \in R$, $r \circ a \in I$ [19]. The hyperring R is said to be *commutative*, if R is commutative with respect to hyperoperation "$\cdot$". A hyperring R has identity element 1, if, for all $r \in R$, it is satisfied that $r \in 1 \cdot r$.

Let $(R, \uplus, \circ)$ be a hyperring and $(M, +)$ be a hypergroup. If there exists an external hyperoperation $\cdot \colon R \times M \to P^*(M)$ such that, for all $a, b \in M$ and $r, s \in R$, we have $r \cdot (a + b) = (r \cdot a) + (r \cdot b)$ and $(r \circ s) \cdot a = r \cdot (s \cdot a)$, then $(M, +, \cdot)$ is called a *left hypermodule over* R [4]. Similarly, a right hypermodule over R can be defined. M is called a *hypermodule* over R, if it is both left and right hypermodule over R. If $(M, +)$ is a canonical hypergroup and $(R, \uplus, \circ)$ is a Krasner hyperring, then M is said to be a *canonical R-hypermodule*. In addition, M is called *Krasner R-hypermodule*, if it is a canonical R-hypermodule, where "$\cdot$" is an external operation, that is $\cdot \colon R \times M \to M$ by $(r, m) \to r \cdot m \in M$, and $m \cdot 0 = 0$. A non-empty subset N of an R-hypermodule M is called a *subhypermodule*, if N is a hypermodule over R. A hypermodule M is called *unitary* if $1 \cdot m = m$ for all $m \in M$. In this work, all R-hypermodules are left unitary Krasner R-hypermodules unless otherwise stated.

Throughout this work, we admit that every hypermodule M is a Krasner R-hypermodule thereby $\{0\}$ is a subhypermodule of M. We denote a subhypermodule N of M by $N \leq M$. It can be seen that for a Krasner R-hypermodule M and $N \leq M$, we can construct the *quotient Krasner R-hypermodule* M/N, endowed with $(a + N) \oplus (b + N) = \{c + N : c \in a + b\}$ and $r \odot (a + N) = r \cdot a + N$ for all $a + N,\ b + N \in \dfrac{M}{N}$ and $r \in R$. A hypermodule M is said to be *generated by* X such that $M = \cap Y$ for all subhypermodules $X \subset Y \subset M$. If X is finite set, then a hypermodule M is called a *finitely generated hypermodule.*

Let M_1 and M_2 be two R-hypermodules and $f \colon M_1 \to M_2$ be a function. Then f is called a *homomorphism* if $f(a + b) \subseteq f(a) + f(b)$ and $f(ra) = r \cdot f(a)$ for all $a, b \in M_1$ and $r \in R$. Also, f is called a *strong homomorphism* if $f(a + b) = f(a) + f(b)$ and $f(ra) = r \cdot f(a)$ for all $a, b \in M_1$ and $r \in R$. In this case, we have $f(0_{M_1}) = 0_{M_2}$. If a strong homomorphism f is one-to-one and surjective function, it is called a *strong isomorphism* [11].

A subhypermodule N of M is called a *maximal subhypermodule* if there is no proper subhypermodule of M contains N [1].

The Zariski topology was introduced primarily by Oskar Zariski. The Zariski topology is very different from the usual Euclidean topology on $\mathbb{R}^n$ or $\mathbb{C}^n$, which is defined by open sets with a union or finite intersection of basic open sets which are balls. In the usual Euclidean topology, the ball with radius $\varepsilon > 0$ centered at $x \in \mathbb{A}^n$ is

$$B(x,r) = \left\{ y \in \mathbb{A}^n \mid \sum_{i=1}^{n} |y_i - x_i| < \varepsilon \right\},$$

where $\mathbb{A}^n$ is an affine space. Let $\mathbb{F}$ be an algebraically closed field and $\mathbb{F}[X] = \mathbb{F}[X_1, \ldots, X_n]$ the polynomial ring in n variables over $\mathbb{F}$. A subset

$$Z = \{\, y \in \mathbb{A}^n \mid f_\alpha \in \mathbb{F}[X] \ \ \forall \alpha \in A \,\} \subset \mathbb{A}^n$$

is called an *algebraic set*, if it is the set of common zeros of some collection of polynomials. It also is written by $Z = Z(\{f_\alpha\}_{\alpha \in A})$. For a single nonconstant $f \in \mathbb{F}[X]$, the set $Z(f)$ is called a *hypersurface* in $\mathbb{A}^n$. In Zariski topology, open sets are obtained by the complements of hypersurfaces that are said principal open sets:

$$D(f) = \mathbb{A}^n - Z(f) = \{\, y \in \mathbb{A}^n \mid f(y) \neq 0 \,\}.$$

In other words, a topology is defined on A^n and it is called a *Zariski topology*. If Z is any algebraic set, then the Zariski topology on Z is the topology induced on it from $\mathbb{A}^n$. Closed (open) sets in $\mathbb{A}^n$ are intersections of $\mathbb{A}^n$ with closed (open) sets in $\mathbb{A}^n$ [18]. The Zariski topology is not Hausdorff; the Zariski closed sets in $\mathbb{A}^1$ are the empty set, finite collections of points in affine line and $\mathbb{A}^1$ itself. If $\mathbb{F}$ is infinite, the separation property of Hausdorff spaces fails when we meet any two nonempty open sets. As it turns out, thanks to this topology, tools from topology are used to study algebraic varieties which are the central objects of study in algebraic geometry. When there is an algebraic variety on complex numbers, the Zariski topology is coarser than the usual topology, since every algebraic set is closed for the usual topology [12]. In the literature, the Zariski topology was generalized for structures such as the spectrum of a ring.

In [16] a topology on $X = \operatorname{Spec}(M)$ is called the *Zariski topology*, in which closed sets are varieties $V(N) = \{P \in X : P \supseteq N\}$ for all submodules N of M. We transfer this topology to hypermodules structures, we construct Zariski topology on $X = \operatorname{Spec}(M)$, where M is a hypermodule and it consists of closed sets are varieties $V(N) = \{P \in X : P \supseteq N\}$ for all subhypermodules N of M. In other words, we obtain that the Zariski topology is a specific construction equipping $\operatorname{Spec}(M)$ with a topology. In [9, 15 – 17] the definition of Zariski topology on prime spectrum of a module M is different.

In this paper, we define the concept of multiplication hypermodules over a commutative hyperring with identity and we topologize the spectrum of multiplication hypermodules and investigate the properties of the induced topology. In addition we generalize the known results of Zariski topology on $\operatorname{Spec}(R)$ to $\operatorname{Spec}(M)$. We prove that $\operatorname{Spec}(M)$ is a T_0 space and it is compact if and only if M is finitely generated. In particular, we study cooperation between $\operatorname{Spec}(M)$ and $R/\operatorname{Ann}(N)$ and obtain some important results.

2. Multiplication hypermodules. In this section we obtain respectable properties of multiplication hypermodules. These results will be used frequently in the next section.

Definition 2.1. (i) *We call a unitary hypermodule M a multiplication R-hypermodule over a commutative with identity hyperring R, if for every subhypermodule N of M, there exists a hyperideal I of R such that $N = I \cdot M$.*

(ii) *We call a proper subhypermodule N of an R-hypermodule M a prime hypermodule if $r \cdot a \in N$ for all $r \in R$ and $a \in M$ then either $r \cdot M \subseteq N$ or $a \in N$. We will show that if N is a prime subhypermodule of M then $P = \mathrm{Ann}\,(M \setminus N)$ is necessarily a prime hyperideal of R. We concentrate upon the notion of prime subhypermodules and a new kind of Zariski topology on* $\mathrm{Spec}(M)$, *prime spectrum of a hypermodule M, containing the set of all prime subhypermodules of M.*

(iii) *Let M be an R-hypermodule and N be a subhypermodule of M such that $N = I \cdot M$ for some hyperideal I of R. Then, we call I a presentation hyperideal of N, or for short a presentation of N. Here, it is not always true that a hypermodule N has presentation. So it is easily proven that every subhypermodule of M has a presentation hyperideal if and only if M is a multiplication hypermodule.*

Now let us continue the section by classifying the properties of multiplication hypermodule.

Proposition 2.1. *Let M be a non-zero multiplication R-hypermodule. Then every proper subhypermodule of M is contained in a maximal subhypermodule of M.*

Proof. Let N be any proper subhypermodule of M. By the hypothesis, there exists an ideal I of R such that $N = I \cdot M$. We define a set Ψ of ideals in which all elements contain I. By Zorn's lemma, there exists a maximal element P of Ψ. So, $N = I \cdot M \subseteq P \cdot M$. It is shown easily that $P \cdot M$ is a maximal subhypermodule M.

Proposition 2.2. *Let M be a non-zero multiplication R-hypermodule. Then N is a maximal subhypermodule of M if and only if there exists a maximal hyperideal P of R such that $N = P \cdot M \neq M$.*

Proof. $(\Rightarrow)$ Clear by Proposition 2.1.

$(\Leftarrow)$ Let U be a subhypermodule of M contains N. Since M is multiplication R-hypermodule, there exists an ideal I of R such that $U = I \cdot M$. Since $N \subseteq U$, $P \cdot M \subseteq I \cdot M$ and using maximality, we have $P = I$. Hence, N is a maximal subhypermodule of M.

Theorem 2.1. *The following statements are equivalent for a proper subhypermodule N of a multiplication R-hypermodule M:*

(i) *N is a prime subhypermodule of M;*

(ii) $\mathrm{Ann}\,(M \setminus N)$ *is a prime hyperideal of R;*

(iii) *$N = P \cdot M$ for some prime hyperideal P of R with* $\mathrm{Ann}\,(M) \subseteq P$.

Proof. (i) $\Rightarrow$ (ii) Let N be a prime subhypermodule of M. It is easily shown that $\mathrm{Ann}\,(M \setminus N) = \{a \in R \colon a \cdot x = 0 \text{ for all } x \in M \setminus N\}$ is a hyperideal of R. Since N is a prime subhypermodule of M, either $a \cdot M \subseteq N$ or $x \in N$ for all $a \in R$ and $x \in N$. So, $a \cdot x = 0$ for all $x \in M \setminus N$, $a \in \mathrm{Ann}\,(M \setminus N)$. Hence, $\mathrm{Ann}\,(M \setminus N)$ is a prime hyperideal of R.

(ii) $\Rightarrow$ (iiii) Let $\mathrm{Ann}\,(M \setminus N)$ be a prime hyperideal of R. Suppose that $N = P \cdot M$ for a hyperideal P of R. So $a \cdot M \subseteq N$ for all $a \in P$. Consider the hyperideal $\mathrm{Ann}\,(M) = \{a \in R \colon a \cdot M = \{0\}\}$ of R. By the definition, $\mathrm{Ann}\,(M) \subseteq P$ and $P = \mathrm{Ann}\,(M \setminus N)$. Therefore, P is a prime hyperideal of R.

(iii) $\Rightarrow$ (i) Let $N = P \cdot M$ for some prime hyperideal P of R with $\mathrm{Ann}\,(M) \subseteq P$. Let $a \cdot x \in N$ for all $a \in R$ and $x \in M$. Since $N = P \cdot M$, then $a \in M$, $a \cdot M \subseteq P \cdot M = N$. Therefore, N is a prime subhypermodule of M.

Proposition 2.3. *Let N be a proper subhypermodule of a multiplication R-hypermodule M. Then N is a prime subhypermodule of M if and only if $AB \subseteq N$ implies $A \subseteq N$ or $B \subseteq N$, where A and B are subhypermodules of M.*

Proof. $(\Rightarrow)$ Let N be a prime subhypermodule of M and $A \cdot B \subseteq N$, but $A \nsubseteq N$ and $B \nsubseteq N$ for some subhypermodules A and B of M. Suppose that I and J are presentations of A and B, respectively. Then we have $A \cdot B = I \cdot (J \cdot M) \subseteq N$. Hence, there exist elements $s \cdot x \in A \setminus N$ and $s' \cdot x' \in A \setminus N$ for some $s \in I$ and $s' \in J$. Note that $s \cdot (s' \cdot x') \in N$. Hence, $s \cdot M \subseteq N$, that is, $s \cdot x \in N$, which is a contradition.

$(\Leftarrow)$ Let $r \cdot x \in N$ for some $r \in R$ and $x \in M \setminus N$, but $r \cdot M \nsubseteq N$. Then $r, m \notin H$ for some $m \in M$. Let I and J be presentation hyperideals of rx and m, respectively. Then

$$R \cdot (r \cdot x)\,(R \cdot m) = (R \cdot x)\,(R \cdot r \cdot m) = (I \cdot m)\,(J \cdot m) = I \cdot (J \cdot M) = (I \circ J) \cdot M \subseteq N.$$

By the hypothesis, we have $R \cdot x \subseteq N$ or $R \cdot r \cdot m \subseteq N$. It follows that $x \in P$ or $r \cdot x \in N$, which is a contradiction. Hence, N is a prime subhypermodule of M.

***Corollary* 2.1.** *Let N be a proper subhypermodule of M. Then N is a prime subhypermodule of M if and only if whenever $x, x^{'} \in M$, $xx' \subseteq N$ implies $x \in N$ or $x' \in N$.*

Proof. $(\Rightarrow)$ If N is a prime subhypermodule of M, it follows from $xx' \subseteq N$ that $x \in N$ or $x' \in N$ for every $x,\ x' \in M$.

$(\Leftarrow)$ Suppose that $xx' \subseteq N$ implies $x \in N$ or $x' \in N$, where $x,\ x' \in M$, and $AB \subseteq N$ for subhypermodules A and B of M, but $A \nsubseteq N$ and $B \nsubseteq N$. Thus there exist elements $a \in A \setminus N$ and $b \in B \setminus N$. Then $ab \in (R \cdot a)\,(R \cdot b) \subseteq AB \subseteq N$ and, hence, $a \in A$ or $b \in B$, which is a contradiction. Therefore, N is a prime subhypermodule of M.

3. Prime spectrum. In this section, we present the relationship of Zariski topology and multiplicative hypermodules. In order for authors to better understand this concept, we need the following definition.

Definition 3.1. *We denote by* $\operatorname{Spec}^*(M)$ *the collection of all prime subhypermodules of M. It is not always true that* $\operatorname{Spec}^*(M)$ *is not be empty for any hypermodule M. We call such a hypermodule primeless. It is clear that zero hypermodule is primeless. Now, we give a non-trival example.*

***Example* 3.1.** (i) Consider the Krasner hyperring $\mathbb{Z}$ and a multiplicatively closed subset S of $\mathbb{Z}$ such that $0 \notin S$. The equivalence relation $\sim$ is defined on the set $\mathbb{Z} \times S$ as follows: $(a, s) \sim (b, t)$ if and only if there exists $u \in S$ such that $uta = usb$. The equivalence class of (a, s) is denoted by a/s and $S^{-1}\mathbb{Z}$ be the quotient set. By using [8] (Example 3.1.3 (10)), we see that $S^{-1}\mathbb{Z}$ is a hyperring taking the hyperoperation of addition and multiplication. By [5] (Proposition 2.11), $S^{-1}P$ is a prime subhypermodule of $S^{-1}\mathbb{Z}$ where $P = \langle p \rangle$, p is a prime integer. So, $\operatorname{Spec}^*(S^{-1}\mathbb{Z})$ is non-trivial. Therefore the hypermodule $S^{-1}\mathbb{Z}$ is primeless.

(ii) Consider the Prüfer group $M = \mathbb{Z}(p^{\infty})$ for a prime integer p. Here the hypermodule M is $\mathbb{Z}$-hypermodule. Since $\left\langle \dfrac{1}{p^n} + \mathbb{Z} \right\rangle$ is a prime subhypermodule of M for every integer n, $\operatorname{Spec}^*(M)$ is non-trival.

(iii) Let $R = K[\{X_n\}_{n=1}^{\infty}]$ be a hyperring where K is a hyperfield and $I = (X_i^2, X_i - X_i X_j \mid j > > i \geq 1)$, $\overline{R} = R/I$ and $= (\{X_n\}_{n=1}^{\infty})/I$. Then it can be seen in a similarly way from [3] (Example 2.5) that M is a multiplication R-hypermodule.

We denote by $V^*(N)$ the set of all prime subhypermodule of M containing N. Here $V^*(N)$ is just the empty set and $V^*(\{0_M\})$ is $\operatorname{Spec}^*(M)$.

Let I be any index set and N_i be any subhypermodules of M. Then note that $\bigcap_{i\in I} V^*(N_i) = V^*\left(\sum_{i\in I} N_i\right)$. Thus, if $\xi^*(M)$ denotes the collection of all subsets $V^*(N)$ of $\operatorname{Spec}^*(M)$, then $\xi^*(M)$ contains the empty set and $\operatorname{Spec}^*(M)$ and is closed under arbitrary intersection. If also $\xi^*(M)$ is closed under finite union, i.e., for any subhypermodules H and N of M such that $V^*(H) \cup V^*(N) = V^*(J)$, in this case $\xi^*(M)$ satisfies the axioms of closed subsets of a topological space, which is Zariski topology.

Now we define a hypermodule with Zariski topology is called a *top** *hypermodule* in the following theorem and investigate the interesting properties of this topology. Firstly, we need the following definition.

Definition 3.2. (i) *We call the subhypermodule N of M a semiprime if N is as an intersection of prime subhypermodules of M, and the subhypermodule L of M an extraordinary if whenever U and V are semiprime subhypermodules of M with $U \cap V \subseteq L$, then $U \subseteq L$ or $V \subseteq L$, and the element m of a R-hypermodule M a unit provided that m is not contained in any maximal subhypermodule of M.*

(ii) *We call the set $r(N) = \{ x \in M \mid \exists n \in \mathbb{Z}^+ : x^n \in N \}$ radical hypermodule of N such that $r(N)$ is a subhypermodule of hypermodule M.*

Theorem 3.1. *Let M be an R-hypermodule. Then the following statements hold*:

(i) *M is a top** *hypermodule*;

(ii) *every prime subhypermodule of M is extraordinary*;

(iii) $V^*(N) \cup V^*(L) = V^*(N \cap L)$ *for some semiprime subhypermodules N, and L of M.*

Proof. Theorem can be proved similarly way in [17] (Lemma 2.1).

Proposition 3.1. *Let M be a multiplication R-hypermodule. Then $m \in M$ is unit if and only if $\langle m \rangle = M$.*

Proof. $(\Leftarrow)$ Is clear.

$(\Rightarrow)$ Suppose that an element m of M is unit. Then m is not contain in any maximal subhypermodule of M. So, every proper subhypermodule of M is contained in a maximal subhypermodule, a contradiction. Thus, we must have $\langle m \rangle = M$, by Proposition 2.1.

The following theorem shows that the Zariski topology is the topological space whose fundamental set is $\operatorname{Spec}^*(M)$.

Theorem 3.2. *Let M be a multiplication R-hypermodule. Then the following statements hold for a subset X of M, for subhypermodules N, L of M and, for every $i \in I$, a subhypermodule N_i of M*:

(i) $V^*(X) = V^*(\langle X \rangle)$;

(ii) $V^*(N) \cup V^*(L) = V^*(NL) = V^*(N \cap L)$;

(iii) $\bigcap_{i\in I} V^*(N_i) = V^*\left(\sum_{i\in I} N_i\right)$;

(iv) $V^*(N) = V^*(r(N))$;

(v) *if* $V^*(N) \subseteq V^*(L)$, *then* $L \subseteq r(N)$;

(vi) $V^*(N) = V^*(L)$ *if and only if* $r(N) = r(L)$;

(vii) $V^*(N) = \bigcup_{P\in V^{*R}(\operatorname{Ann}(M/N))} \operatorname{Spec}^{*P}(M)$, *where*

$$\mathrm{Spec}^{*P}(M) = \{P \in \mathrm{Spec}^*(M) : \mathrm{Ann}(M/P) = p\}.$$

Proof. (i) Since $X \subseteq \langle X \rangle$, we have $V^*(X) = V^*(\langle X \rangle)$. The converse follows that $\langle X \rangle$ is a smallest hypermodule contains X of M.

(ii) Since $NL \subseteq N$ and $NL \subseteq L$, then we have $V^*(N) \cup V^*(L) \subseteq V^*(NL)$. Conversely, suppose that $P \in V^*(NL)$. It follows from $NL \subseteq P$ that $N \subseteq P$ or $L \subseteq P$ by Proposition 2.3. Then $P \in V^*(N) \cup V^*(L)$. Thus, $V^*(NL) \subseteq V^*(N) \cup V^*(L)$. So, $V^*(N) \cup V^*(L) = V^*(NL)$. The second equality immediately follows from Theorem 3.1 (iii).

(iii) It is clear that $\bigcap_{i\in I} V^*(N_i) \supseteq V^*\left(\sum_{i\in I} N_i\right)$. For the converse inclusion, suppose that $P \in V^*\left(\sum_{i\in I} N_i\right)$. Then $\sum_{i\in I} N_i \subseteq P$. So, we have $N_i \subseteq P$ for every $i \in I$. Therefore, $P \in V^*(N_i)$ for every $i \in I$. Conversely, $V^*\left(\sum_{i\in I} N_i\right) \subseteq \bigcap_{i\in I} V^*(N_i)$.

(iv) It is clear that $H \subseteq r(H)$. Then we have $V^*(r(N)) \subseteq V^*(N)$. The converse inclusion is obvious.

(v) It is clear.

(vi) It follows from (iii).

(vii) Suppose that $P \in V^*(N)$. Since $N \subseteq P$, $\mathrm{Ann}(M/N) \subseteq \mathrm{Ann}(M/P) = P$. It follows from Propositions 2.1 and 2.2 that $P \in \mathrm{Spec}^{*P}(M)$ for $P \in V^{*R}(\mathrm{Ann}(M/N))$. Conversely, suppose that $P \in \mathrm{Spec}^{*P}(M)$ for $P \in V^{*R}(\mathrm{Ann}(M/N))$. Then we have $\mathrm{Ann}(M/N) \subseteq P$ and $P = \mathrm{Ann}(M/N)$ for some $P \in \mathrm{Spec}^*(M)$. Since $N = \mathrm{Ann}(M/N)M \subseteq \mathrm{Ann}(M/P) = P$, we get $P \in V^*(N)$.

***Corollary* 3.1.** *Every multiplication hypermodule is extraordinary.*

For each subset S of M, by $D(N)$ or X_N^* we mean $X - V^*(N)$. In particular, if $S = \{a\}$, we denote $D^*(a)$ by D_a^* or X_a^*. Here, the sets X_a^* are open, and they are called basic open sets and $X = \mathrm{Spec}^*(M)$.

Theorem 3.3. *Let M be a multiplication R-hypermodule. Then the following statements are hold*:

(i) $X_a^* \cap X_b^* = X_{ab}^*$;

(ii) $D^*(I \cdot M) = D^*(J \cdot M) = D^*(I \cdot (J \cdot M))$ *for every hyperideal I and J of R*;

(iii) $X_a^* = \varnothing \Leftrightarrow$ *a is nilpotent*;

(iv) $X_a^* = X \Leftrightarrow$ *a is unit in M.*

Proof. (i) It follows from Theorem 3.2 (ii).

(ii) By Proposition 2.3.

(iii) It is obtained using Corollary 2.1.

(iv) Clear by Propositions 2.1, 2.2 and 3.1.

In the following proposition, we have obtained a basis in the Zariski topology to built on multiplicative hypermodules.

Proposition 3.2. *Let M be a multiplication R-hypermodule. Then the sets $\{ X_a^* \mid a \in M \}$ form a basis for the Zariski topology on* $\mathrm{Spec}^*(M)$.

Proof. Suppose that A is an open set in $X = \mathrm{Spec}^*(M)$. Then there exists a subhypermodule N of M such that $A = X \setminus V^*(N)$. Let $\{a_i : i \in I\}$ be a generator set of N, that is, $N = \langle \{a_i : i \in I\} \rangle$. It follows from Theorem 3.2 (iii) that $V^*(N) = V^*(\langle \{a_i : i \in I\} \rangle) = V^*\left(\sum_{i\in I} R \cdot a_i\right) = \bigcap_{i\in I} V^*(a_i)$. Thus, $A = X \setminus V^*(N) = X \setminus \bigcap_{i\in I} V^*(a_i) = \bigcup_{i\in I} X(a_i)$. Consequently, $\{X_a^* : a \in M\}$ is a basis for the Zariski topology on X.

Compactness is a very important property of a topological space. For this reason, we will present following theorem and proposition that give compactness properties.

Theorem 3.4. *Let M be a multiplication R-hypermodule. Then every basic open set of X is compact.*

Proof. It is enough to show that every cover of basic open sets has a finite subcover. Suppose that $D^*(a) \subseteq \bigcup_{i\in I} D^*(a_i)$, and let N be the subhypermodule of M generated by a_i's. Note that $V^*(N) = \bigcap_{i\in I} V^*(a_i) \subseteq D^*(a)$. By Theorem 3.2 (iv) and Theorem 3.1 (iii), $V^*(r(N)) \subseteq \subseteq V^*(r(\langle a\rangle))$, and so $r(\langle a\rangle) \subseteq r(N)$. In addition, $N = \sqrt{B}\cdot M$, where $B = \operatorname{Ann}(M/N)$. Then $a = \sum_{\lambda\in\Lambda} s_\lambda \cdot a_\lambda,\ s_\lambda \in \sqrt{B}$, for a finite subset Λ of I. For $s_\lambda \in \sqrt{B}$, we have $s_\lambda^{\ell_\lambda} \in B$ for some positive integer ℓ_i. Let $\ell = \sum_{\lambda\in\Lambda} \ell_\lambda$, then $s_\lambda^{\ell} \in B$ for every $\lambda \in \Lambda$, that is, $R\cdot a_\lambda = I_\lambda \cdot M$. It follows from $a = \sum_{\lambda\in\Lambda} s_\lambda\cdot a_\lambda$ such that $a \in \sum_{\lambda\in\Lambda} s_\lambda\cdot I_\lambda\cdot M = \left(\sum_{\lambda\in\Lambda}(s_\lambda\cdot I_\lambda)\right)\cdot M$. Therefore, $a^\ell \subseteq \left(\sum_{\lambda\in\Lambda} s_\lambda\cdot I_\lambda\right)^\ell \cdot M \subseteq B\cdot M$. Since $\bigcap_{i\in I} V^*(a_i) = V^*(N) \subseteq V^*\left(a^\ell\right) = V^*(a)$, we have $X_a^* \subseteq \bigcup_{i\in I} X^*(f_i)$.

Corollary **3.2.** *Let M be a multiplication R-hypermodule. An open set X is compact if and only if it is finite union of basic open sets.*

Proof. It follows from Proposition 3.2 and Theorem 3.4.

Proposition 3.3. *Let M be a multiplication R-hypermodule. Then M is finitely generated if and only if X is compact.*

Proof. $(\Rightarrow)$ Since M is finitely generated, then $M = \langle a_1, a_2, \ldots, a_n\rangle$. Then we have $V^*(\langle a_1, a_2, \ldots, a_n\rangle) = \varnothing$. Hence, $D^*(a_1, a_2, \ldots, a_n) = X$. Since $\bigcup_{i=1}^{n} X_{a_i}^* = X$, X is compact.

$(\Leftarrow)$ Since X is compact, then $X = \bigcup_{i=1}^{n} X_{a_i}^*$ by Corollary 3.2. Then we have $V^*(\langle a_1, a_2, \ldots \ldots, a_n\rangle) = \bigcap_{i=1}^{n} V^*(a_i) = \varnothing$. It follows from Proposition 2.1 that $M = \langle a_1, a_2, \ldots, a_n\rangle$. So, M is finitely generated.

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