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ON THE GROWTH OF DERIVATIVES OF ALGEBRAIC POLYNOMIALS IN A WEIGHTED LEBESGUE SPACE

ПРО ЗРОСТАННЯ ПОХІДНИХ АЛГЕБРАЇЧНИХ ПОЛІНОМІВ У ВАГОВОМУ ПРОСТОРІ ЛЕБЕГА

We study growth rates of derivatives of an arbitrary algebraic polynomial in bounded and inbounded regions of the complex plane in weighted Lebesgue spaces.

Вивчається зростання похідних довільних алгебраїчних поліномів у обмежених і необмежених областях комплексної площини у вагових просторах Лебега.

1. Introduction. Let $\mathbb{C}$ be a complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region with boundary $L := \partial G$ (without loss of generality, let $0 \in G$); $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = \operatorname{ext} L$. For $t \in \mathbb{C}$ and $\delta > 0$, let $\Delta(t, \delta) := \{w \in \mathbb{C} \colon |w - t| > \delta\}$, $\Delta := \Delta(0, 1)$. Let $\Phi : \Omega \to \Delta$ be the univalent conformal mapping normalized by $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \dfrac{\Phi(z)}{z} > 0$, $\Psi := \Phi^{-1}$. For $R > 1$, we take $L_R := \{z \colon |\Phi(z)| = R\}$, $G_R := \operatorname{int} L_R$ and $\Omega_R := \operatorname{ext} L_R$.

Let $\wp_n$ denotes the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $\{z_j\}_{j=1}^{l} \in L$ be the fixed system of distinct points. For some fixed R_0, $1 < R_0 < \infty$, and $z \in \overline{G}_{R_0}$, consider generalized Jacobi weight function $h(z)$, which is defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^{l} |z - z_j|^{\gamma_j}, \tag{1}$$

where $\gamma_j > -1$ for all $j = 1, 2, \ldots, l$, and h_0 is uniformly separated from zero, i.e., there exists a constant $c_0(L) > 0$ such that, for all $z \in G_{R_0}$, $h_0(z) \geq c_0(L) > 0$.

For each $0 < p \leq \infty$ and rectifiable Jordan curve $L = \partial G$, we introduce

$$\|P_n\|_p := \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) \, |P_n(z)|^p \, |dz| \right)^{1/p} < \infty, \quad 0 < p < \infty, \tag{2}$$

$$\|P_n\|_\infty := \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|, \quad p = \infty,$$

$$\mathcal{L}_p(1, L) =: \mathcal{L}_p(L).$$

As is known, in the theory of approximations on the complex plane, a special place is occupied by the following well-known Bernstein – Walsh inequality [42]:

$$\|P_n\|_{C(\overline{G}_R)} \leq |\Phi(z)|^n \, \|P_n\|_{C(\overline{G})} \quad \forall P_n \in \wp_n. \tag{3}$$

So, for the points $z \in \overline{G}_{1+n^{-1}}$, the $\|P_n\|_\infty$ have the same order of growth in $\overline{G}_R$ and $\overline{G}$ with respect to n. An analogue of this inequality in space $\mathcal{L}_p(L)$ is the following inequality [28]:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq |\Phi(z)|^{n+\frac{1}{p}} \, \|P_n\|_{\mathcal{L}_p(L)} \quad \forall P_n \in \wp_n, \quad p > 0. \tag{4}$$

The estimate (4) has been generalized in [9] (Lemma 2.4) for weight function $h(z)$ defined as in (1) and was obtained

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \, \|P_n\|_{\mathcal{L}_p(h,L)}, \quad \gamma^* = \max\{0; \gamma_j : 1 \leq j \leq \, l\}. \tag{5}$$

If we replace the curve L with the region G and define two-dimensional analogs of the quantities (2) (we denote them by $\|P_n\|_{A_p(h,G)}$, $\|P_n\|_{A_p(1,G)}$ and $A_p(G)$, respectively), then for them we can also indicate the corresponding estimate of the type (5). For this, first of all, we will give the following definition.

For any $\delta > 0$ and arbitrary $t, w \in \mathbb{C}$, let $B(t,\delta) := \{t\colon\, |t-w| < \delta\}$ and $\varphi\colon G \to B := := B(0,1) := \{w\colon\, |w| < 1\}$ be a conformal and univalent map which is normalized by $\varphi(0) = 0$ and $\varphi'(0) > 0$, $\psi := \varphi^{-1}$.

Following to [36, p. 286], a bounded Jordan region G is called a κ-quasidisk, $0 \leq \kappa < 1$, if any conformal mapping ψ can be extended to a K-quasiconformal, $K = \dfrac{1+\kappa}{1-\kappa}$, homeomorphism of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$. In that case the curve $L := \partial G$ is called a κ -quasicircle. The region G (curve L) is called a quasidisk (quasicircle), if it is κ-quasidisk (κ-quasicircle) with some $0 \leq \kappa < 1$.

We denote this class as $Q(\kappa)$, $0 \leq \kappa < 1$, and say that $L = \partial G \in Q(\kappa)$, if $G \in Q(\kappa)$, $0 \leq \kappa < < 1$. Further, we denote that $G \in Q$ $(L \in Q)$, if $G \in Q(\kappa)$ $(L \in Q(\kappa))$ for some $0 \leq \kappa < 1$. It is well-known that quasicircles can be non-rectifiable (see, for example, [26; 29, p. 104]). Additionally, we say that $L \in \widetilde{Q}(\kappa), 0 \leq \kappa < 1$, if $L \in Q(\kappa)$ and L is rectifiable.

In [2] given an analog of the estimates (3) and (5) for the quasidisks and $h(z)$ defined as in (1) for the $\|P_n\|_{A_p(h,G)}$ as follows:

$$\|P_n\|_{A_p(h,G_R)} \leq c_1 {R^*}^{n+\frac{1}{p}} \, \|P_n\|_{A_p(h,G)}, \quad R > 1, \quad p > 0,$$

where $R^* := 1 + c_2(R-1)$, $c_2 > 0$ and $c_1 := c_1(G,p,c_2) > 0$ are constants, independent from n and R.

Further, for arbitrary Jordan region G and any $P_n \in \wp_n$ in [4] (Theorem 1.1) was proved that

$$\|P_n\|_{A_p(G_R)} \leq cR^{n+\frac{2}{p}} \, \|P_n\|_{A_p(G_{R_1})}, \quad p > 0,$$

is true for arbitrary $R > R_1 = 1 + \dfrac{1}{n}$, where $c = \left(\dfrac{2}{e^p - 1}\right)^{\frac{1}{p}} \left[1 + O\left(\dfrac{1}{n}\right)\right]$, $n \to \infty$, is asymptotically sharp constant.

N. Stylianopoulos in [39] replaced the norm $\|P_n\|_{C(\overline{G})}$ with norm $\|P_n\|_{A_2(G)}$ on the right-hand side of (3) and found a new version of the Bernstein – Walsh lemma: *Assume that L is quasiconformal and rectifiable. Then there exists a constant $c = c(L) > 0$ depending only on L such that*

$$|P_n(z)| \leq c\frac{\sqrt{n}}{d(z,L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega,$$

where $d(z,L) := \inf\{|\zeta - z| : \zeta \in L\}$, *holds for every* $P_n \in \wp_n$.

In this paper, we continue the study of the problem on uniform and pointwise estimates of the derivatives $|P_n'(z)|$ in bounded ($\overline{G_R}$) and unbounded (Ω_R) regions of the complex plane for each $R \geq 1$ and obtained estimates of the following type:

$$\left|P_n'(z)\right| \leq c_4 \|P_n\|_p \begin{cases} \lambda_n(L,h,p), & z \in \overline{G_R}, \\ \eta_n(L,h,p,z), & z \in \Omega_R, \end{cases} \tag{6}$$

where $\lambda_n(.)$ and $\eta_n(.) \to \infty$ as $n \to \infty$, depending on the properties of the L, h.

Analogous results of (6)-type for $|P_n(z)|$, different weight function h, in unbounded region were obtained in [5–17, 19, 20, 27, p. 418–428, 31, 35, 39].

Estimates of the (6)-type on points $z \in \overline{G}$ (also $z \in \overline{G_R}$), respect to norm $\|P_n\|_{\mathcal{L}_p(h,L)}$ or $\|P_n\|_{A_p(h,G)}$, $p > 0$, for some $h(z) \equiv 1$ or $h(z) \neq 1$ was studied since the beginning of the 20th century in, for example, [25, 40], and has been studied by in [2, 3, 18, 22, 23, 24, 27, p. 418–428, 31], [32] (Sect. 5.3), [33, 34, p. 122–133, 38] (see also the references cited therein).

2. Definitions and main results. Throughout this paper, $c, c_0, c_1, c_2, \ldots$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. For any $k \geq 0$ and $m > k$, notation $i = \overline{k,m}$ means $i = k, k+1, \ldots, m$.

In this work, we will try to get the result for more general curves, also including the above class of curves. For this we need to give the following definitions of regions with some general functional condition.

Definition. *We say that* $L = \partial G = \partial\Omega \in Q_\alpha$, *if* L *is a quasicircle and* $\Phi \in H^\alpha(\overline{\Omega})$ *for some* $0 < \alpha \leq 1$, *i.e.,* $|\Phi(w) - \Phi(\tau)| \leq M|w - \tau|^\alpha, 0 < \alpha \leq 1$, *for all* $|w| \geq 1, |\tau| \geq 1$, *and* $M > 0$ *constant independent of* w *and* τ.

Additionally, we say that $L \in \widetilde{Q}_\alpha, 0 < \alpha \leq 1$, if $L \in Q_\alpha$ and L is rectifiable.

We note that the class Q_α is sufficiently large. A detailed account on it and the related topics are contained in [30, 37, 41] (see also the references cited therein). We consider only some cases:

a) If L is a piecewise Dini-smooth curve and largest exterior angle on L has opening $\alpha\pi$, $0 < \alpha \leq 1$, [37, p. 52], then $L \in \widetilde{Q}_\alpha$.

b) If $L =: \partial G$ is a smooth curve having continuous tangent line, then $L \in \widetilde{Q}_\alpha$ for all $0 < \alpha < 1$.

c) If G is "L-shaped" region, then $L \in \widetilde{Q}_{\frac{2}{3}}$.

d) If L is quasismooth (in the sense of Lavrentiev), then $L \in \widetilde{Q}_\alpha$ for $\alpha = \dfrac{1}{2}\left(1 - \dfrac{1}{\pi}\arcsin\dfrac{1}{c}\right)^{-1}$ and $c > 1$ [41].

e) If L is "c-quasiconformal", then $L \in Q_\alpha$ for $\alpha = \dfrac{\pi}{2\left(\pi - \arcsin\dfrac{1}{c}\right)}$; if L is an asymptotic conformal curve, then $L \in Q_\alpha$ for all $0 < \alpha < 1$ [30].

For $0 < \delta_j < \delta_0 := \dfrac{1}{4}\min\{|z_i - z_j| : i, j = 1, 2, \ldots, l, \ i \neq j\}$, let $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$, $\delta := \min_{1 \leq j \leq l}\delta_j$. For $L = \partial G$, we set $U_\infty(L,\delta) := \bigcup_{\zeta \in L} U(\zeta, \delta)$ is infinite open cover of the curve L;

$U_N(L,\delta) := \bigcup_{j=1}^{N} U_j(L,\delta) \subset U_\infty(L,\delta)$ is finite open cover of the curve L; $\Omega(\delta) := \Omega(L,\delta) :=$ $:= \Omega \cap U_N(L,\delta), \widehat{\Omega} := \Omega \backslash \Omega(\delta)$; $\Omega_R(\delta) := \Omega(L_R,\delta) := \Omega_R \cap U_N(L_R,\delta)$, $\widehat{\Omega}_R := \Omega_R \backslash \Omega_R(\delta)$.

Now, we start to formulate the new results.

Theorem 1. *Let $p > 1$, $L \in \widetilde{Q}_\alpha$ for some $\frac{1}{2} \le \alpha \le 1$, and $h(z)$ be defined by* (1). *Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have*

$$\left|P_n'(z)\right| \le c_1 \left|\Phi^{n+1}(z)\right| \left\{ \frac{\|P_n\|_p}{d(z,L)} A_{n,p}^1(z) + B_{n,1}^1(z) \left|P_n(z)\right| \right\}, \tag{7}$$

where $c_1 = c_1(L,\gamma,p) > 0$ is constant independent from n and z;

$$A_{n,p}^1(z) := \begin{cases} n^{\frac{\gamma^*+1}{\alpha p}}, & 1 < p < 1 + \dfrac{\gamma^*+1}{\alpha}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \dfrac{\gamma^*+1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1 + \dfrac{\gamma^*+1}{\alpha}, \end{cases} \qquad B_{n,1}^1(z) := n^{\frac{1}{\alpha}}, \quad \textit{if} \quad z \in \Omega(\delta);$$

$$A_{n,p}^1(z) := \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & 1 < p < 1 + \dfrac{\gamma}{1+\alpha}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \dfrac{\gamma}{1+\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1 + \dfrac{\gamma^*}{1+\alpha}, \end{cases} \qquad B_{n,1}^1(z) := n, \quad \textit{if} \quad z \in \widehat{\Omega}(\delta),$$

and $\gamma^* := \max\{0;\gamma\}$.

Theorem 2. *Let $p > 1$, $L \in \widetilde{Q}_\alpha$ for some $\frac{1}{2} \le \alpha \le 1$, and $h(z)$ be defined by* (1). *Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and $z \in \Omega_R$, we have*

$$\left|P_n(z)\right| \le c_2 \frac{\left|\Phi^{n+1}(z)\right|}{d(z,L)} A_{n,p}^2 \|P_n\|_p, \tag{8}$$

where $c_2 = c_2(L,\gamma,p) > 0$ is constant independent from n and z;

$$A_{n,p}^2 := \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & 1 < p < 1 + \dfrac{\gamma}{1+\alpha}, & \gamma > 1+\alpha, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \dfrac{\gamma}{1+\alpha}, & \gamma > 1+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1 + \dfrac{\gamma}{1+\alpha}, & \gamma > 1+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1, & -1 < \gamma \le 1+\alpha. \end{cases}$$

Now, from Theorems 1 and 2, we can give an estimates for the $|P_n'(z)|$ for $z \in \Omega_R$.

Let $p > 1$, $L \in \widetilde{Q}_\alpha$ for some $\frac{1}{2} \le \alpha \le 1$, and $h(z)$ be defined by (1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have

$$\left|P_n'(z)\right| \le c_3 \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} \|P_n\|_p A_{n,p}^3(z), \tag{9}$$

where $c_3 = c_3(L,\gamma,p) > 0$ is constant independent from n and z;

$$A^3_{n,p}(z) := \begin{cases} n^{\frac{\gamma+1}{p\alpha}}, & 1 < p < 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 0, \\ n^{1-\frac{1}{p}+\frac{1}{\alpha}} (\ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 0, \\ n^{1-\frac{1}{p}+\frac{1}{\alpha}}, & p > 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 0, \\ n^{1-\frac{1}{p}+\frac{1}{\alpha}}, & p > 1, \quad -1 < \gamma \leq 0, \end{cases} \quad \text{if} \quad z \in \Omega(\delta);$$

$$A^3_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}+1}, & 1 < p < 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 1+\alpha, \\ n^{2-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 1+\alpha, \\ n^{2-\frac{1}{p}}, & p > 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 1+\alpha, \\ n^{2-\frac{1}{p}}, & p > 1, \quad -1 < \gamma \leq 1+\alpha, \end{cases} \quad \text{if} \quad z \in \widehat{\Omega}(\delta).$$

Now, we will give estimate for $|P'_n(z)|$ for bounded regions of the class $\widetilde{Q}_\alpha$.

Theorem 3. *Let $p > 1,\ L \in \widetilde{Q}_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by* (1). *Then, for any $P_n \in \wp_n,\ n \in \mathbb{N}$, we have*

$$\left\|P'_n\right\|_\infty \leq c_4 n^{\left(\frac{\gamma^*+1}{p}+1\right)\frac{1}{\alpha}} \left\|P_n\right\|_p, \tag{10}$$

where $c_4 = c_4(L,\gamma,p) > 0$ is constant independent from n and z.

***Remark* 1.** The inequalities (10) is sharp.

According to (3) (applied to the polynomial $Q_{n-1}(z) := P'_n(z)$), the estimation (10) is true also for the points $z \in \overline{G}_{1+\varepsilon_0 n^{-1}}$ with a different constant. Therefore, combining estimations (9) and (10) (for the $z \in \overline{G}_R$,) we will obtain estimation on the growth of $|P'_n(z)|$ in the whole complex plane.

Theorem 4. *Let $p > 1,\ L \in \widetilde{Q}_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by* (1). *Then, for any $P_n \in \wp_n,\ n \in \mathbb{N}$, we have*

$$\left|P'_n(z)\right| \leq c_5 \left\|P_n\right\|_p \begin{cases} n^{\left(\frac{\gamma^*+1}{p}+1\right)\frac{1}{\alpha}}, & z \in \overline{G}_R, \\ \dfrac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} A^3_{n,p}(z), & z \in \Omega_R, \end{cases}$$

where $c_5 = c_5(L,\gamma,p) > 0$ is constant independent from n and z, $A^3_{n,p}(z)$ defined as in Theorem 2 *for each $z \in \Omega_R$.*

3. Some auxiliary results. Throughout this paper we denote "$a \preceq b$" and "$a \asymp b$" are equivalent to $a \leq cb$ and $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2, respectively.

Lemma 1 [1]. *Let G be a quasidisk, $z_1 \in L,\ z_2, z_3 \in \Omega \cap \{z\colon |z-z_1| \preceq d(z_1, L_{r_0})\},\ w_j = \Phi(z_j),\ j = 1,2,3$. Then*:

a) *The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. So, are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.*

b) *If $|z_1 - z_2| \preceq |z_1 - z_3|$, then*

$$\left|\frac{w_1-w_3}{w_1-w_2}\right|^{c_1} \preceq \left|\frac{z_1-z_3}{z_1-z_2}\right| \preceq \left|\frac{w_1-w_3}{w_1-w_2}\right|^{c_2},$$

where $0<r_0<1$ *a constant, depending on* G.

***Corollary* 1.** *Under the conditions of Lemma* 1, *we have*

$$|w_1-w_2|^{c_1} \preceq |z_1-z_2| \preceq |w_1-w_2|^{\varepsilon},$$

where $\varepsilon=\varepsilon(G)<1$.

Lemma 2. *Let* $L\in Q_\alpha$ *for some* $\dfrac{1}{2}\leq\alpha\leq 1$. *Then, for all* $w_1,w_2\in\overline{\Omega}\,'$, *we have*

$$|\Psi(w_1)-\Psi(w_2)| \succeq |w_1-w_2|^{\frac{1}{\alpha}}.$$

This fact it follows from of an appropriate result for the mapping $f\in\sum(\kappa)$ [36, p. 287] and estimation for the Ψ' [21] (Theorem 2.8):

$$d(\Psi(\tau),L) \asymp \left|\Psi'(\tau)\right|(|\tau|-1). \tag{11}$$

4. Proof of theorems. ***Proof of Theorem* 1.** Let $L\in\widetilde{Q}_\alpha$, $\dfrac{1}{2}\leq\alpha\leq 1$, $0<\beta\leq 1$ and $R=1+\dfrac{1}{n}$, $R_1:=1+\dfrac{R-1}{2}$. For $z\in\Omega$, let us set

$$H_n(z):=\frac{P_n(z)}{\Phi^{n+1}(z)}.$$

Let us represent the derivative of $H_n(z)$ as follows:

$$H_n'(z)=\frac{P_n'(z)}{\Phi^{n+1}(z)}+P_n(z)\left(\Phi^{-(n+1)}(z)\right)', \quad z\in\Omega.$$

Then

$$\left|P_n'(z)\right| \leq \left|\Phi^{n+1}(z)\right|\left\{\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)'\right|+|P_n(z)|\left|\left(\Phi^{-(n+1)}(z)\right)'\right|\right\}. \tag{12}$$

Therefore, to estimate $|P_n'(z)|$ we must evaluate

$$\text{A)}\ \left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)'\right| \quad \text{and} \quad \text{B)}\ \left|\left(\Phi^{-(n+1)}(z)\right)'\right|, \quad z\in\Omega.$$

A. Since the function $H_n(z):=\dfrac{P_n(z)}{\Phi^{n+1}(z)}$, $H_n(\infty)=0$, is analytic in Ω, continuous on $\overline{\Omega}$, then Cauchy integral representation for the derivatives gives

$$H_n'(z)=-\frac{1}{2\pi i}\int\limits_{L_{R_1}} H_n(\zeta)\frac{d\zeta}{(\zeta-z)^2}, \quad z\in\Omega_R.$$

Then

$$\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)'\right| \le \frac{1}{2\pi}\int\limits_{L_{R_1}}\left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta-z|^2} \le \frac{1}{2\pi d(z,L_{R_1})}\int\limits_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|}. \tag{13}$$

Denote by

$$A_n(z) := \int\limits_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|}, \tag{14}$$

and estimate this integral. For this we give some notations.

Let $w_j := \Phi(z_j), \varphi_j := \arg w_j$. Without loss of generality, we will assume that $\varphi_l < 2\pi$. For $\eta := \min\left\{\eta_j, j=\overline{1,l}\right\}$, where $\eta_j = \min_{t\in\partial\Phi(\Omega(z_j,\delta_j))}|t-w_j| > 0$, let us set

$$\Delta(\eta_j) := \{t\colon\ |t-w_j|\le\eta_j\}\subset\Phi(\Omega(z_j,\ \delta_j)),$$

$$\Delta(\eta) := \bigcup_{j=1}^{l}\Delta_j(\eta),\quad \widehat{\Delta}_j = \Delta\backslash\Delta(\eta_j),\quad \widehat{\Delta}(\eta) := \Delta\backslash\Delta(\eta),\quad \Delta_1' := \Delta_1'(1),$$

$$\Delta_1'(\rho) := \left\{t = Re^{i\theta}\colon R\ge\rho>1,\ \frac{\varphi_0+\varphi_1}{2}\le\theta<\frac{\varphi_1+\varphi_2}{2}\right\},$$

$$\Delta_j' := \Delta_j'(1),\quad \Delta_j'(\rho) := \left\{t = Re^{i\theta}\colon R\ge\rho>1,\ \frac{\varphi_{j-1}+\varphi_j}{2}\le\theta<\frac{\varphi_j+\varphi_0}{2}\right\},\ \ j=2,3,\ldots,l,$$

where $\varphi_0 = 2\pi-\varphi_l$, $\Omega_j := \Psi(\Delta_j')$, $L_{R_1}^j := L_{R_1}\cap\Omega_j$, $\Omega = \bigcup_{j=1}^{l}\Omega_j$.

For simplicity of calculations, we can limit ourselves to only one point on the boundary, which the weight function has singularity, i.e., let $h(z)$ be defined as in (1) for $l=1$ and we put $\gamma_1 =: \gamma$. To estimate $A_n(z)$, multiplying the numerator and denominator of the integrand by $h^{\frac{1}{p}}(\zeta)$ and applying the Hölder inequality and (5), we obtain

$$A_n(z) = \int\limits_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|} = \int\limits_{L_{R_1}}h^{\frac{1}{p}}(\zeta)\,|P_n(\zeta)|\frac{|d\zeta|}{h^{\frac{1}{p}}(\zeta)\,|\zeta-z|} \le$$

$$\le \|P_n\|_p\left(\int\limits_{L_{R_1}}\frac{|d\zeta|}{h^{\frac{q}{p}}(\zeta)\,|\zeta-z|^q}\right)^{\frac{1}{q}},\quad \frac{1}{p}+\frac{1}{q}=1.$$

Replacing the variable $\tau = \Phi(\zeta)$, we get

$$A_n(z) \preceq \|P_n\|_p\left(\int\limits_{|\tau|=R_1}\frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}\,|\Psi(\tau)-\Psi(w)|^q}\right)^{\frac{1}{q}},\quad z=\Psi(w). \tag{15}$$

To estimate the integral on the right-hand side, we put

$$F_{R_1}^1 := \Phi(L_{R_1}) := \Delta_1'\cap\{\tau\colon\ |\tau|=R_1\},$$

$$E_{R_1}^{11} := \left\{\tau : \tau \in F_{R_1}^1,\ |\tau - w_1| < c_1(R_1 - 1)\right\},$$
$$E_{R_1}^{12} := \left\{\tau : \tau \in F_{R_1}^1,\ c_1(R_1 - 1)\ \le |\tau - w_1| < \eta\right\},$$
$$E_{R_1}^{13} := \{\tau : \tau \in \Phi(L_{R_1}), \eta \le |\tau - w_1| < \eta^*\},$$

where $0 < c_1 < \eta$ is chosen so that $\{\tau\colon |\tau - w_1| < c_1(R_1 - 1)\} \cap \Delta \neq \varnothing$ and $\Phi(L_{R_1}) = \bigcup_{k=1}^3 E_{R_1}^{1k}$. Taking into consideration these notations, from (15) we have

$$A_n(z) \preceq \|P_n\|_p \sum_{k=1}^{3} J_n^k(z), \tag{16}$$

where

$$\left(J_n^k(z)\right)^q := \int\limits_{E_{R_1}^{1k}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}\,|\Psi(\tau) - \Psi(w)|^q}, \quad k = 1, 2, 3.$$

For any $k = 1, 2$, denote by

$$E_{R_1,1}^{1k} := \left\{\tau \in E_{R_1}^{1k} :\ |\Psi(\tau) - \Psi(w_1)| \ge |\Psi(\tau) - \Psi(w)|\right\}, \quad E_{R_1,2}^{1k} := E_{R_1}^{1k} \backslash E_{R_1,1}^{1k},$$

$$\left(I(E_{R_1,1}^{1k})\right)^q := \begin{cases} \displaystyle\int_{E_{R_1,1}^{1k}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+q}}, & \text{if} \quad \gamma \ge 0, \\ \displaystyle\int_{E_{R_1,1}^{1k}} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma)(q-1)}\,|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau) - \Psi(w)|^q}, & \text{if} \quad \gamma < 0, \end{cases} \tag{17}$$

$$\left(I(E_{R_1,2}^{1k})\right)^q := \int\limits_{E_{R_1,2}^{1k}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)+q}}, \quad k = 1, 2,$$

and estimate the last integrals. Given the possible values γ ($-1 < \gamma < 0$ and $\gamma \ge 0$), we will consider the cases separately.

1. Let $\gamma \ge 0$. If $z \in \Omega(\delta)$, applying Lemma 2 and (11), we get

$$\begin{gathered} \left(I(E_{R_1,1}^{11})\right)^q \preceq n \int\limits_{E_{R_1,1}^{11}} \frac{|d\tau|}{|\tau - w|^{\frac{[\gamma(q-1)+q-1]}{\alpha}}} \preceq n^{1+\frac{\gamma(q-1)+q-1}{\alpha}} \operatorname{mes} E_{R_1,1}^{11} \preceq n^{\frac{\gamma(q-1)+q-1}{\alpha}}, \\ I(E_{R_1,1}^{11}) \preceq n^{\frac{\gamma+1}{p\alpha}}, \\ \left(I(E_{R_1,2}^{11})\right)^q \preceq n \int\limits_{E_{R_1,2}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)+q-1}{\alpha}}} \preceq n^{1+\frac{\gamma(q-1)+q-1}{\alpha}} \operatorname{mes} E_{R_1,2}^{11} \preceq n^{\frac{\gamma(q-1)+q-1}{\alpha}}, \\ I(E_{R_1,2}^{11}) \preceq n^{\frac{\gamma+1}{p\alpha}}, \end{gathered} \tag{18}$$

$$\left(I(E^{12}_{R_1,1})\right)^q \preceq \int\limits_{E^{12}_{R_1,1}} \frac{d(\Psi(\tau), L)\,|d\tau|}{(|\tau|-1)\,|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+q}} \preceq$$

$$\preceq n \begin{cases} n^{\frac{\gamma(q-1)+q-1}{\alpha}-1}, & [\gamma(q-1)+q-1] > \alpha, \\ \ln n, & [\gamma(q-1)+q-1] = \alpha, \\ 1, & [\gamma(q-1)+q-1] < \alpha, \end{cases} \tag{19}$$

$$I(E^{12}_{R_1,1}) \preceq \begin{cases} n^{\frac{\gamma+1}{\alpha p}}, & p < 1+\dfrac{\gamma+1}{\alpha}, \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma+1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{\gamma+1}{\alpha}, \end{cases}$$

$$\left(I(E^{12}_{R_1,2})\right)^q \preceq n \begin{cases} n^{\frac{\gamma(q-1)+q-1}{\alpha}-1}, & [\gamma(q-1)+q-1] > \alpha, \\ \ln n, & [\gamma(q-1)+q-1] = \alpha, \\ 1, & [\gamma(q-1)+q-1] < \alpha, \end{cases} \tag{20}$$

$$I(E^{12}_{R_1,2}) \preceq \begin{cases} n^{\frac{\gamma+1}{\alpha p}}, & p < 1+\dfrac{\gamma+1}{\alpha}, \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma+1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{\gamma+1}{\alpha}. \end{cases}$$

For $\tau \in E^{13}_{R_1}$ we see that $\eta < |\tau - w_1| < 2\pi \dot{R}_1$, $|\tau - w| \geq \eta - c_1$. Therefore, $|\Psi(\tau) - \Psi(w_1)| \succeq 1$ from Lemma 1 and, for $|\tau - w_1| \geq \eta$, $|\Psi(\tau) - \Psi(w)| \succeq |\tau - w|^{\frac{1}{\alpha}}$ from Lemma 2. Then, for $w \in \Delta(w_1, \eta)$, applying (11), we have

$$\left(J^3_2(z)\right)^q \preceq \int\limits_{E^{13}_{R_1}} \frac{d(\Psi(\tau), L)\,|d\tau|}{(|\tau|-1)\,|\Psi(\tau)-\Psi(w)|^q} \preceq \begin{cases} n^{\frac{q-1}{\alpha}}, & q-1 > \alpha, \\ n\ln n, & q-1 = \alpha, \\ n, & q-1 < \alpha, \end{cases} \preceq n \int\limits_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau - w|^{\frac{q-1}{\alpha}}},$$

$$J^3_2(z) \preceq \begin{cases} n^{\frac{1}{p\alpha}}, & p < 1+\dfrac{1}{\alpha}, \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{1}{\alpha}, \quad z \in \Omega(\delta), \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{1}{\alpha}, \end{cases} \tag{21}$$

$$J^3_2(z) \preceq n^{\left(\frac{1}{\alpha}-1\right)\left(1-\frac{1}{p}\right)}, \quad z \in \widehat{\Omega}(\delta).$$

Combining (18)–(21), for $p > 1, \gamma \geq 0$ and $z \in \Omega(\delta)$, we get

$$\sum_{k=1}^{3} J_n^k(z) \preceq \begin{cases} n^{\frac{\gamma+1}{\alpha p}}, & 1 < p < 1+\dfrac{\gamma+1}{\alpha}, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma+1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{\gamma+1}{\alpha}. \end{cases} \tag{22}$$

If $z \in \widehat{\Omega}(\delta)$, then

$$\left(J_2^1(z)\right)^q \preceq n \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)-1}{\alpha}}} \preceq n^{1+\frac{\gamma(q-1)-1}{\alpha}} \operatorname{mes} E_{R_1}^{11} \preceq n^{\frac{\gamma(q-1)-1}{\alpha}}, \quad J_2^1(z) \preceq n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, \tag{23}$$

$$\left(J_2^2(z)\right)^q \preceq n \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)-1}{\alpha}}} \preceq \begin{cases} n^{\frac{\gamma(q-1)-1}{\alpha}}, & \gamma(q-1)-1 > \alpha, \\ n \ln n, & \gamma(q-1)-1 = \alpha, \\ n, & \gamma(q-1)-1 < \alpha, \end{cases}$$

$$J_2^2(z) \preceq \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & 1 < p < 1+\dfrac{\gamma}{1+\alpha}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma}{1+\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{\gamma}{1+\alpha}, \end{cases} \tag{24}$$

$$\left(J_2^3(z)\right)^q \preceq \int\limits_{E_{R_1}^{13}} \frac{d(\Psi(\tau), L)}{|\tau| - 1} |d\tau| \preceq n^{\left(\frac{1}{\alpha}-1\right)}, \quad J_2^3(z) \preceq n^{\left(\frac{1}{\alpha}-1\right)\left(1-\frac{1}{p}\right)},$$

and, from (16) – (24), we obtain

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\frac{\gamma+1}{\alpha p}}, & 1 < p < 1+\dfrac{\gamma+1}{\alpha}, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma+1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{\gamma+1}{\alpha}, \end{cases} \quad \text{if} \quad z \in \Omega(\delta), \tag{25}$$

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & 1 < p < 1+\dfrac{\gamma}{1+\alpha}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma}{1+\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{\gamma}{1+\alpha}, \end{cases} \quad \text{if} \quad z \in \widehat{\Omega}(\delta). \tag{26}$$

2. If $\gamma < 0$, for $w \in \Delta(w_1, \eta) \cap \Omega_R(\delta)$ such that $|\Psi(\tau) - \Psi(w_1)| \preceq |\Psi(\tau) - \Psi(w)|$, according to Lemma 1, analogously we have

$$\left(I(E^{11}_{R_1,1})\right)^q \preceq n \int\limits_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\frac{q-1}{\alpha}}} \preceq n^{1+\frac{q-1}{\alpha}} \operatorname{mes} E^{11}_{R_1} \preceq n^{\frac{q-1}{\alpha}}, \quad I(E^{11}_{R_1,1}) \preceq n^{\frac{1}{p\alpha}}, \tag{27}$$

$$\left(I(E^{11}_{R_1,2})\right)^q \preceq n \int\limits_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma(q-1)+q-1}{\alpha}}} \preceq n^{1+\frac{\gamma(q-1)+q-1}{\alpha}} \operatorname{mes} E^{11}_{R_1,2} \preceq n^{\frac{\gamma(q-1)+q-1}{\alpha}},$$

$$I(E^{11}_{R_1,2}) \preceq n^{\frac{\gamma+1}{p\alpha}}. \tag{28}$$

For $\tau \in E^{12}_{R_1}$ we see that $|\tau - w_1| < \eta$ and, from Lemma 1, $|\Psi(\tau) - \Psi(w_1)| \preceq 1$. Then, for $w \in \Delta(w_1,\eta) \cap \Omega_R(\delta)$ such that $|\Psi(\tau) - \Psi(w_1)| \preceq |\Psi(\tau) - \Psi(w)|$, applying Lemma 2, we get

$$\left(I(E^{12}_{R_1,1})\right)^q \preceq n \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\frac{q-1}{\alpha}}} \preceq \begin{cases} n^{\frac{q-1}{\alpha}}, & q-1>\alpha, \\ (n\ln n)^{1-\frac{1}{p}}, & q-1=\alpha, \\ n^{1-\frac{1}{p}}, & q-1<\alpha, \end{cases}$$

$$I(E^{12}_{R_1,1}) \preceq \begin{cases} n^{\frac{1}{\alpha p}}, & 1<p<1+\dfrac{1}{\alpha}, \\ (n\ln n)^{1-\frac{1}{p}}, & p=1+\dfrac{1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p>1+\dfrac{1}{\alpha}, \end{cases} \tag{29}$$

$$\left(I(E^{12}_{R_1,2})\right)^q \preceq n \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\tau-w|^{\frac{q+\gamma(q-1)-1}{\alpha}}} \preceq \begin{cases} n^{\frac{q+\gamma(q-1)-1}{\alpha}}, & q+\gamma(q-1)-1>\alpha, \\ n\ln n, & q+\gamma(q-1)-1=\alpha, \\ n, & q+\gamma(q-1)-1<\alpha, \end{cases}$$

$$I(E^{12}_{R_1,2}) \preceq \begin{cases} n^{\frac{\gamma+1}{p\alpha}}, & 1<p<1+\dfrac{\gamma+1}{\alpha}, \\ (n\ln n)^{1-\frac{1}{p}}, & p=1+\dfrac{\gamma+1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p>1+\dfrac{\gamma+1}{\alpha}. \end{cases} \tag{30}$$

For $\tau \in E^{13}_{R_1}$ and each $w \in \Delta(w_1,\eta) \cap \Omega_R(\delta)$ we see that $\eta < |\tau - w_1| < 2\pi R_1$. Therefore, from Lemma 1 and applying (11), we obtain

$$\left(I(E^{13}_{R_1})\right)^q \preceq n \int\limits_{E^{13}_{R_1}} \frac{|d\tau|}{|\tau-w|^{\frac{q-1}{\alpha}}} \preceq \begin{cases} n^{\frac{q-1}{\alpha}}, & q-1>\alpha, \\ (n\ln n), & q-1=\alpha, \\ n, & q-1<\alpha, \end{cases}$$

$$I(E^{13}_{R_1}) \preceq \begin{cases} n^{\frac{1}{\alpha p}}, & 1<p<1+\dfrac{1}{\alpha}, \\ (n\ln n)^{1-\frac{1}{p}}, & p=1+\dfrac{1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p>1+\dfrac{1}{\alpha}. \end{cases} \tag{31}$$

Therefore, combining (27)–(31), in case of $\gamma < 0$ for $z \in \Omega(\delta)$, we have

$$\sum_{k=1}^{3} J_n^k(z) \preceq \begin{cases} n^{\frac{1}{\alpha p}}, & 1 < p < 1 + \dfrac{1}{\alpha}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \dfrac{1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1 + \dfrac{1}{\alpha}. \end{cases} \tag{32}$$

If $z \in \widehat{\Omega}(\delta)$, then $|w - w_1| \geq \eta$. From Lemma 2 and from (11), we get

$$\left(J_2^1(z)\right)^q \preceq n \int\limits_{E_{R_1}^{11}} |\Psi(\tau) - \Psi(w_1)|^{(-\gamma)(q-1)+1} |d\tau| \preceq n \operatorname{mes} E_{R_1}^{11} \preceq 1, \quad J_2^1(z) \preceq 1,$$

$$\left(J_2^2(z)\right)^q \preceq \int\limits_{E_{R_1}^{12}} \frac{d(\Psi(\tau), L)}{|\tau| - 1} |d\tau| \preceq n, \quad J_2^2(z) \preceq n^{1-\frac{1}{p}}, \tag{33}$$

$$\left(J_2^3(z)\right)^q \preceq \int\limits_{E_{R_1}^{13}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\tau| - 1} \preceq n, \quad J_2^3(z) \preceq n^{1-\frac{1}{p}}.$$

Therefore, combining the last tree estimates, in case of $\gamma < 0$, for $z \in \widehat{\Omega}(\delta)$, we obtain

$$\sum_{k=1}^{3} J_n^k(z) \preceq n^{1-\frac{1}{p}}. \tag{34}$$

Then, for $-1 < \gamma < 0$, from (16), (32) and (34), we have

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\frac{1}{\alpha p}}, & 1 < p < 1 + \dfrac{1}{\alpha}, \quad z \in \Omega(\delta), \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \dfrac{1}{\alpha}, \quad z \in \Omega(\delta), \\ n^{1-\frac{1}{p}}, & p > 1 + \dfrac{1}{\alpha}, \quad z \in \Omega(\delta), \\ n^{1-\frac{1}{p}}, & p > 1, \quad z \in \widehat{\Omega}(\delta). \end{cases} \tag{35}$$

Therefore, combining (22) and (35), for any $\gamma > -1$, $p > 1$, we obtain

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\frac{\gamma^*+1}{\alpha p}}, & 1 < p < 1 + \dfrac{\gamma^*+1}{\alpha}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \dfrac{\gamma^*+1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1 + \dfrac{\gamma^*+1}{\alpha}, \end{cases} \quad \text{if} \quad z \in \Omega(\delta), \tag{36}$$

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & 1 < p < 1 + \dfrac{\gamma}{1+\alpha}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \dfrac{\gamma}{1+\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1 + \dfrac{\gamma^*}{1+\alpha}, \end{cases} \quad \text{if} \quad z \in \widehat{\Omega}(\delta), \tag{37}$$

$\gamma^* := \max\{0; \gamma\}$.

B. Now, we begin to estimate the $\left|\left(\Phi^{-(n+1)}(z)\right)'\right|$.

Since $\Phi(\infty)=\infty$, then Cauchy integral representation for the region Ω_R gives

$$\left(\Phi^{-(n+1)}(z)\right)'=-\frac{1}{2\pi i}\int\limits_{L_{R_1}}\frac{\Phi^{-(n+1)}(\zeta)d\zeta}{(\zeta-z)^2},\quad z\in\Omega_R.$$

Replacing the variable $\tau=\Phi(\zeta)$ and according to (11), we have

$$\left|\left(\Phi^{-(n+1)}(z)\right)'\right|\preceq n\int\limits_{|\tau|=R_1}\frac{|d\tau|}{|\tau-w|^{\frac{1}{\alpha}}}\preceq\begin{cases}n^{\frac{1}{\alpha}}, & \text{if}\quad z\in\Omega(\delta),\\ n, & \text{if}\quad z\in\widehat{\Omega}(\delta).\end{cases}\tag{38}$$

Combining estimates (12)–(16), (36), (37) and (38), we get

$$\left|P_n'(z)\right|\leq\left|\Phi^{n+1}(z)\right|\left[\frac{A_n(z)}{d(z,L)}+|P_n(z)|\begin{cases}n^{\frac{1}{\alpha}}, & \text{if}\quad z\in\Omega(\delta),\\ n, & \text{if}\quad z\in\widehat{\Omega}(\delta)\end{cases}\right],\tag{39}$$

where for any $\gamma>-1, p>1,\ z\in\Omega(\delta)$ and $z\in\widehat{\Omega}(\delta)$ $A_n(z)$ defined as in (36), (37), respectively.

Theorem 1 is proved.

***Proof of Theorem* 2.** Now let us start the evaluations of $|P_n(z)|$. Let $L\in\widetilde{Q}_\alpha$ for some $\frac{1}{2}\leq\alpha\leq 1$.

Since the function $H_n(z):=\dfrac{P_n(z)}{\Phi^{n+1}(z)}$, $H_n(\infty)=0$, is analytic in Ω, continuous on $\overline{\Omega}$, then Cauchy integral representation for the region Ω_{R_1} gives

$$H_n(z)=-\frac{1}{2\pi i}\int\limits_{L_{R_1}}H_n(\zeta)\frac{d\zeta}{\zeta-z},\quad z\in\Omega_R.$$

Then

$$\left|\frac{P_n(z)}{\Phi^{n+1}(z)}\right|\leq\frac{1}{2\pi}\int\limits_{L_{R_1}}\left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta-z|}\leq\frac{1}{2\pi d(z,L_{R_1})}\int\limits_{L_{R_1}}|P_n(\zeta)|\,|d\zeta|,$$

and, so,

$$|P_n(z)|\preceq\frac{\left|\Phi^{n+1}(z)\right|}{d(z,L_{R_1})}\int\limits_{L_{R_1}}|P_n(\zeta)|\,|d\zeta|.\tag{40}$$

Denote by

$$A_n:=\int\limits_{L_{R_1}}|P_n(\zeta)|\,|d\zeta|,\tag{41}$$

and repeating estimate (14) for $A_n(z)$, for any $\gamma>-1,\ p>1,\ z\in\Omega_R$, for A_n, we obtain

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & 1 < p < 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 1+\alpha, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 1+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 1+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1, \quad -1 < \gamma \le 1+\alpha. \end{cases} \tag{42}$$

Combining estimates (40)–(42), we have

$$|P_n(z)| \preceq \frac{\left|\Phi^{n+1}(z)\right|}{d(z, L_{R_1})} A_n,$$

$$\text{where} \quad A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & 1 < p < 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 1+\alpha, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 1+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{\gamma}{1+\alpha}, \quad \gamma > 1+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1, \quad -1 < \gamma \le 1+\alpha. \end{cases}$$

Theorem 2 is proved.

***Proof of Theorem* 3.** From (36), (37) and (39), we get

$$\left|P_n'(z)\right| \preceq \frac{\left|\Phi^{n+1}(z)\right|}{d(z, L)} \left[A_n(z) + |P_n(z)| \begin{cases} n^{\frac{1}{\alpha}}, & \text{if} \quad z \in \Omega(\delta), \\ n, & \text{if} \quad z \in \widehat{\Omega}(\delta) \end{cases}\right],$$

where, for any $\gamma > -1$, $p > 1$,

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\frac{\gamma^*+1}{\alpha p}}, & 1 < p < 1+\dfrac{\gamma^*+1}{\alpha}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma^*+1}{\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{\gamma^*+1}{\alpha}, \end{cases} \quad \text{if} \quad z \in \Omega(\delta),$$

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & 1 < p < 1+\dfrac{\gamma}{1+\alpha}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1+\dfrac{\gamma}{1+\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1+\dfrac{\gamma^*}{1+\alpha}, \end{cases} \quad \text{if} \quad z \in \widehat{\Omega}(\delta),$$

and $\gamma^* := \max\{0; \gamma\}$.

Taking into account estimates for $|P_n(z)|$ from (8) and combining with (38) gives a proofs of need estimates

$$\left|P_n'(z)\right| \preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)}\begin{cases} n^{\frac{\gamma+1}{p\alpha}}, & 1<p<1+\dfrac{\gamma}{1+\alpha}, \quad \gamma>0,\\ n^{1-\frac{1}{p}+\frac{1}{\alpha}}(\ln n)^{1-\frac{1}{p}}, & p=1+\dfrac{\gamma}{1+\alpha}, \quad \gamma>0,\\ n^{1-\frac{1}{p}+\frac{1}{\alpha}}, & p>1+\dfrac{\gamma}{1+\alpha}, \quad \gamma>0,\\ n^{1-\frac{1}{p}+\frac{1}{\alpha}}, & p>1, \quad -1<\gamma\le 0,\end{cases} \quad \text{if } z\in\Omega(\delta),$$

$$\left|P_n'(z)\right| \preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)}\begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}+1}, & 1<p<1+\dfrac{\gamma}{1+\alpha}, \quad \gamma>1+\alpha,\\ n^{2-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\dfrac{\gamma}{1+\alpha}, \quad \gamma>1+\alpha,\\ n^{2-\frac{1}{p}}, & p>1+\dfrac{\gamma}{1+\alpha}, \quad \gamma>1+\alpha,\\ n^{2-\frac{1}{p}}, & p>1, \quad -1<\gamma\le 1+\alpha,\end{cases} \quad \text{if } z\in\widehat{\Omega}(\delta).$$

In conclusion, note that in proofs everywhere there is a quantity $d(z,L_{R_1})$. Let us show that $d(z,L_{R_1}) \succeq d(z,L)$ holds for all $z\in\Omega_R$. For the points $z\notin\Omega(L_{R_1},d(L_{R_1},L_R))$, we have $d(z,L_{R_1})\succeq\delta\succeq d(z,L)$. Now, let $z\in\Omega(L_{R_1},d(L_{R_1},L_R))$. Denote by $\xi_1\in L_{R_1}$ the point such that $d(z,L_{R_1})=|z-\xi_1|$ and point $\xi_2\in L$ such that $d(z,L)=|z-\xi_2|$, and for $w=\Phi(z)$, $t_1=\Phi(\xi_1), t_2=\Phi(\xi_2)$, we get $|w-w_1|\ge ||w-w_2|-|w_2-w_1||\ge \left||w-w_2|-\dfrac{1}{2}|w-w_2|\right|\ge$ $\ge\dfrac{1}{2}|w-w_2|$. Then, according to Lemma 1, we obtain $d(z,L_{R_1})\succeq d(z,L)$.

Theorem 3 is proved.

***Proof of Theorem* 4.** Let $p>1, U:=B(z,d(z,L_{R_1}))$ and $z\in L$ is an arbitrary fixed point. By Cauchy integral formulas for derivatives, we have

$$P_n'(z)=\frac{1}{2\pi i}\int\limits_{\partial U}\frac{P_n(t)}{(t-z)^2}dt.$$

Then, and applying (3), we obtain

$$\left|P_n'(z)\right|\le\frac{1}{2\pi}\max_{z\in\partial U}|P_n(t)|\int\limits_{\partial U}\frac{|dt|}{|t-z|^2}\preceq\max_{t\in\overline{G}}|P_n(t)|\frac{1}{d(z,L_{R_1})}.$$

Now, applying [11] (Theorem 2.4) and using Lemma 2, we get

$$\left|P_n'(z)\right|\preceq n^{\frac{\gamma^*+1}{p\alpha}}\|P_n\|_p n^{\frac{1}{\alpha}}\preceq n^{\left(\frac{\gamma^*+1}{p}+1\right)\frac{1}{\alpha}}\|P_n\|_p.$$

Sharpness of the inequality (10) can be argued as follows.

These inequalities can be interpreted as a combination of the well known sharp Markov inequalities $\|P_n'\|_\infty\preceq n\|P_n\|_\infty$ whit inequality for $\|P_n\|_\infty$ in terms of the norm $\|P_n\|_p$. The sharpness of the last inequality can be verified in the following examples: for the polynomial $T_n(z)=1+z+\ldots+z^n$, $h^*(z)=h_0(z)$, $h^{**}(z)=|z-1|^\gamma$, $\gamma>0$, $L:=\{z\colon |z|=1\}$ and any $n\in\mathbb{N}$ there exist $c_6=c_6(h^*,p)>0$, $c_7=c_7(h^{**},p)>0$ such that

a) $\|T\|_\infty\ge c_6 n^{\frac{1}{p}}\|T\|_{\mathcal{L}_p(h^*,\,L)}$, $p>1$,

b) $\|T\|_\infty\ge c_7 n^{\frac{\gamma+1}{p}}\|T\|_{\mathcal{L}_p(h^{**},\,L)}$, $p>\gamma+1$.

Really, if $L := \{z\colon\ |z| = 1\}$, then $L \in \widetilde{Q}_1$. Let a) $h^*(z) \equiv 1$; b) $h^{**}(z) = |z-1|^{\gamma}$, $\gamma > 0$.

Obviously,

$$|T(z)| \le \sum_{j=0}^{n-1} \left|z^j\right| = n, \quad |z| = 1, \quad |T(1)| = n.$$

Then $\|T\|_{\infty} = n$.

On the other hand, according to [40, p. 236], we have

$$\|T\|_{\mathcal{L}_p(h^*,L)} \asymp n^{1-\frac{1}{p}}, \quad p > 1, \quad \text{and} \quad \|T\|_{\mathcal{L}_p(h^{**},L)} \asymp n^{1-\frac{\gamma+1}{p}}, \quad p > \gamma + 1.$$

Therefore,

a) $\|T\|_{\infty} = n \asymp n^{\frac{1}{p}} \|T\|_{\mathcal{L}_p(h^*,L)}$, $p > 1$;

b) $\|T\|_{\infty} = n = n \cdot n^{1-\frac{\gamma+1}{p}} \cdot n^{\frac{\gamma+1}{p}-1} \asymp n^{\frac{\gamma+1}{p}} \|T\|_{\mathcal{L}_p(h^{**},L)}$, $p > \gamma + 1$.

Theorem 4 is proved.

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