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ON SOME IDENTITIES INVOLVING CERTAIN HARDY SUMS AND KLOOSTERMAN SUM

ПРО ДЕЯКІ ТОТОЖНОСТІ ІЗ ПЕВНИМИ СУМАМИ ГАРДІ ТА СУМОЮ КЛООСТЕРМАНА

We give a new reciprocity theorem for the Hardy sum $s_5(h,p)$. Also, a hybrid mean value problem involving the Hardy sum $s_4(h,p)$ and Kloosterman sum is studied and two exact computational formulae are obtained.

Запропоновано нову теорему взаємності для суми Гарді $s_5(h,p)$. Крім цього, вивчається гібридна задача про середні значення, яка містить суму Гарді $s_4(h,p)$ і суму Клоостермана, та отримано дві точні обчислювальні формули.

1. Introduction. Let

$$((x)) = \begin{cases} x - [x] - 1/2, & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Z}, \\ 0, & \text{if} \quad x \in \mathbb{Z}, \end{cases}$$

with $[x]$ being the largest integer $\leq x$. For positive integer p and integer h the classical Dedekind sum $s(h,p)$, arising in the theory of Dedekind η-function, were introduced by R. Dedekind in 1892 by

$$s(h,p) = \sum_{a=1}^{p} \left(\left(\frac{a}{p}\right)\right) \left(\left(\frac{ha}{p}\right)\right).$$

Perhaps the most important property of Dedekind sums is the reciprocity theore

$$s(h,p) + s(p,h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{p} + \frac{p}{h} + \frac{1}{hp}\right), \tag{1.1}$$

when $(h,p) = 1$ (for basic properties see [9]). The arithmetic properties of Dedekind sums were investigated by many authors (see, for example, [12, 14, 15]). J. B. Conrey et al. [4] dealt with the mean value distribution of Dedekind sums and achieved an asymptotic formula for $\sum_{h=1}^{p}{}' |s(h,p)|^{2m}$, where the dash denotes the summation over all $1 \leq h \leq p$ such that $(h,p) = 1$. Moreover, H. Valum [11] derived a relation between the mean square value of $s(h,p)$ and the fourth power mean of Dirichlet L-function.

Similar arithmetic sums arise in the theory of logarithms of the classical theta functions. They are studied by Hardy and Berndt, and for this reason they are called Hardy or Hardy – Berndt sums. There are six such sums, two of which are [2, 6]

$$s_4(h,p) = \sum_{a=1}^{p-1} (-1)^{[ha/p]},$$

$$s_5(h,p) = \sum_{a=1}^{p-1} (-1)^{a+[ha/p]} \left(\left(\frac{a}{p}\right)\right).$$

Goldberg [6] showed that these sums also arise in the theory of $r_m(n)$, the number of representations of n as a sum of m integral squares and in the study of the Fourier coefficients of the reciprocals of the classical theta functions. Like Dedekind sums, these Hardy sums also satisfy a reciprocity (or reciprocity-like) formula [2, 6]. R. Sitaramachandrarao [10] expressed these sums in terms of classical Dedekind sums. For example,

$$s_4(h,p) = -4s(h,p) + 8s(h,2p), \quad \text{if} \quad h \quad \text{is odd.} \tag{1.2}$$

Recently, Du and Zhang [5] have studied the computational problem of Dedekind sums and established a new reciprocity formula by using analytic method and the properties of Dirichlet L-function. That is, they gave the following theorem.

Theorem 1.1 ([5], Theorem 1)**.** *Let h and p are two positive odd numbers with $(h,p) = 1$. Then*

$$s\left(2\bar{p},h\right) + s\left(2\bar{h},p\right) = \frac{h^2+p^2+4}{24hp} - \frac{1}{4}, \tag{1.3}$$

where $\bar{p}$ and $\bar{h}$ satisfy the congruence $p\bar{p} \equiv 1 \pmod{h}$ and $h\bar{h} \equiv 1 \pmod{p}$.

On the other hand, the mean value of Hardy sums or hybrid mean value involving Hardy sums and other celebrated sums are intensively studied. For example, the authors of [13] discussed the hybrid mean value involving certain Hardy sums and Kloosterman sum, defined for any positive integer $p > 1$ and integer n by

$$K(n,p) = {\sum_{a=1}^{p}}' e\left(\frac{na+\bar{a}}{p}\right),$$

where $\bar{a}$ denotes the solution of the congruence $xa \equiv 1 \pmod{p}$, the dash denotes the summation over all $1 \le a \le p$ such that $(a,p) = 1$ and $e(x) = e^{2\pi i x}$. They obtained exact computational formulas

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} K(m,p)K(n,p)S\left(2m\bar{n},p\right)$$

and

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} |K(m,p)|^2\,|K(n,p)|^2\, S\left(2m\bar{n},p\right),$$

where $S(h,p)$ is one of the Hardy sums. Some elementary properties of $K(n,p)$ can be found in [3, 7]. Peng and Zhang [8] investigated the hybrid mean value involving $s_5(h,p)$ in order to help to achieve several identities between Hardy sums and Kloosterman sums.

As mentioned in [8], little about $s_5(h,p)$ is known. Thus, it is meaningful to continue to study the properties of $s_5(h,p)$.

In this paper, firstly, we give following new reciprocity theorem for Hardy sum $s_5(h,p)$ by applying rather elementary method.

Theorem 1.2. *Let h and p are odd primes. Then we have*

$$s_5\left(2\bar{p}, h\right) + s_5\left(2\bar{h}, p\right) = \frac{1}{2} - \frac{h^2 + p^2}{4hp},$$

where $\bar{p}$ and $\bar{h}$ satisfy the congruence $p\bar{p} \equiv 1 \pmod{h}$ and $h\bar{h} \equiv 1 \pmod{p}$.

Secondly, using the properties of Gauss sums and mean value theorems of Dirichlet L-function, we obtain the following conclusions for Hardy sum $s_4(h,p)$ and Kloosterman sum in order to help to obtain further relations between these sums.

Theorem 1.3. *Let p be odd prime, then we have*

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} K(m,p)K(n,p)s_4\left(m\bar{n}, p\right) = p^2(p-1).$$

Theorem 1.4. *Let p be odd prime, then we obtain*

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} \left|K(m,p)\right|^2 \left|K(n,p)\right|^2 s_4\left(m\bar{n}, p\right) =$$

$$= \begin{cases} p^3(p-1), & \textit{if} \quad p \equiv 1 \pmod{4}, \\ p^3(p-1) - 36p^2h_p^2, & \textit{if} \quad p \equiv 3 \pmod{8}, \\ p^3(p-1) - 4p^2h_p^2, & \textit{if} \quad p \equiv 7 \pmod{8}, \end{cases}$$

where h_p denotes the class number of the quadratic field $\mathbb{Q}\left(\sqrt{-p}\right)$.

2. Preliminaries. In order to prove our theorems, we will need some lemmas. Hereinafter, we shall use many properties of Gauss sums, all of which can be found in [1].

Lemma 2.1. *Let p be an odd prime. Then, for any odd number h with $(h,p)=1$, we have*

$$s_5(h,p) = 2s(h,p) - 4s\left(\bar{2}h, p\right),$$

where $\bar{2}$ satisfies the congruence $2\bar{2} \equiv 1 \pmod{p}$.

Proof. See [8] (Lemma 2.3).

Lemma 2.2. *Let $p > 2$ be an integer, then, for any integer h with $(h,p)=1$, we obtain*

$$s(h,p) = \frac{1}{\pi^2 p}\sum_{d|p}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h)\left|L(1,\chi)\right|^2,$$

where $L(1,\chi)$ is the Dirichlet L-function corresponding to the character $\chi \bmod d$ and $\phi(p)$ is the Euler function.

Proof. See Lemma 2 of [14].

Lemma 2.3. *Let p be an odd prime. Then, for any nonprincipal character $\chi \bmod p$, we have*

$$\sum_{n=1}^{p-1} \chi(n)\left|K(n,p)\right|^2 = \bar{\chi}(-1)\frac{\tau^3(\chi)\tau\left(\bar{\chi}^2\right)}{\tau\left(\bar{\chi}\right)},$$

where $\tau(\chi)$ denotes the Gauss sum.

Proof. This is Lemma 1 of [13].

Lemma 2.4. *Let p be an odd prime. Then we obtain*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{(p-1)^2(p-2)}{p^2}$$

and

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)\,|L(1,\chi)|^2 = \frac{\pi^2}{24}\frac{(p-1)^2(p-5)}{p^2}.$$

Proof. See [13] (Lemma 5).

3. Proofs. This section is devoted to complete the proof of theorems.

***3.1. Proof of Theorem* 1.2.** Employing Lemma 2.1 repeatedly, one can write

$$s_5\left(2\bar{h},p\right) = 2s\left(2\bar{h},p\right) - 4s(h,p) \tag{3.1}$$

and

$$s_5\left(2\bar{p},h\right) = 2s\left(2\bar{p},h\right) - 4s(p,h), \tag{3.2}$$

where we have used the fact that if positive integers n and q satisfying $(n,q)=1$. Then $s(n,q) = s\left(\bar{n},q\right)$, where $\bar{n}$ satisfies the congruence $n\bar{n} \equiv 1 \bmod q$.

Adding (3.1) and (3.2), then applying reciprocity formulas (1.1) and (1.3) give the desired result.

***3.2. Proof of Theorem* 1.3.** Before beginning the proof, we should prove the following relation.

Lemma 3.1. *For odd prime p and any odd number h with $(h,p)=1$, we have the identity*

$$s_4(h,p) = 20s(h,p) - 8s(2h,p) - 8s\left(\bar{2}h,p\right),$$

where $\bar{2}$ satisfies the congruence $2\bar{2} \equiv 1 \pmod p$.

Proof. From (1.2) and Lemma 2.2, one has

$$s_4(h,p) = -4s(h,p) + 8s(h,2p) =$$

$$= -4s(h,p) + \frac{4}{\pi^2 p}\sum_{d|2p}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h)\,|L(1,\chi)|^2. \tag{3.3}$$

Since the divisors of $2p$ are $1,\ 2,\ p,\ 2p$ and

$$s(h,p) = \frac{1}{\pi^2}\frac{p}{p-1}\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(h)\,|L(1,\chi)|^2,$$

the right-hand side of (3.3) becomes

$$-4s(h,p) + \frac{4p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(h)\,|L(1,\chi)|^2 +$$

$$+\frac{16p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod 2p\\ \chi(-1)=-1}}\chi(h)\left|L(1,\chi)\right|^2=$$

$$=\frac{16p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(h)\lambda(h)\left|L\left(1,\chi\lambda\right)\right|^2, \tag{3.4}$$

where λ denotes the character $\bmod 2$. Now, from the Euler product formula, we obtain

$$\left|L\left(1,\chi\lambda\right)\right|^2=\prod_{p_1}\left|1-\frac{\chi(p_1)\lambda(p_1)}{p_1}\right|^{-2}=$$

$$=\prod_{p_1>2}\left|1-\frac{\chi(p_1)}{p_1}\right|^{-2}=\left|1-\frac{\chi(2)}{2}\right|^2\prod_{p_1}\left|1-\frac{\chi(p_1)}{p_1}\right|^{-2}=$$

$$=\left(\frac{5}{4}-\frac{\chi(2)}{2}-\frac{\bar{\chi}(2)}{2}\right)\left|L(1,\chi)\right|^2. \tag{3.5}$$

Thus, substituting (3.5) in (3.4) completes the proof.

We proceed to the proof of Theorem 1.3. Notice that if χ is nonprincipal character $\bmod p$, then $|\tau(\chi)|=\sqrt{p}$ and

$$\sum_{m=1}^{p-1}\chi(m)K(m,p)=\sum_{a=1}^{p-1}\sum_{m=1}^{p-1}\chi(m)e\left(\frac{ma+\bar{a}}{p}\right)=\left|\tau^2(\chi)\right|=p.$$

So, it follows from Lemmas 2.4 and 3.1 that

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}K(m,p)K(n,p)s_4\left(m\bar{n},p\right)=$$

$$=\frac{20p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{n=1}^{p-1}\chi(n)K(n,p)\right|^2\left|L(1,\chi)\right|^2-$$

$$-\frac{8p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\left|\sum_{n=1}^{p-1}\chi(n)K(n,p)\right|^2\left|L(1,\chi)\right|^2-$$

$$-\frac{8p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\bar{\chi}(2)\left|\sum_{n=1}^{p-1}\chi(n)K(n,p)\right|^2\left|L(1,\chi)\right|^2=$$

$$=\frac{20p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2-\frac{8p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\left|L(1,\chi)\right|^2-$$

$$-\frac{8p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\bar{\chi}\,(2)\,|L(1,\chi)|^2=$$

$$=p^2(p-1),$$

which completes the proof.

***3.3. Proof of Theorem* 1.4.** If $p\equiv 1 \pmod 4$, in view of Lemmas 2.3, 2.4 and 3.1, we have

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,p)|^2\,|K(n,p)|^2\,s_4\,(m\bar{n},p)=$$

$$=\frac{20p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{n=1}^{p-1}\chi(n)\,|K(n,p)|^2\right|^2|L(1,\chi)|^2-$$

$$-\frac{8p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\left|\sum_{n=1}^{p-1}\chi(n)\,|K(n,p)|^2\right|^2|L(1,\chi)|^2-$$

$$-\frac{8p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\bar{\chi}(2)\left|\sum_{n=1}^{p-1}\chi(n)\,|K(n,p)|^2\right|^2|L(1,\chi)|^2=$$

$$=p^3(p-1).$$

If $p\equiv 3 \pmod 4$, then we have the Legendre symbol $\left(\frac{-1}{p}\right)=\chi_2(-1)=-1,\ L(1,\chi_2)=\pi h_p/\sqrt{p}$ and

$$\tau\left(\chi_2^2\right)=\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)^2 e\left(\frac{a}{p}\right)=-1.$$

Hence, using Lemma 2.3 and proceeding as in the proof of Theorem 1.3 yield that

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|K(m,p)|^2\,|K(n,p)|^2\,s_4\,(m\bar{n},p)=$$

$$=\frac{20p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{n=1}^{p-1}\chi(n)\,|K(n,p)|^2\right|^2|L(1,\chi)|^2-$$

$$-\frac{8p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\left|\sum_{n=1}^{p-1}\chi(n)\,|K(n,p)|^2\right|^2|L(1,\chi)|^2-$$

$$-\frac{8p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\bar{\chi}(2)\left|\sum_{n=1}^{p-1}\chi(n)\,|K(n,p)|^2\right|^2|L(1,\chi)|^2=$$

$$=\frac{20p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2-\frac{16p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\,|L(1,\chi)|^2-$$

$$-\frac{20p^3}{\pi^2}\,|L(1,\chi_2)|^2+\frac{16p^3}{\pi^2}\left(\frac{2}{p}\right)|L(1,\chi)|^2=$$

$$=p^3(p-1)-20p^2h_p^2+16p^2\left(\frac{2}{p}\right)h_p^2=$$

$$=\begin{cases}p^3(p-1)-36p^2h_p^2, & \text{if} \quad p\equiv 3\,(\operatorname{mod} 8)\,,\\ p^3(p-1)-4p^2h_p^2, & \text{if} \quad p\equiv 7\,(\operatorname{mod} 8)\,,\end{cases}$$

where we have used that $\left(\frac{2}{p}\right)=-1$ if $p\equiv 3\,(\operatorname{mod} 8)$ and $\left(\frac{2}{p}\right)=1$ if $p\equiv 7(\operatorname{mod} 8)$. So, the proof is completed.

References

1. T. M. Apostol, *Introduction to analytic number theory,* Undergrad. Texts Math., Springer-Verlag, New York (1976).
2. B. C. Berndt, *Analytic Eisenstein series, theta functions and series relations in the spirit of Ramanujan,* J. reine und angew. Math., **303/304**, 332 – 365 (1978).
3. S. Chowla, *On Kloosterman's sum,* Nor. Vidensk. Selsk. Fak. Frondheim, **40**, 70 – 72 (1967).
4. J. B. Conrey, E. Fransen, R. Klein, C. Scott, *Mean values of Dedekind sums,* J. Number Theory, **56**, № 2, 214 – 226 (1996).
5. X. Du, L. Zhang, *On the Dedekind sums and its new reciprocity formula,* Miskolc Math. Notes, **19**, 235 – 239 (2018).
6. L. A. Goldberg, *Transformations of theta-functions and analogues of Dedekind sums,* Ph. D. Thesis, Univ. Illinois, Urbana (1981).
7. A. V. Malyshev, *A generalization of Kloosterman sums and their estimates,* Vestn. Leningr. Univ., **15**, 59 – 75 (1960) (in Russian).
8. W. Peng, T. Zhang, *Some identities involving certain Hardy sum and Kloosterman sum,* J. Number Theory, **165**, 355 – 362 (2016).
9. H. Rademacher, E. Grosswald, *Dedekind sums,* Math. Assoc. America, Washington, D.C. (1972).
10. R. Sitaramachandrarao, *Dedekind and Hardy sums,* Acta Arith., **48**, № 4, 325 – 340 (1987).
11. H. Walum, *An exact formula for an average of L-series,* Ill. J. Math., **26**, 1 – 3 (1982).
12. Z. Wenpeng, L. Yanni, *A hybrid mean value related to the Dedekind sums and Kloosterman sums,* Sci. China Math., **53**, 2543 – 2550 (2010).
13. H. Zhang, W. Zhang, *On the identity involving certain Hardy sums and Kloosterman sums,* J. Inequal. and Appl., **52** (2014), 9 p.
14. W. Zhang, *On the mean values of Dedekind sums,* J. Théor. Nombres Bordeaux, **8**, 429 – 442 (1996).
15. W. Zhang, *A note on the mean square value of the Dedekind sums,* Acta Math. Hung., **86**, 119 – 135 (2000).

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