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B. Hu, J. Huang (School Math. and Statistics, Jiangsu Normal Univ., Xuzhou, China),
N. M. Adarchenko (Francisk Skorina Gomel State Univ., Belarus)

On Π-PERMUTABLE SUBGROUPS IN FINITE GROUPS*

ПРО Π-ПЕРЕСТАВНІ ПІДГРУПИ СКІНЧЕННИХ ГРУП

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$ and let Π be a nonempty subset of the set σ. A set $\mathcal{H}$ of subgroups of a finite group G is said to be a *complete Hall* Π*-set* of G if every member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \Pi$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \Pi$ such that $\sigma_i \cap \pi(G) \neq \varnothing$. A subgroup A of G is called (i) $\mathcal{H}^G$*-permutable* if $AH^x = H^x A$ for $H \in \mathcal{H}$ and $x \in G$; (ii) Π*-permutable in* G if A is $\mathcal{H}^G$-permutable for some complete Hall Π-set $\mathcal{H}$ of G.

We study the influence of Π-permutable subgroups on the structure of G. In particular, we prove that if $\pi = \bigcup_{\sigma_i \in \Pi} \sigma_i$ and $G = AB$, where A and B are $\mathcal{H}^G$-permutable π-separable (respectively, π-closed) subgroups of G, then G is also π-separable (respectively, π-closed). Some known results are generalized.

Нехай $\sigma = \{\sigma_i | i \in I\}$ — деяке розбиття множини всіх простих чисел $\mathbb{P}$ і Π — непорожня підмножина множини σ. Множина $\mathcal{H}$ підгруп скінченної групи G називається *повною холлівською* Π*-множиною* в G, якщо кожен член з $\mathcal{H}$ є холлівською σ_i-підгрупою в G для деякого $\sigma_i \in \Pi$ і $\mathcal{H}$ містить точно одну холлівську σ_i-підгрупу з G для кожного $\sigma_i \in \Pi$ такого, що $\sigma_i \cap \pi(G) \neq \varnothing$. Підгрупа A з G називається: (i) $\mathcal{H}^G$*-переставною*, якщо $AH^x = H^x A$ для всіх $H \in \mathcal{H}$ і $x \in G$; (ii) Π*-переставною в* G, якщо A є $\mathcal{H}^G$-переставною для деякої повної Π-множини $\mathcal{H}$ в G.

У цій статті вивчено вплив Π-переставних підгруп на структуру групи G. Зокрема, доведено таке твердження: якщо $\pi = \bigcup_{\sigma_i \in \Pi} \sigma_i$ та $G = AB$, де A і B є $\mathcal{H}^G$-переставними π-сепарабельними (відповідно, π-замкненими) підгрупами G, то G також має бути π-сепарабельною (відповідно, π-замкненою). Крім того, узагальнено деякі відомі результати.

1. Introduction. Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi = \{p_1, \ldots, p_n\} \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. They say that n is a π*-number* provided $\pi(n) \subseteq \pi$.

Before continuing, we recall some concepts of the theory of σ-properties in [1, 2].

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for $i \neq j$; Π is always supposed to be a nonempty subset of the set σ, $\Pi' = \sigma \backslash \Pi$ and $\pi(\Pi) = \bigcup_{\sigma_i \in \Pi} \sigma_i$. The group G is called Π*-primary* if G is a σ_i-group for some $\sigma_i \in \Pi$.

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By the analogy with the notations $\pi(n)$ and $\pi(G)$, we write $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \varnothing\}$, $\sigma(G) = \sigma(|G|)$. The group G is called: a Π-*group* if $\sigma(G) \subseteq \Pi$; Π-*soluble* if every chief factor of G is either Π-primary or a Π'-group; Π-*closed* if G is $\pi(\Pi)$-closed in the usual sense, that is, G has a normal Hall $\pi(\Pi)$-subgroup; *strongly* Π-*closed* if G has a normal Hall σ_i-subgroup for $\sigma_i \in \Pi$.

A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall* Π-*set* of G if every member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \Pi$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \Pi \cap \sigma(G)$.

Let $\mathcal{L}$ be some nonempty set of subgroups of G. Then a subgroup A of G is called $\mathcal{L}$-*permutable* [4] if $AH = HA$ for $H \in \mathcal{L}$; $\mathcal{L}^G$-*permutable* [1] if $AH^x = H^xA$ for $H \in \mathcal{L}$ and $x \in G$.

Definition 1.1. *Let $\mathcal{H}$ be a complete Hall Π-set of G. Then we say that a subgroup A of G is Π-permutable in G* [5], *if A is $\mathcal{H}^G$-permutable for some complete Hall Π-set $\mathcal{H}$ of G.*

***Example* 1.1.** (i) Let $\sigma^1 = \{\{2\}, \{3\}, \ldots\}$ (we use here the notations in [3]) and $\Pi = \{\{p_1\}, \ldots$ $\ldots, \{p_n\}\}$, that is, $\pi(\Pi) = \{p_1, \ldots, p_n\}$. Then a subgroup A of G is Π-permutable in G if and only if it is π-*permutable* or π-*quasinormal* in G in the sense of Kegel [6], that is, A permutes with all Sylow p-subgroups of G for $p \in \pi$. Moreover, G is Π-soluble if and only if it is π-soluble, and G is strongly Π-closed if and only if it has a normal nilpotent Hall π-subgroup.

(ii) Let $\sigma^\pi = \{\pi, \pi'\}$ [3] and $\Pi = \{\pi\}$. Then a subgroup A of G is Π-permutable in G provided G has a Hall π-subgroup H such that $AH^x = H^xA$ for $x \in G$. It is clear also that G is σ^π-soluble if and only if it is π-separable.

Note that in the case when $\pi = \{2, 3\}$ and $G = A_5$ is the alternating group of degree 5, every subgroup A of G with $5 \in \pi(A)$ is Π-permutable in G, and in this case every subgroup A of G with $1 < A < G$ is not σ^π-permutable in G.

(iii) G is Π-soluble if and only if it is σ^*-soluble, where $\sigma^* = \Pi \cup \{\pi'\}$ and $\pi = \pi(\Pi)$.

Our first observation is the following theorem.

Theorem A. *Let $\pi = \pi(\Pi)$ and $G = AB$, where A and B are $\mathcal{H}^G$-permutable subgroups of G for some complete Hall Π-set $\mathcal{H}$ of G. If A and B are π-separable* (*respectively*, Π-*soluble*, (*strongly*) Π-*closed*), *then G is also π-separable* (*respectively*, Π-*soluble*, (*strongly*) Π-*closed*).

***Corollary* 1.1.** *The group G is π-separable if and only if G possesses a Hall π-subgroup* (*a Hall π'-subgroup*) *H and $G = AB$ for some π-separable subgroups A and B which permute with all conjugates of H.*

Proof. *Sufficiency*. We can assume without loss of generality that H is a Hall π-subgroup of G. Then A and B are $\mathcal{H}^G$-permutable, where $\mathcal{H} = \{H\}$ is a complete Hall Π-set of G and $\Pi = \{\pi\} \subseteq \{\pi, \pi'\} = \sigma^\pi$ (see Example 1.1 (ii)), so G is π-separable by Theorem A.

Necessity. We can take $A = 1$ and $B = G$ and use the well-known properties of the Hall subgroups of π-separable groups [7] (VI, Hauptsatz 1.7).

In the case when $\sigma = \sigma^1$ we get from Theorem A also the following results.

***Corollary* 1.2** (see Theorem 1 in [8] or Theorem 1 in [9])**.** *Let A and B be π-permutable subgroups of G and $G = AB$. If A and B are π-separable*, *then G is also π-separable.*

***Corollary* 1.3** (see Theorem 1 in [10])**.** *Let A and B be π-permutable subgroups of G and $G =$ $= AB$. If A and B are π-soluble*, *then G is also π-soluble.*

***Corollary* 1.4.** *Let $G = AB$, where A and B are π-permutable in G. If the groups A and B possess normal nilpotent Hall π-subgroups, then G also possesses a normal nilpotent Hall π-subgroup.*

***Corollary* 1.5.** *Let $G = AB$, where A and B are p'-permutable subgroups of G, that is, A and B permute with all Sylow q-subgroups of G for primes $q \neq p$. If A and B are p-nilpotent, then G is also p-nilpotent.*

Now fix some ordering ϕ on $\mathbb{P}$. The record $p\phi q$ means that p precedes q in ϕ and $p \neq q$. The group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$ is called ϕ-dispersive whenever $p_1 \phi p_2 \phi \ldots \phi p_n$ and for every i there is a normal subgroup of G of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i}$.

***Corollary* 1.6.** *Let $\pi(G) = \{p_1, , \ldots, p_n\}$, where $p_1 \phi p_2 \phi \ldots \phi p_n$, and let $p = p_n$. Suppose also that $G = AB$, where A and B are p'-permutable subgroups of G. If A and B are ϕ-dispersive, then G is also ϕ-dispersive.*

***Corollary* 1.7** (see Theorem 3.2 in [11], Ch. 4)**.** *Let $G = AB$, where A and B are normal subgroups of G. If A and B are ϕ-dispersive, then G is also ϕ-dispersive.*

***Example* 1.2.** Let p, q be primes, where q divides $p - 1$, and let P be a group of order p and Q a non-Abelian group of order q^3 of exponent q. Finally, let $V \neq W$ be maximal subgroups of Q and $G = P \wr Q = K \rtimes Q$, where K is the base group of the regular wreath product G. Then $G = AB$, where $A = KV$ and $B = KW$ are supersoluble normal subgroups of G with A', $B' \leq K$ and $K = F(G) = O_{p',p}(G) = O_p(G)$. Hence, G' is not p-nilpotent, so G is not p-supersoluble.

Example 1.2 shows that we can not obtain an analogue of Corollary 1.5 for the groups $G = AB$ with p-nilpotent derived subgroups A' and B'. Nevertheless, we prove the following theorem.

Theorem B. *Let $G = AB$, where A and B are p'-permutable subgroups of G with p-nilpotent derived subgroups A' and B'. If $G/O_{p',p}(G)$ is nilpotent and $(|G : A|, |G : B|) = 1$, then G' is p-nilpotent.*

Since the product of any two meta-nilpotent normal subgroups is clearly meta-nilpotent, we get from Theorem B the following known result.

***Corollary* 1.8** (see Theorem 3.5 in [11], Ch. 4)**.** *Let $G = AB$, where A and B are normal subgroups of G with nilpotent derived subgroups A' and B'. If $(|G : A|, |G : B|) = 1$, then G' is nilpotent.*

It is well-known (see also Example 1.2) that the product $G = AB$ of two normal supersoluble subgroups A and B of G need not be supersoluble. Nevertheless, such a product is supersoluble if either the derived subgroup G' is nilpotent or $(|G : A|, |G : B|) = 1$. Using Theorem B we prove the following result which allows us to get the local versions of these two results.

Theorem C. *Let $G = AB$, where A and B are p'-permutable p-supersoluble subgroups of G. If either G' is p-nilpotent or $(|G : A|, |G : B|) = 1$ and $G/O_{p',p}(G)$ is nilpotent, then G is p-supersoluble.*

***Corollary* 1.9** (see [12] or [11], Ch. 4, Theorem 3.4)**.** *Let $G = AB$, where A and B are supersoluble normal subgroups of G. If $(|G : A|, |G : B|) = 1$, then G is supersoluble.*

***Corollary* 1.10** (R. Baer [11], Theorem 1.13)**.** *Let $G = AB$, where A and B are supersoluble normal subgroups of G. If G' is nilpotent, then G is supersoluble.*

2. Basic lemmas.

Lemma 2.1. *Suppose that G has a complete Hall Π-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that the subgroups $A \leq E$ and B of G are $\mathcal{H}^G$-permutable. Let R be a normal subgroup of G. Then*:

(1) *$\mathcal{H}^0 = \{H_1R/R, \ldots, H_tR/R\}$ is a complete Hall Π-set of G/R and AR/R is $(\mathcal{H}^0)^{G/R}$-permutable*;

(2) *$\mathcal{H}_0 = \{H_1 \cap E, \ldots, H_t \cap E\}$ is a complete Hall Π-set of E and A is ${\mathcal{H}_0}^E$-permutable*;

(3) *if $BE = EB$, then $B \cap E$ is ${\mathcal{H}_0}^E$-permutable.*

Proof. Without loss of generality we can assume that H_i is a σ_i-group for $i = 1, \ldots, t$.

(1) $\mathcal{H}^0$ is a complete Hall Π-set of G/R such that

$$(AR/R)(H_iR/R)^{xR} = AH_i^xR/R = H_i^xAR/R = (H_iR/R)^{xR}(AR/R)$$

for $xR \in G/R$ and $i = 1, \ldots, t$, that is, AR/R is $(\mathcal{H}^0)^{G/R}$-permutable.

(2) Since $EH_i = H_iE$ is a subgroup of G and H_i is a Hall σ_i-subgroup of G, $|EH_i : H_i| = |E : E \cap H_i|$ is a σ_i'-number. Hence $E \cap H_i$ is a Hall σ_i-subgroup of E for $i = 1, \ldots, t$, so $\mathcal{H}_0$ is a complete Hall Π-set of E. Now, for any $x \in E$, we have $AH_i^x = H_i^xA$, which implies that

$$E \cap AH_i^x = A(E \cap H_i^x) = A(E \cap H_i)^x = (E \cap H_i)^xA,$$

that is, A is ${\mathcal{H}_0}^E$-permutable.

(3) In view of part (2), we obtain only to show that for any i and for $x \in E$ the following holds: $E \cap H_i^xB = (E \cap H_i^x)(E \cap B) = (E \cap H_i)^x(E \cap B) = (E \cap B)(E \cap H_i)^x$. But first we show that $D = (D \cap H_i^x)(D \cap B)$, where $D = E \cap H_i^xB$. Note that $DH_i^x = EH_i^x \cap H_i^xB$ is a subgroup of G, so $DH_i^x = H_i^xD$ and, hence, $D \cap H_i^x$ is a Hall σ_i-subgroup of D. Similarly, $DB = BD$ is a subgroup of G. On the other hand, $|H_i^xB : B| = |H_i^x : H_i^x \cap B|$ is a σ_i-number and so $|BD : B| = |D : D \cap B|$ is a σ_i-number since $BD \leq H_i^xB$. But then $D = (D \cap H_i^x)(D \cap B)$. Finally, we have

$$E \cap H_i^xB = D = (D \cap H_i^x)(D \cap B) = (E \cap H_i^xB \cap H_i^x)(E \cap H_i^xB \cap B) = (E \cap H_i^x)(E \cap B).$$

The lemma is proved.

In fact, the following lemma can be proved by the direct calculations.

Lemma 2.2. *Let A, B and H be subgroups of G. If $HA = AH$ and $HB = BH$, then $H\langle A, B\rangle = \langle A, B\rangle H$.*

We say that an $\mathcal{H}^G$-permutable subgroup A of G is a *maximal $\mathcal{H}^G$-permutable subgroup* of G if $A < G$ and for every $\mathcal{H}^G$-permutable subgroup B of G with $A \leq B < G$ we have $A = B$.

Lemma 2.3. *Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a complete Hall Π-set of G and A a maximal $\mathcal{H}^G$-permutable subgroup of G. Then one of the following statements is true*:

(1) *A is normal in G*;

(2) *$H_i^G \leq A$ for $i = 1, \ldots, t$*;

(3) *there exists i such that $G = AH_i$ and $H_j^G \leq A$ for $j \neq i$.*

Proof. Assume that A is not normal in G, and, for some i, we get $H_i^G \nleq A$. Then $A < AH_i^G$ and AH_i^G is $\mathcal{H}^G$-permutable, so $G = AH_i^G$ by the maximality of A. It follows that, for some $x \in G$ and $y \in H_i^x$, we have $y \notin N_G(A)$. Then A^y is $\mathcal{H}^G$-permutable and $A < \langle A, A^y \rangle$. Moreover, $\langle A, A^y \rangle$ is also $\mathcal{H}^G$-permutable by Lemma 2.2 and, hence, $\langle A, A^y \rangle = G$. But $A^y \leq AH_i^x$ and so $G = AH_i^x = AH_i$, which implies that, for every $z \in G$ and for every $j \neq i$, we obtain $H_j^z \leq A$, that is, $H_j^G \leq A$.

The lemma is proved.

Recall that $O_\Pi(G)$ [1] denotes the product of all normal Π-subgroups of G.

Lemma 2.4. *Let N, $R \leq H$ be normal subgroups of G. Then*:

(1) *all quotients and all subgroups of a (strongly) Π-closed group are (strongly) Π-closed*;

(2) *if G/N and G/R are (strongly) Π-closed, then $G/(N \cap R)$ is (strongly) Π-closed*;

(3) *if $R \leq \Phi(G)$ and H/R is (strongly) Π-closed, then H is (strongly) Π-closed*.

Proof. (1) This assertion directly follows from properties of Hall subgroups.

(2) Let A and B be any two (strongly) Π-closed groups. Then $O_\Pi(A)$ is a Hall $\pi(\Pi)$-subgroup of A and $O_\Pi(B)$ is a Hall $\pi(\Pi)$-subgroup of B. Hence, $O_\Pi(A \times B) = O_\Pi(A) \times O_\Pi(B)$ is a Hall $\pi(\Pi)$-subgroup of $A \times B$, so $A \times B$ is (strongly) Π-closed. Finally, $G/(N \cap R)$ is isomorphic to some subgroup of $(G/N) \times (G/R)$ by [7] (Ch. I, Hilfssatz 9.6), so we have (2).

(3) It is enough to prove that if H/R has a normal Hall σ_i-group V/R for some $\sigma_i \in \Pi$, then H also has a normal Hall σ_i-subgroup.

First note that V is normal in G since V/R is characteristic in H/R. Let $D = O_{\sigma_i{}'}(V)$. Then, since $R \leq \Phi(G)$, D is a Hall $\sigma_i{}'$-subgroup of V. Hence, by the Schur–Zassenhaus theorem, V has a Hall σ_i-subgroup, say E. It is clear that V is σ_i'-soluble, so any two Hall σ_i-subgroups of V are conjugated in V. Therefore, by the Frattini argument we have $G = VN_G(E) = (RE)N_G(E) = N_G(E)$. Thus, E is a normal Hall σ_i-subgroup of H.

The lemma is proved.

Lemma 2.5. (1) *If $G/\Phi(G)$ is p-supersoluble, then G is p-supersoluble* (*see* [7], IV, *Satz* 8.6).

(2) *Let N and R be distinct minimal normal subgroups of G. If G/N and G/R are p-supersoluble, then G is p-supersoluble.*

(3) *Let $A = G/O_{p'}(G)$. Then G is p-supersoluble if and only if $A/O_p(A)$ is an Abelian group of exponent dividing $p-1$, p is the largest prime dividing $|A|$ and $F(A) = O_p(A)$ is a normal Sylow subgroup of A.*

Proof. (2) This follows from the G-isomorphism $NR/N \simeq R$.

(3) Since G is p-supersoluble if and only if $G/O_{p'}(G)$ is p-supersoluble, we may assume without loss of generality that $O_{p'}(G) = 1$.

First assume that G is p-supersoluble. In this case $G/C_G(H/K)$ is an Abelian group of exponent dividing $p-1$ for any chief factor H/K of G with $|H/K| = p$. On the other hand,

$$O_{p',p}(G) = O_p(G) = \bigcap \Big\{ C_G(H/K) \mid H/K \text{ is a chief factor of } G \text{ with } |H/K| = p \Big\}$$

by [11] (Appendixes, Theorem 3.2). Hence $G/O_p(G)$ is an Abelian group of exponent dividing $p-1$. Thus, p is the largest prime dividing $|G|$ and $F(G) = O_p(G)$ is a normal Sylow p-subgroup of G.

Finally, if $G/O_p(G)$ is an Abelian group of exponent dividing $p-1$, then every chief factor H/K of G below $O_p(G)$ is cyclic by [11] (Ch. 1, Theorem 1.4). Hence, G is supersoluble.

The lemma is proved.

3. Proofs of the results.

Proposition 3.1. *If A is a Π-permutable subgroup of G, then $O_\Pi(A) \leq O_\Pi(G)$.*

Proof. Assume that this proposition is false and G be a counterexample with $|G| + |G : A|$ minimal. Then $O_\Pi(A) \neq 1$ and A is not normal in G. Moreover, $\Pi \cap \sigma(G) \neq \varnothing$. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a complete Hall Π-set of G such that A is $\mathcal{H}^G$-permutable. We can assume without loss of generality that H_i is a σ_i-group of G for $i = 1, \ldots, t$.

Let R be a minimal normal subgroup of G. Then the hypothesis holds for $(G/R, RA/R)$ by Lemma 2.1(1), so the choice of G implies that $O_\Pi(A)R/R \leq O_\Pi(G/R)$. If $R \leq O_\Pi(G)$, then $O_\Pi(G)/R = O_\Pi(G/R)$ and so $O_\Pi(A) \leq O_\Pi(G)$. Therefore, $O_\Pi(G) = 1$.

Now let E be a maximal $\mathcal{H}^G$-permutable subgroup of G containing A. Then A is Π-permutable in E by Lemma 2.1(2), so $O_\Pi(A) \leq O_\Pi(E)$ by the choice of G. On the other hand, in the case when $A < E$ we have $O_\Pi(E) \leq O_\Pi(G)$ by the choice of $|G| + |G : A|$, which implies that $O_\Pi(A) \leq O_\Pi(G)$. Hence, $A = E$.

If $D := H_1^G \ldots H_t^G \leq A$, then $O_\Pi(A) \leq O_\Pi(D) \leq O_\Pi(G) = 1$ since $O_\Pi(D)$ is characteristic in D and so normal in G. Finally, assume that $D \nleq A$. Then, by Lemma 2.3, there exists i such that $V := H_1^G \ldots H_{i-1}^G H_{i+1}^G \ldots H_t^G \leq A$ and $G = AH_i$. Hence, $O_\Pi(A) \cap V \leq O_\Pi(V) \leq O_\Pi(G) = 1$, so $O_\Pi(A) = O_{\sigma_i}(A)$. Then $O_{\sigma_i}(A) \leq H_i^x$ for $x \in G$, so $O_{\sigma_i}(A) \leq (H_i)_G \leq O_\Pi(G) = 1$. Therefore, $O_\Pi(A) = 1$, a contradiction.

The proposition is proved.

***Corollary* 3.1.** *Let $G = AB$, where A and B are $\mathcal{H}^G$-permutable subgroups of G for some complete Hall Π-set $\mathcal{H}$ of G. If A and B are Π-closed, then G is also Π-closed.*

Proof. By Proposition 3.1, $O_\Pi(A) \leq O_\Pi(G)$, where $O_\Pi(A)$ is a Hall $\pi(\Pi)$-subgroup of A by hypothesis. Then $A/O_\Pi(A) = A/(A \cap O_\Pi(G))$ is a Π'-group. Similarly, $B/O_\Pi(B) = B/(B \cap \cap O_\Pi(G))$ is a Π'-group. Hence, $G/O_\Pi(G) = (AO_\Pi(G)/O_\Pi(G))(O_\Pi(G)B/O_\Pi(G))$ is a Π'-group.

The corollary is proved.

***Proof of Theorem* A.** In view of Corollary 3.1, it is enough to show that if A and B are π-separable (respectively, Π-soluble, strongly Π-closed), then G is also π-separable (respectively, Π-soluble, strongly Π-closed). Assume that this is false and let G be a counterexample with $|G| + |G : A| + |G : B|$ minimal. Then $A \neq 1 \neq B$ and $\Pi \cap \sigma(G) \neq \varnothing$. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group of G for $i = 1, \ldots, t$. Let R be a minimal normal subgroup.

(1) A and B are maximal $\mathcal{H}^G$-permutable subgroups of G.

It is clear that $A < G$, so, for some maximal $\mathcal{H}^G$-permutable subgroup E of G, we have $A \leq E$. Since $G = AB$, we get $E = A(B \cap E)$, where $B \cap E$ is π-separable (respectively, Π-soluble, strongly Π-closed (see Lemma 2.4 (1))). Moreover, $\mathcal{H}_0 = \{H_1 \cap E, \ldots, H_t \cap E\}$ is a complete Hall Π-set of E and the subgroups A and $B \cap E$ are $\mathcal{H}_0^E$-permutable by Lemma 2.1 (2), (3). Therefore, the hypothesis holds for $(E, A, B \cap E)$. Note also that $|E : B \cap E| = |A : A \cap B \cap E| = |A : A \cap B| = |G : B|$ and so $|E| + |E : A| + |E : B \cap E| < |G| + |G : A| + |G : B|$, which implies that E is π-separable (respectively, Π-soluble, strongly Π-closed). Therefore, if $A < E$, then the choice of (G, A, B) implies that G is π-separable (respectively, Π-soluble, strongly Π-closed), a contradiction. Hence, $A = E$ is a maximal $\mathcal{H}^G$-permutable subgroup of G. Similarly, B is a maximal $\mathcal{H}^G$-permutable subgroup of G.

(2) G/R is π-separable (respectively, Π-soluble, strongly Π-closed). Therefore, R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$ (this follows from Lemmas 2.1(1), 2.4 and the choice of G).

(3) G is π-separable (respectively, Π-soluble).

Assume that this is false. Then $O_\Pi(G) = 1 = O_{\Pi'}(G)$ (respectively, $O_{\Pi'}(G) = 1 = O_{\sigma_i}(G)$ for $\sigma_i \in \Pi$) by Claim (2), so $A_G = 1 = B_G$. Therefore from Lemma 2.3 and Claim (1) it follows that $t = 1$ and $G = AH_1 = BH_1$. In this case we get also that $O_\Pi(G) = O_{\sigma_1}(G) = 1$. On the other hand, we have $O_\Pi(A) \leq O_\Pi(G)$ by Proposition 3.1. Therefore $O_\Pi(A) = O_{\sigma_1}(A) = 1$ and so we have $W := O_{\sigma_1'}(A) \neq 1$ since $A \neq 1$ is π-separable, where $\pi = \pi(\Pi)$. From $G = AH_1 = BH_1$ it follows that $|G: B| = |A: B\cap A|$ is a σ_1-number and hence $1 < W \leq B\cap A$, so $W^G = W^{AB} = W^B \leq B$. Therefore, $B_G \neq 1$. This contradiction completes the proof of (3).

Now assume that A and B are strongly Π-closed.

(4) G is strongly Π-closed.

Assume that this is false. First note that by Corollary 3.1, G is Π-closed, that is, $O_\Pi(G)$ is a Hall π-subgroup of G. Moreover, $O_\Pi(G) \neq 1$ since $\Pi\cap\sigma(G) \neq \varnothing$. On the other hand, G is Π-soluble by Claim (3) since every strongly Π-closed group is Π-soluble. Therefore $O_\Pi(G)$ is σ-soluble. Hence R is a σ_i-group for some $\sigma_i \in \Pi$ and $O_{\sigma_j}(G) = 1$ for $j \neq i$ by Claim (2). Hence $O_{\sigma_j}(A) = 1$ for every $\sigma_j \in \Pi \setminus \sigma_i$ by Proposition 3.1, so $O_\Pi(A) = O_{\sigma_i}(A)$ is a Hall π-subgroup of A since A is Π-closed by hypothesis. Similarly, $O_\Pi(B) = O_{\sigma_i}(B)$ is a Hall π-subgroup of B. Therefore, $O_\Pi(G) = O_{\sigma_i}(G)$ is a Hall π-subgroup of G and so G is strongly Π-closed, a contradiction.

The theorem is proved.

***Proof of Theorem* B.** Assume that this theorem is false and let G be a counterexample with $|G|+|G: A|+|G: B|$ minimal. Then $A \neq 1 \neq B$ and p divides $|G|$. Let P be a Sylow p-subgroup of G and Q_i a Sylow q_i-subgroup of G for $i = 1, \ldots, t$, where $\{q_1, \ldots, q_t\} = \pi(G) \setminus \{p\}$. Let R be a minimal normal subgroup. Since $G/O_{p',p}(G)$ is nilpotent by hypothesis, G is p-soluble and so R is either a p-group or a p'-group.

(1) A and B are maximal p'-permutable subgroups of G.

It is clear that $A < G$, so, for some maximal p'-permutable subgroup E of G, we have $A \leq E$. First we show that E' is p-nilpotent. Since $E = A(B \cap E)$, where $|E| + |E: A| + |E: B \cap E| < |G| + |G: A| + |G: B|$ (see Claim (1) in the proof of Theorem A), it is enough to show that the hypothesis holds for $(E, A, B \cap E)$.

Let $O = O_{p',p}(G)$. Then $O \cap E \leq O_{p',p}(E)$ by Lemma 2.4 (1) and $OE/O \simeq E/(O \cap E)$ is nilpotent since G/O is nilpotent by hypothesis. Therefore $E/O_{p',p}(E)$ is nilpotent. Similarly, $(B \cap E)'$ is p-nilpotent. It is clear also that $(|E: A|, |E: B \cap E|) = 1$. Finally, the subgroups A and $B \cap E$ are p'-permutable (see Claim (1) in the proof of Theorem A). Hence, the hypothesis holds for $(E, A, B \cap E)$ and so the choice of (G, A, B) implies that E' is p-nilpotent.

If $A < E$, then $|G|+|G: E|+|G: B| < |G|+|G: A|+|G: B|$. On the other hand, the hypothesis holds for (G, E, B), so the choice of (G, A, B) implies that G' is p-nilpotent, a contradiction. Hence, $A = E$ is a maximal p'-permutable subgroup of G. Similarly, it can be proved that B is a maximal p'-permutable subgroup of G.

(2) The derived subgroup $(G/R)'$ of G/R is p-nilpotent for every minimal normal subgroup R of G.

Note that $G/R = (AR/R)(BR/R)$, where AR/R and BR/R are p'-permutable in G/R by Lemma 2.1(1). It is clear also that $(|G/R: AR/R|, |G/R: BR/R|) = 1$. Also, by Lemma 2.4 (2), we get $O_{p',p}(G)R/R \leq O_{p',p}(G/R)$, so $(G/R)/O_{p',p}(G/R)$ is nilpotent. Finally, $(AR/R)' =$

$= A'R/R \simeq A'/(A' \cap R)$ and $(BR/R)' \simeq B'/(B' \cap R)$ are p-nilpotent. Therefore, the hypothesis holds for G/R, so we have (2) by the choice of G.

(3) $R = C_G(R) = O_p(G) = O_{p',p}(G)$ is the unique minimal normal subgroup of G. Hence G/R is nilpotent.

Claim (2) implies that $(G/R)' = G'R/R \simeq G'/(G' \cap R)$ is p-nilpotent. On the other hand, G' is not p-nilpotent. Hence $R \leq G'$ and G'/R is p-nilpotent. Moreover, R is not a p'-group, so R is a p-group. Now note that if $N \neq R$ is a minimal normal subgroup of G, then $G' \simeq G'/(R \cap N) = G'/1$ is p-nilpotent by Lemma 2.4 (2). Hence R is the unique minimal normal subgroup of G. Moreover, $R \nleq \Phi(G)$ by Lemma 2.4 (3). Therefore for some maximal subgroup M of G we have $G = R \rtimes M$ and $M_G = 1$. But $C_G(R) \cap M$ is clearly normal in G and so $R = C_G(R) = O_p(G) = O_{p',p}(G)$ since $O_{p',p}(G) \leq C_G(R)$ by [11] (Appendixes, Theorem 3.2). Hence, $G/R = G/O_{p',p}(G)$ is nilpotent by hypothesis.

(4) $O_{p'}(A) = 1 = O_{p'}(B)$. Hence the subgroups A and B are p-closed (this follows from Proposition 3.1 and Claim (3)).

(5) $R \leq A \cap B$. Hence A and B are normal in G.

Assume, for example, that $R \nleq A$. Then $G = AR$ by the maximality of A, so $Q_1^G \leq A$ by Lemma 2.1(2) since R is a p-group by Claim (3). But then $R \leq Q_1^G \leq A$, again by Claim (3), a contradiction. Hence $R \leq A \cap B$, so A and B are subnormal in G since G/R is nilpotent by Claim (3). But then the maximality of A and B implies that A and B are normal in G.

The final contradiction. Since $(|G\colon A|, |G\colon B|) = 1$, we have either $P \leq A$ or $P \leq B$ by Claim (5). Then $P = R$ is normal in G since A and B are normal p-closed subgroups of G by Claims (4) and (5). Now for any i we obtain either $Q_i \leq A$ or $Q_i \leq B$. It follows that $Q_iR/R \simeq Q_i$ is Abelian by Claim (4). Hence every Sylow subgroup of the nilpotent group G/R is Abelian. Hence, G/R is Abelian, so G' is nilpotent, a contradiction.

The theorem is proved.

***Proof of Theorem* C.** Assume that this theorem is false and let G be a counterexample of minimal order. Then p divides $|G|$. Let P be a Sylow p-subgroup of G.

By Lemma 2.5, $A/O_{p',p}(A)$ and $B/O_{p',p}(B)$ are Abelian groups of exponent dividing $p-1$. Hence G' is p-nilpotent by Theorem B, so G is p-soluble.

Now let R be a minimal normal subgroup of G. The hypothesis holds for G/R by Lemma 2.1(1), so the choice of G implies that G/R is p-supersoluble. It follows that R is not a p'-group and R is the unique minimal normal subgroup of G with $R \nleq \Phi(G)$. Hence $O_{p'}(G) = 1$ and $R = C_G(R) = O_p(G) = O_{p',p}(G)$ (see the proof of Theorem B). Therefore, G' is a p-group since $O_{p'}(G')$ is characteristic in G' and so normal in G. But then P is normal in G. Hence $P = R$ and so G/R is an Abelian irreducible automorphism group of R, which implies that $G/R = (AR/R)(BR/R)$ is cyclic. Hence, for every Sylow q-subgroup Q of G, where $q \neq p$, we have either $QR/R \leq AR/R \simeq A/(A \cap R) = A/(A \cap P)$ or $QR/R \leq BR/R \simeq B/(B \cap R) = B/(B \cap P)$.

From Proposition 3.1 it follows that $O_{p',p}(A) = O_p(A) = A \cap P$ is a Sylow p-subgroup of A and $O_{p',p}(B) = O_p(B) = B \cap P$ is a Sylow p-subgroup of B. Therefore $A/(A \cap P)$ and $B/(B \cap P)$ are Abelian groups of exponent dividing $p-1$ by Lemma 2.5. Hence, for every Sylow q-subgroup Q of G, where $q \neq p$, $Q \simeq QR/R$ is a cyclic group of exponent dividing $p-1$. But then G/R is a cyclic group of exponent dividing $p-1$ and so $|R| = p$ [11] (Ch. 1, Theorem 1.4). Therefore G is p-supersoluble, a contradiction.

The theorem is proved.

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