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CLASSES OF HARMONIC FUNCTIONS DEFINED BY EXTENDED SĂLĂGEAN OPERATOR*

КЛАСИ ГАРМОНІЧНИХ ФУНКЦІЙ, ЯКІ ВИЗНАЧЕНІ РОЗШИРЕНИМ ОПЕРАТОРОМ САЛАДЖАНА

The object of the present paper is to investigate classes of harmonic functions defined by the extended Sălăgean operator. By using the extreme points theory we obtain coefficients estimates and distortion theorems for these classes of functions. Some integral mean inequalities are also pointed out.

Досліджуються класи гармонічних функцій, які визначені розширеним оператором Саладжана. За допомогою теорії екстремальних точок отримано оцінки для коефіцієнтів та теореми деформації для класів функцій. Також наведено деякі нерівності для інтегральних середніх.

1. Introduction. A complex-valued harmonic mapping f in the open unit disk $\mathbb{U} := \mathbb{U}(1)$, where $\mathbb{U}(r) := \{z \in \mathbb{C} : |z| < r\}$, has a canonical decomposition

$$f = h + \overline{g}, \tag{1}$$

where h and g are analytic functions in $\mathbb{U}$. We call h the analytic part and g the coanalytic part of f, respectively. Throughout this paper, we will discuss harmonic mappings that are sense-preserving in $\mathbb{U}$. By a theorem of Lewy [16], necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{U}$ is

$$\left|h'(z)\right| > \left|g'(z)\right|, \quad z \in \mathbb{U}. \tag{2}$$

Let $\mathcal{H}$ denote the class of sense-preserving harmonic functions in the unit disc $\mathbb{U}$. Any function $f \in \mathcal{H}$ can be written in the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}, \quad z \in \mathbb{U}. \tag{3}$$

Let $\mathbb{N}_l := \{l, l+1, \ldots\}$, $\mathbb{N} := \mathbb{N}_1$, $k \in \mathbb{N}_2$. We denote by $\mathcal{S}_{\mathcal{H}}(k)$ the class of function $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{n=k}^{\infty} \left(a_n z^n + \overline{b_n z^n}\right), \quad z \in \mathbb{U}, \tag{4}$$

which are univalent in $\mathbb{U}$.

We say that a function $f \in \mathcal{S}_{\mathcal{H}}(k)$ is harmonic starlike in $\mathbb{U}(r)$ if $\dfrac{\partial}{\partial t}\left(\arg f\left(re^{it}\right)\right) > 0$, $0 \le t \le 2\pi$, i.e., f maps the circle $\partial\mathbb{U}(r)$ onto a closed curve that is starlike with respect to the origin. It is easy to verify, that the condition (5) is equivalent to the following:

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$$\operatorname{Re}\frac{D_{\mathcal{H}}f(z)}{f(z)}>0,\quad |z|=r,$$

where

$$D_{\mathcal{H}}f(z):=zh'(z)-\overline{zg'(z)},\quad z\in\mathbb{U}.$$

For $\lambda,\tau\in\mathbb{C},\ |\tau|=1$ and $f=h+\overline{g}\in\mathcal{H}$ of the form (1), we consider the linear operator $J_{\mathcal{H}}^{\lambda,\tau}:\mathcal{H}\to\mathcal{H}$ defined by

$$J_{\mathcal{H}}^{\lambda,\tau}f(z):=z+\sum_{n=2}^{\infty}n^{\lambda}a_nz^n+\tau\sum_{n=2}^{\infty}\overline{n^{\lambda}b_n}\overline{z}^n,\quad z\in\mathbb{U}.$$

For the analytic definition of the above case, see Sălăgean operator [20]. The operator $J_{\mathcal{H}}^{\lambda,\tau}$ for $\lambda=n\in\mathbb{N},\ \tau=(-1)^n$ was investigated in [17] (see also [5, 9, 11, 22]).

We say that a function $f\in\mathcal{H}$ is *subordinate* to a function $F\in\mathcal{H}$, and write $f(z)\prec F(z)$ (or simply $f\prec F$) if there exists a complex-valued function ω which maps $\mathbb{U}$ into oneself with $\omega(0)=0$ such that $f(z)=F(\omega(z)),\ z\in\mathbb{U}$.

Let A,B be complex parameters, $A\neq B$. We denote by $\mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(k;A,B)$ the class of functions $f\in\mathcal{S}_{\mathcal{H}}(k)$ such that

$$\frac{J_{\mathcal{H}}^{\lambda+1,-\tau}f(z)}{J_{\mathcal{H}}^{\lambda,\tau}f(z)}\prec\frac{1+Az}{1+Bz}. \tag{5}$$

Also, by $\mathcal{R}_{\mathcal{H}}^{\lambda,\tau}(k;A,B)$ we denote the class of functions $f\in\mathcal{S}_{\mathcal{H}}(k)$ such that

$$\frac{J_{\mathcal{H}}^{\lambda,\tau}f(z)}{z}\prec\frac{1+Az}{1+Bz}.$$

In particular, if we put $\lambda=n\in\mathbb{N}_0,\ \tau=(-1)^n$, then we obtain the classes

$$\mathcal{S}_{\mathcal{H}}^{n}(k;A,B):=\mathcal{S}_{\mathcal{H}}^{n,(-1)^n}(k;A,B),\ \mathcal{R}_{\mathcal{H}}^{n}(k;A,B):=\mathcal{R}_{\mathcal{H}}^{n,(-1)^n}(k;A,B)$$

studied in [8]. The classes $\mathcal{S}_{\mathcal{H}}(k;A,B):=\mathcal{S}_{\mathcal{H}}^{0}(k;A,B),\ \mathcal{K}_{\mathcal{H}}(k;A,B):=\mathcal{S}_{\mathcal{H}}^{1}(k;A,B)$ and $\mathcal{R}_{\mathcal{H}}(k;A,B):=\mathcal{R}_{\mathcal{H}}^{1}(k;A,B)$ are defined in [5] (see also) with restrictions $-B\leq A<B\leq 1$, $k=2$.

In this paper, we obtain some necessary and sufficient conditions for defined classes of functions. Some topological properties and extreme points of the classes are also considered. By using extreme points theory we obtain coefficients estimates, distortion theorems, integral mean inequalities for these classes of functions.

2. Dual sets. For functions $f_1,f_2\in\mathcal{H}$ of the form

$$f_l(z)=z+\sum_{n=k}^{\infty}a_{l,n}z^n+\sum_{n=k}^{\infty}\overline{b_{l,n}z^n},\quad z\in\mathbb{D},\quad l\in\mathbb{N}, \tag{6}$$

we define *the Hadamard product or convolution* of f_1 and f_2 by

$$(f_1*f_2)(z)=\sum_{k=0}^{\infty}\left(a_{1,k}a_{2,k}z^k+\overline{b_{1,k}b_{2,k}z^k}\right),\quad z\in\mathbb{U}.$$

Let $\mathcal{V} \subset \mathcal{H}$, $\mathbb{U}_0 := \mathbb{U} \setminus \{0\}$. Motivated by Ruscheweyh [19] we define the dual set of $\mathcal{V}$ by

$$\mathcal{V}^* := \left\{ f \in \mathcal{S}_{\mathcal{H}}(k) : \bigwedge_{q \in \mathcal{V}} (f * q)(z) \neq 0,\ z \in \mathbb{U}_0 \right\}.$$

The object of the section is to show that the defined classes of functions can be presented as dual sets.

Theorem 1.

$$\mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(k; A, B) = \left\{ J_{\mathcal{H}}^{\lambda,\tau}(\psi_\xi) : |\xi| = 1 \right\}^*,$$

where

$$\psi_\xi(z) := z\frac{(B-A)\xi + (1+A\xi)z}{(1-z)^2} - \overline{z}\frac{2 + (A+B)\xi - (1+A\xi)\overline{z}}{(1-\overline{z})^2}, \quad z \in \mathbb{U}. \tag{7}$$

Proof. Let $f \in \mathcal{H}$. Then $f \in \mathcal{H}^\lambda(A, B)$ if and only if and the condition (5) holds or equivalently

$$\frac{J_{\mathcal{H}}^{\lambda+1} f(z)}{J_{\mathcal{H}}^{\lambda} f(z)} \neq \frac{1 + A\zeta}{1 + B\zeta}, \quad \zeta \in \mathbb{C}, \quad |\zeta| = 1. \tag{8}$$

Now for $J_{\mathcal{H}}^{\lambda+1} h(z) = J_{\mathcal{H}}^{\lambda} h(z) * z/(1-z)^2$ and $J_{\mathcal{H}}^{\lambda} h(z) = J_{\mathcal{H}}^{\lambda} h(z) * z/(1-z)$, the above inequality (8) yields

$$\begin{aligned}
&(1 + B\zeta) J_{\mathcal{H}}^{\lambda+1} f(z) - (1 + A\zeta) J_{\mathcal{H}}^{\lambda} f(z) = \\
&= (1 + B\zeta) J_{\mathcal{H}}^{\lambda+1} h(z) - (1 + A\zeta) J_{\mathcal{H}}^{\lambda} h(z) - \\
&- (-1)^\lambda \left[(1 + B\zeta) J_{\mathcal{H}}^{\lambda+1} \overline{g(z)} + (1 + A\zeta) J_{\mathcal{H}}^{\lambda} \overline{g(z)} \right] = \\
&= J_{\mathcal{H}}^{\lambda} h(z) * \left(\frac{(1 + B\zeta) z}{(1-z)^2} - \frac{(1 + A\zeta) z}{1 - z} \right) - \\
&- (-1)^{\lambda+1} J_{\mathcal{H}}^{\lambda} \overline{g(z)} * \left(\frac{(1 + B\zeta)\overline{z}}{(1-\overline{z})^2} + \frac{(1 + A\zeta)\overline{z}}{1 - \overline{z}} \right) = \\
&= J_{\mathcal{H}}^{\lambda} f(z) * \phi(z; \zeta) \neq 0.
\end{aligned}$$

Let $f \in \mathcal{S}_{\mathcal{H}}(k)$ be of the form (1). Then $f \in \mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(k; A, B)$ if and only if it satisfies (5) or equivalently

$$\frac{J_{\mathcal{H}}^{\lambda+1,-\tau} f(z)}{J_{\mathcal{H}}^{\lambda,\tau} f(z)} \neq \frac{1 + A\xi}{1 + B\xi}, \quad z \in \mathbb{U}_0, \quad |\xi| = 1. \tag{9}$$

Since

$$J_{\mathcal{H}}^{\lambda+1,-\tau} h(z) = J_{\mathcal{H}}^{\lambda,\tau} h(z) * \frac{z}{(1-z)^2}, \quad J_{\mathcal{H}}^{\lambda,\tau} h(z) = J_{\mathcal{H}}^{\lambda,\tau} h(z) * \frac{z}{1-z},$$

the above inequality yields

$$0 \neq (1+B\xi)\, J_{\mathcal{H}}^{\lambda+1,-\tau} f(z) - (1+A\xi)\, J_{\mathcal{H}}^{\lambda,\tau} f(z) =$$

$$= (1+B\xi)\, J_{\mathcal{H}}^{\lambda+1,-\tau} h(z) - (1+A\xi)\, J_{\mathcal{H}}^{\lambda,\tau} h(z) -$$

$$-\tau \left[(1+B\xi)\, \overline{J_{\mathcal{H}}^{\lambda+1,-\tau} g(z)} + (1+A\xi)\, \overline{J_{\mathcal{H}}^{\lambda,\tau} h(z)} \right] =$$

$$= J_{\mathcal{H}}^{\lambda,\tau} h(z) * \left(\frac{(1+B\xi)\, z}{(1-z)^2} - \frac{(1+A\xi)\, z}{1-z} \right) -$$

$$-\tau \overline{g(z)} * \left(\frac{(1+B\zeta)\, \overline{z}}{(1-\overline{z})^2} + \frac{(1+A\zeta)\, \overline{z}}{1-\overline{z}} \right) =$$

$$= f(z) * J_{\mathcal{H}}^{\lambda,\tau} \psi_\xi(z), \quad z \in \mathbb{U}_0, \quad |\xi| = 1.$$

Thus, $f \in \mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(k; A, B)$ if and only if $f(z) * J_{\mathcal{H}}^{\lambda,\tau} \psi_\xi(z) \neq 0$ for $z \in \mathbb{U}_0$, $|\xi| = 1$, i.e., $\mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(k; A, B) = \left\{ J_{\mathcal{H}}^{\lambda,\tau}(\psi_\xi) : |\xi| = 1 \right\}^*$.

Similarly as Theorem 1 we prove the following theorem.

Theorem 2.

$$\mathcal{R}_{\mathcal{H}}^{\lambda,\tau}(k; A, B) = \left\{ J_{\mathcal{H}}^{\lambda,\tau}(\delta_\xi) : |\xi| = 1 \right\}^*,$$

where

$$\delta_\xi(z) := z \frac{1 + B\xi - (1+A\xi)(1-z)}{1-z} + \tau \overline{z} \frac{1+B\xi}{1-\overline{z}}, \quad z \in \mathbb{U}. \tag{10}$$

If we put $\lambda = 0$ or $\lambda = 1$, $\tau = (-1)^n$ in Theorems 1 and 2 we obtain the following results (see [5]).

Theorem 3.

$$\mathcal{S}_{\mathcal{H}}(k; A, B) = \{\psi_\xi : |\xi| = 1\}^*,$$

where ψ_ξ is defined by (7).

Theorem 4.

$$\mathcal{K}_{\mathcal{H}}(k; A, B) = \{\psi_\xi : |\xi| = 1\}^*,$$

where

$$\psi_\xi(z) := z \frac{(B-A)\xi + (2 + A\xi + B\xi) z}{(1-z)^3} - \overline{z} \frac{2 + (A+B)\xi + (B-A)\xi \overline{z}}{(1-\overline{z})^3}, \quad z \in \mathbb{U}.$$

Theorem 5.

$$\mathcal{R}_{\mathcal{H}}(k; A, B) = \{\delta_\xi : |\xi| = 1\}^*,$$

where

$$\delta_\xi(z) := z \frac{1 + B\xi - (1+A\xi)(1-z)^2}{(1-z)^2} - \overline{z} \frac{1+B\xi}{(1-\overline{z})^2}, \quad z \in \mathbb{U}.$$

3. Correlated coefficients. Let us consider the function $\varphi \in \mathcal{H}$ of the form

$$\varphi = u + \overline{v}, \quad u(z) = \sum_{n=0}^{\infty} u_n z^n, \quad v(z) = \sum_{n=1}^{\infty} v_n z^n m, \quad z \in \mathbb{U}. \tag{11}$$

We say that a function $f \in \mathcal{H}$ of the form (4) has coefficients correlated with the function φ if

$$u_n a_n = -\left|u_n\right| \left|a_n\right|, \; v_n b_n = \left|v_n\right| \left|b_n\right|, \quad n \in \mathbb{N}_k. \tag{12}$$

In particular, if there exists a real number η such that

$$\varphi(z) = \frac{z}{1 - e^{i\eta} z} + \frac{e^{2i\eta}\overline{z}}{1 - e^{i\eta}\overline{z}} = \sum_{n=1}^{\infty} \left(e^{i(n-1)\eta} z^n + e^{i(n+1)\eta} \overline{z}^n \right), \quad z \in \mathbb{U},$$

then we obtain functions with varying coefficients defined by Jahangiri and Silverman [12] (see also [8]). Moreover, if we take

$$\varphi(z) = 2\mathrm{Re}\frac{z}{1 - z} = \sum_{n=1}^{\infty} \left(z^n + \overline{z}^n \right), \quad z \in \mathbb{U},$$

then we obtain functions with negative coefficients introduced by Silverman [21]. These functions were intensively investigated by many authors (see, for example, [4, 5, 8 – 10, 12, 14, 24]).

Let $\mathcal{T}^{\lambda,\tau}(k, \eta)$ denote the class of functions $f \in \mathcal{H}$ with coefficients correlated with respect to the function

$$\varphi(z) = J_{\mathcal{H}}^{\lambda,\tau} \left(\frac{z}{1 - e^{i\eta} z} + \frac{e^{2i\eta}\overline{z}}{1 - e^{i\eta}\overline{z}} \right) = \sum_{n=1}^{\infty} e^{i\eta(n-1)} n^{\lambda} z^n + \tau \sum_{n=2}^{\infty} e^{i\eta(n+1)} \overline{n^{\lambda} z^n}, \quad z \in \mathbb{U}. \tag{13}$$

Moreover, let us define

$$\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k, \eta; A, B) := \mathcal{T}^{\lambda,\tau}(k, \eta) \cap \mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(k; A, B), \; \mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(k, \eta; A, B) := \mathcal{T}^{\lambda,\tau}(k, \eta) \cap \mathcal{R}_{\mathcal{H}}^{\lambda,\tau}(k; A, B),$$

where η, A, and B are real parameters with $B > \max\{0, A\}$. Finally, let us assume

$$\left| n^{\lambda} \right| \geq \left| k^{\lambda} \right| \geq 1, \quad n \in \mathbb{N}_k. \tag{14}$$

Theorem 6. *If a function $f \in \mathcal{H}$ of the form* (4) *satisfies the condition*

$$\sum_{n=k}^{\infty} \left(\left|\alpha_n\right| \left|a_n\right| + \left|\beta_n\right| \left|b_n\right| \right) \leq B - A, \tag{15}$$

where

$$\alpha_n = n^{\lambda} \left\{ n\left(1 + B\right) - \left(1 + A\right) \right\}, \quad \beta_n = n^{\lambda} \left\{ n\left(1 + B\right) + \left(1 + A\right) \right\}, \tag{16}$$

then $f \in \mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(k; A, B)$.

Proof. It is clear that the theorem is true for the function $f(z) \equiv z$. Let $f \in \mathcal{H}$ be a function of the form (4) and let there exist $n \in \mathbb{N}_k$ such that $a_n \neq 0$ or $b_n \neq 0$. By (14) we have

$$\frac{|\alpha_n|}{B-A} \geq n, \quad \frac{|\beta_n|}{B-A} \geq n, \quad n \in \mathbb{N}_k, \tag{17}$$

Thus, by (15) we get

$$\sum_{n=k}^{\infty} (n|a_n| + n|b_n|) \leq 1 \tag{18}$$

and

$$\left|h'(z)\right| - \left|g'(z)\right| \geq 1 - \sum_{n=k}^{\infty} n\,|a_n|\,|z|^n - \sum_{n=k}^{\infty} n|b_n|\,|z|^n \geq 1 - |z| \sum_{n=k}^{\infty} (n|a_n| + n|b_n|) \geq$$

$$\geq 1 - \frac{|z|}{B-A} \sum_{n=k}^{\infty} (|\alpha_n||a_n| + |\beta_n||b_n|) \geq 1 - |z| > 0, \quad z \in \mathbb{U}.$$

Therefore, by (2) the function f is locally univalent and sense-preserving in $\mathbb{U}$. Moreover, if $z_1, z_2 \in$ $\in \mathbb{U},\ z_1 \neq z_2$, then

$$\left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| = \left|\sum_{l=1}^{n} z_1^{l-1} z_2^{n-l}\right| \leq \sum_{l=1}^{n} |z_1|^{l-1}\,|z_2|^{n-l} < n, \quad n \in \mathbb{N}_k.$$

Hence, by (18) we have

$$|f(z_1) - f(z_2)| \geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \geq$$

$$\geq \left|z_1 - z_2 - \sum_{n=k}^{\infty} a_n (z_1^n - z_2^n)\right| - \left|\sum_{n=k}^{\infty} \overline{b_n (z_1^n - z_2^n)}\right| \geq$$

$$\geq |z_1 - z_2| - \sum_{n=k}^{\infty} |a_n|\,|z_1^n - z_2^n| - \sum_{n=k}^{\infty} |b_n|\,|z_1^n - z_2^n| =$$

$$= |z_1 - z_2| \left(1 - \sum_{n=k}^{\infty} |a_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| - \sum_{n=k}^{\infty} |b_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right|\right) >$$

$$> |z_1 - z_2| \left(1 - \sum_{n=k}^{\infty} n|a_n| - \sum_{n=k}^{\infty} n|b_n|\right) \geq 0.$$

This leads to the univalence of f, i.e., $f \in \mathcal{S}_{\mathcal{H}}$. Therefore, $f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k; A, B)$ if and only if there exists a complex-valued function ω, $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in \mathbb{U}$, such that

$$\frac{J_{\mathcal{H}}^{\lambda+1,-\tau}(z)}{J_{\mathcal{H}}^{\lambda,\tau} f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad z \in \mathbb{U},$$

or equivalently

$$\left|\frac{D_{\mathcal{H}}\left(J_{\mathcal{H}}^{\lambda,\tau}f\right)(z)-J_{\mathcal{H}}^{\lambda,\tau}f(z)}{BD_{\mathcal{H}}\left(J_{\mathcal{H}}^{\lambda\tau}f\right)(z)-A\left(J_{\mathcal{H}}^{\lambda\tau}f(z)\right)(z)}\right|<1,\quad z\in\mathbb{U}. \tag{19}$$

Thus, it is suffice to prove that

$$\left|D_{\mathcal{H}}\left(J_{\mathcal{H}}^{\lambda\tau}f\right)(z)-J_{\mathcal{H}}^{\lambda\tau}f(z)\right|-\left|BD_{\mathcal{H}}\left(J_{\mathcal{H}}^{\lambda\tau}f\right)(z)-AJ_{\mathcal{H}}^{\lambda\tau}f(z)\right|<0,\quad z\in\mathbb{U}\setminus\{0\}.$$

Indeed, letting $|z|=r,\ 0<r<1$, we have

$$\left|D_{\mathcal{H}}\left(J_{\mathcal{H}}^{\lambda\tau}f\right)(z)-J_{\mathcal{H}}^{\lambda\tau}f(z)\right|-\left|BD_{\mathcal{H}}\left(J_{\mathcal{H}}^{\lambda\tau}f\right)(z)-AJ_{\mathcal{H}}^{\lambda\tau}f(z)\right|=$$

$$=\left|\sum_{n=k}^{\infty}(n-1)n^{\lambda}a_nz^n-\sum_{n=k}^{\infty}(n+1)\tau\overline{n^{\lambda}b_n}\overline{z}^n\right|-$$

$$-\left|(B-A)z+\sum_{n=k}^{\infty}(Bn-A)n^{\lambda}a_nz^n+\sum_{n=k}^{\infty}(Bn+A)\tau\overline{n^{\lambda}b_n}\overline{z}^n\right|\le$$

$$\le\sum_{n=k}^{\infty}(n-1)\left|n^{\lambda}a_n\right|r^n+\sum_{n=k}^{\infty}(n+1)\left|n^{\lambda}b_n\right|r^n-(B-A)r+$$

$$+\sum_{n=k}^{\infty}(Bn-A)\left|n^{\lambda}a_n\right|r^n+\sum_{n=k}^{\infty}(Bn+A)\left|n^{\lambda}b_n\right|r^n\le$$

$$\le r\left\{\sum_{n=k}^{\infty}\left(|\alpha_n|\,|a_n|+|\beta_n|\,|b_n|\right)r^{n-1}-(B-A)\right\}<0,$$

whence $f\in\mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(k;A,B)$.

The next theorem, shows that the condition (15) is also the sufficient condition for a function $f\in\mathcal{H}$ of correlated coefficients to be in the class $\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$.

Theorem 7. *Let $f\in\mathcal{T}^{\lambda,\tau}(k,\eta)$ be a function of the form* (4). *Then $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$ if and only if the condition* (15) *holds true.*

Proof. In view of Theorem 6 we need only show that each function $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$ satisfies the coefficient inequality (15). If $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$, then it is of the form (4) with (12) and it satisfies (19) or equivalently

$$\left|\frac{\sum_{n=k}^{\infty}(n-1)n^{\lambda}a_nz^n-(n+1)\tau\overline{n^{\lambda}b_n}\overline{z}^n}{(B-A)z+\sum_{n=k}^{\infty}\left\{(Bn-A)n^{\lambda}a_nz^n-(Bn+A)\tau\overline{n^{\lambda}b_n}\overline{z}^n\right\}}\right|<1,\quad z\in\mathbb{U}.$$

Therefore, putting $z=re^{i\eta},\ 0\le r<1$, by (13) and (12) we obtain

$$\frac{\sum_{n=k}^{\infty}(n-1)\left|n^{\lambda}\right|\,|a_n|+(n+1)\left|n^{\lambda}\right|\,|b_n|r^{n-1}}{(B-A)-\sum_{n=k}^{\infty}\left\{(Bn-A)\left|n^{\lambda}\right|\,|a_n|+(Bn+A)\left|n^{\lambda}\right|\,|b_n|r^{n-1}\right\}}<1. \tag{20}$$

It is clear that the denominator of the left-hand side cannot vanish for $r \in \langle 0,1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in \langle 0,1)$. Thus, by (20) we have

$$\sum_{n=k}^{\infty} \left(|\alpha_n||a_n| + |\beta_n||b_n|\right) r^{n-1} < B - A, \quad 0 \le r < 1. \tag{21}$$

The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=k}^{\infty} \left(|\alpha_n|\,|a_n| + |\beta_n|\,|b_n|\right)$ is non-decreasing sequence. Moreover, by (21) it is bounded by $B - A$. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=k}^{\infty} \left(|\alpha_n|\,|a_n| + |\beta_n|\,|b_n|\right) = \lim_{n\to\infty} S_n \le B - A,$$

which yields the assertion (15).

The following result may be proved in much the same way as Theorem 7.

Theorem 8. *Let $f \in \mathcal{T}^{\lambda,\tau}(k,\eta)$ be a function of the form* (4). *Then $f \in \mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$ if and only if*

$$\sum_{n=k}^{\infty} \left|n^{\lambda}\right| \left(|a_n| + |b_n|\right) \le \frac{B-A}{1+B}.$$

By Theorems 7 and 8 we have the following corollary.

***Corollary* 1.** *Let $a = \dfrac{1+A}{1+B}$ and*

$$\phi(z) = z + \sum_{n=k}^{\infty} \left(\frac{1}{n-a} z^n + \frac{1}{n+a}\overline{z}^n\right), \quad z \in \mathbb{U},$$

$$\omega(z) = z + \sum_{n=k}^{\infty} \left((n-a)\, z^n + (n+a)\,\overline{z}^n\right), \quad z \in \mathbb{U}.$$

Then

$$f \in \mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B) \Leftrightarrow f * \phi \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B),$$

$$f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B) \Leftrightarrow f * \omega \in \mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B).$$

In particular,

$$\mathcal{R}_{\mathcal{T}}^{\lambda+1,\tau}(k,\eta;-1,B) = \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;-1,B).$$

4. Topological properties. We consider the usual topology on $\mathcal{H}$ defined by a metric in which a sequence $\{f_n\}$ in $\mathcal{H}$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathbb{U}$. It follows from the theorems of Weierstrass and Montel that this topological space is complete.

Let $\mathcal{F}$ be a subclass of the class $\mathcal{H}$. A functions $f \in \mathcal{F}$ is called *an extreme point of* $\mathcal{F}$ if the condition

$$f = \gamma f_1 + (1-\gamma) f_2, \quad f_1, f_2 \in \mathcal{F}, \quad 0 < \gamma < 1,$$

implies $f_1 = f_2 = f$. We shall use the notation $E\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F} \subset \mathcal{F}$.

We say that $\mathcal{F}$ is *locally uniformly bounded* if for each r, $0 < r < 1$, there is a real constant $M = M(r)$ so that

$$|f(z)| \le M, \quad f \in \mathcal{F}, \quad |z| \le r.$$

We say that a class $\mathcal{F}$ is *convex* if

$$\gamma f + (1-\gamma) g \in \mathcal{F}, \quad f, g \in \mathcal{F}, \quad 0 \le \gamma \le 1.$$

Moreover, we define *the closed convex* hull of $\mathcal{F}$ as the intersection of all closed convex subsets of $\mathcal{H}$ that contain $\mathcal{F}$. We denote the closed convex hull of $\mathcal{F}$ by $\overline{co}\mathcal{F}$.

A real-valued functional $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ is called *convex* on a convex class $\mathcal{F} \subset \mathcal{H}$ if

$$\mathcal{J}(\gamma f + (1-\gamma) g) \le \gamma \mathcal{J}(f) + (1-\gamma)\mathcal{J}(g), \quad f, g \in \mathcal{F}, \quad 0 \le \gamma \le 1.$$

The Krein – Milman theorem (see [15]) is fundamental in the theory of extreme points. In particular, it implies the following lemma.

Lemma 1 [5, p. 45]. *Let $\mathcal{F}$ be a nonempty compact convex subclass of the class $\mathcal{H}$ and $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ be a real-valued, continuous and convex functional on $\mathcal{F}$. Then*

$$\max\{\mathcal{J}(f) : f \in \mathcal{F}\} = \max\{\mathcal{J}(f) : f \in E\mathcal{F}\}.$$

Since $\mathcal{H}$ is a complete metric space, Montel's theorem implies the following lemma.

Lemma 2. *A class $\mathcal{F} \subset \mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

Theorem 9. *The class $\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k, \eta; A, B)$ is convex and compact subset of $\mathcal{H}$.*

Proof. Let $f_1, f_2 \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k, \eta; A, B)$ be functions of the form (6), $0 \le \gamma \le 1$. Since

$$\gamma f_1(z) + (1-\gamma) f_2(z) = z + \sum_{n=k}^{\infty} \left\{ (\gamma a_{1,n} + (1-\gamma) a_{2,n}) z^n + \overline{(\gamma b_{1,n} + (1-\gamma) b_{2,n}) z^n} \right\},$$

and by Theorem 7 we have

$$\sum_{n=k}^{\infty} \{|\alpha_n| \, |\gamma a_{1,n} + (1-\gamma) a_{2,n}| + |\beta_n| \, |\gamma b_{1,n} + (1-\gamma) \, b_{2,n} z^n|\} \le$$

$$\le \gamma \sum_{n=k}^{\infty} \{|\alpha_n a_{1,n}| + |\beta_n b_{1,n}|\} + (1-\gamma) \sum_{n=k}^{\infty} \{|\alpha_n a_{2,n}| + |\beta_n b_{2,n}|\} \le$$

$$\le \gamma (B - A) + (1-\gamma)(B - A) = B - A,$$

the function $\phi = \gamma f_1 + (1-\gamma) f_2$ belongs to the class $\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k, \eta; A, B)$. Hence, the class is convex. Furthermore, for $f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k, \eta; A, B)$, $|z| \le r$, $0 < r < 1$, we obtain

$$|f(z)| \le r + \sum_{n=k}^{\infty} (|a_n| + |b_n|) \, r^n \le r + \sum_{n=k}^{\infty} (|\alpha_n| \, |a_n| + |\beta_n| \, |b_n|) \le r + (B - A).$$

Thus, we conclude that the class $\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$ is locally uniformly bounded. By Lemma 2, we only need to show that it is closed, i.e., if $f_l \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$, $l \in \mathbb{N}$, and $f_l \to f$, then $f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$. Let f_l and f are given by (6) and (4), respectively. Using Theorem 7 we get

$$\sum_{n=k}^{\infty} (|\alpha_n a_{l,n}| + |\beta_n b_{l,n}|) \le B - A, \quad l \in \mathbb{N}. \tag{22}$$

Since $f_l \to f$, we conclude that $|a_{l,n}| \to |a_n|$ and $|b_{l,n}| \to |b_n|$ as $l \to \infty$, $n \in \mathbb{N}$. The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=k}^{\infty} (|\alpha_n a_n| + |\beta_n b_n|)$ is nondecreasing sequence. Moreover, by (22) it is bounded by $B - A$. Therefore, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=k}^{\infty} (|\alpha_n a_n| + |\beta_n b_n|) = \lim_{n\to\infty} S_n \le B - A.$$

This gives the condition (15), and, in consequence, $f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$, which completes the proof.

Theorem 10.

$$E\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B) = \{h_n : n \in \mathbb{N}_{k-1}\} \cup \{g_n : n \in \mathbb{N}_k\},$$

where

$$h_{k-1}(z) = z,\ h_n(z) = z - \frac{B-A}{\alpha_n e^{i(n-1)\eta}} z^n, \quad g_n(z) = z + \frac{B-A}{\tau\beta_n e^{-i(n+1)\eta}} \overline{z}^n, \quad z \in \mathbb{U}. \tag{23}$$

Proof. Suppose that $0 < \gamma < 1$ and

$$g_n = \gamma f_1 + (1-\gamma) f_2,$$

where $f_1, f_2 \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$ are functions of the form (6). Then, by (15) we have $|b_{1,n}| = |b_{2,n}| = \dfrac{B-A}{|\beta_n|}$, and, in consequence, $a_{1,l} = a_{2,l} = 0$ for $l \in \mathbb{N}_k$ and $b_{1,l} = b_{2,l} = 0$ for $l \in \mathbb{N}_k \setminus \{n\}$. It follows that $g_n = f_1 = f_2$, and consequently $g_n \in E\mathcal{S}_{\mathcal{T}}^{*}(k,\eta;A,B)$. Similarly, we verify that the functions h_n of the form (23) are the extreme points of the class $\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$. Now, suppose that a function f belongs to the set $E\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$ and f is not of the form (23). Then there exists $m \in \mathbb{N}_k$ such that

$$0 < |a_m| < \frac{B-A}{|\alpha_m|} \ \text{ or } \ 0 < |b_m| < \frac{B-A}{|\beta_m|}.$$

If $0 < |a_m| < \dfrac{B-A}{|\alpha_m|}$, then putting

$$\gamma = \frac{|\alpha_m a_m|}{B-A}, \quad \varphi = \frac{1}{1-\gamma}\left(f - \gamma h_m\right),$$

we have that $0 < \gamma < 1$, $h_m \neq \varphi$ and

$$f = \gamma h_m + (1-\gamma)\varphi.$$

Thus, $f \notin ES_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$. Similarly, if $0 < |b_m| < \dfrac{B-A}{|\beta_n|}$, then putting

$$\gamma = \frac{|\beta_m b_m|}{B-A}, \quad \phi = \frac{1}{1-\gamma}(f - \gamma g_m),$$

we have that $0 < \gamma < 1,\ g_m \neq \phi$ and

$$f = \gamma g_m + (1-\gamma)\phi.$$

It follows that $f \notin ES_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$, and the proof is completed.

5. Applications. It is clear that if the class

$$\mathcal{F} = \{f_n \in \mathcal{H} : n \in \mathbb{N}\},$$

is locally uniformly bounded, then

$$\overline{co}\mathcal{F} = \left\{ \sum_{n=1}^{\infty} \gamma_n f_n : \sum_{n=1}^{\infty} \gamma_n = 1,\ \gamma_n \geq 0,\ n \in \mathbb{N} \right\}.$$

Thus, by Theorem 7 we have the following corollary.

***Corollary* 2.**

$$\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B) = \left\{ \sum_{n=k-1}^{\infty} (\gamma_n h_n + \delta_n g_n) : \sum_{n=k-1}^{\infty} (\gamma_n + \delta_n) = 1\ \ (\delta_{k-1} = 0, \gamma_n, \delta_n \geq 0) \right\},$$

where $h_n,\ g_n$ *are defined by* (23).

For each fixed value of $m, n \in \mathbb{N}_k,\ z \in \mathbb{U}$, the following real-valued functionals are continuous and convex on $\mathcal{H}$:

$$\mathcal{J}(f) = |a_n|, \quad \mathcal{J}(f) = |b_n|, \quad \mathcal{J}(f) = |f(z)|\ \ \mathcal{J}(f) = |D_{\mathcal{H}} f(z)|, \quad f \in \mathcal{H}.$$

Moreover, for $\gamma \geq 1,\ 0 < r < 1$, the real-valued functional

$$\mathcal{J}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^{\gamma} d\theta \right)^{1/\gamma}, \quad f \in \mathcal{H},$$

is also continuous and convex on $\mathcal{H}$.

Therefore, by Lemma 1 and Theorem 7 we have the following corollaries.

***Corollary* 3.** *Let* $f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$ *be a function of the form* (4). *Then*

$$|a_n| \leq \frac{B-A}{|\alpha_n|}, \quad |b_n| \leq \frac{B-A}{|\beta_n|}, \quad n \in \mathbb{N}_k,$$

where α_n, β_n *are defined by* (16). *The result is sharp. The functions* $h_n,\ g_n$ *of the form* (23) *are the extremal functions.*

***Corollary* 4.** *Let* $f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$, $|z|=r<1$. *Then*

$$r-\frac{B-A}{|k^{\lambda}|\,(k-1+kB-A)}r^{k}\leq|f(z)|\leq r+\frac{B-A}{|k^{\lambda}|\,(k-1+kB-A)}r^{k},$$

$$r-\frac{k\,(B-A)}{|k^{\lambda}|\,(k-1+kB-A)}r^{k}\leq|D_{\mathcal{H}}f(z)|\leq r+\frac{k\,(B-A)}{|k^{\lambda}|\,(k-1+kB-A)}r^{k}.$$

The result is sharp. The function h_k *of the form* (23) *is the extremal function.*

***Corollary* 5.** *Let* $0<r<1$, $\gamma\geq 1$. *If* $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$, *then*

$$\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|f(re^{i\theta})\right|^{\gamma}d\theta\leq\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|h_k(re^{i\theta})\right|^{\lambda}d\theta,$$

$$\frac{1}{2\pi}\int\limits_{0}^{2\pi}|D_{\mathcal{H}}f(z)|^{\gamma}\,d\theta\leq\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|D_{\mathcal{H}}h_k(re^{i\theta})\right|^{\gamma}d\theta,$$

where h_k *is the function defined by* (23).

The following covering result follows from Corollary 4.

***Corollary* 6.** *If* $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$, *then* $\mathbb{U}(r)\subset f\left(\mathbb{U}\right)$, *where*

$$r=1-\frac{B-A}{|k^{\lambda}|\,(k-1+kB-A)}.$$

By using Corollary 1 and the results above we obtain corollaries listed below.

***Corollary* 7.** *The class* $\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$ *is convex and compact subset of* $\mathcal{H}$. *Moreover,*

$$E\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)=\{h_n:n\in\mathbb{N}_{k-1}\}\cup\{g_n:n\in\mathbb{N}_k\}$$

and

$$\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)=\left\{\sum_{n=k-1}^{\infty}(\gamma_n h_n+\delta_n g_n):\sum_{n=k-1}^{\infty}(\gamma_n+\delta_n)=1\ \ (\delta_{k-1}=0,\gamma_n,\delta_n\geq 0)\right\},$$

where

$$h_{k-1}(z)=z,\quad h_n(z)=z-\frac{(B-A)\,e^{i(n-1)\eta}}{(1+B)\,n^{\lambda}}z^{n},\quad g_n(z)=z+\frac{(B-A)e^{i(n+1)\eta}}{\tau\,(1+B)\,n^{\lambda}}\overline{z}^{n},\quad z\in\mathbb{U}. \tag{24}$$

***Corollary* 8.** *Let* $f\in\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$ *be a function of the form* (4). *Then*

$$|a_n|\leq\frac{B-A}{(1+B)\,|n^{\lambda}|},\quad |b_n|\leq\frac{B-A}{(1+B)\,|n^{\lambda}|},\quad n\in\mathbb{N}_k,$$

$$r-\frac{B-A}{(1+B)\,|k^{\lambda}|}r^{k}\leq|f(z)|\leq r+\frac{B-A}{(1+B)\,|k^{\lambda}|}r^{k},\quad |z|=r<1,$$

$$r-\frac{k\left(B-A\right)}{\left(1+B\right)\left|k^{\lambda}\right|}r^{k}\leq\left|D_{\mathcal{H}}f(z)\right|\leq r+\frac{k\left(B-A\right)}{\left(1+B\right)\left|k^{\lambda}\right|}r^{k},\quad |z|=r<1,$$

$$\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|f(re^{i\theta})\right|^{\gamma}d\theta\leq\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|h_{k}(re^{i\theta})\right|^{\lambda}d\theta,$$

$$\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|D_{\mathcal{H}}f(re^{i\theta})\right|^{\gamma}d\theta\leq\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|D_{\mathcal{H}}h_{k}(re^{i\theta})\right|^{\gamma}d\theta.$$

The results are sharp. The functions h_n, g_n *of the form* (24) *are the extremal functions.*

***Corollary* 9.** *Let us assume* (14). *If* $f\in\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(k,\eta;A,B)$, *then* $\mathbb{U}(r)\subset f\left(\mathbb{U}\right)$, *where*

$$r=1-\frac{B-A}{k\left(1+B\right)\left|k^{\lambda}\right|}.$$

The class $\mathcal{S}_{\mathcal{H}}^{\lambda}(\tau,n;A,B)$ generalize classes of starlike functions of complex order. The class $\mathcal{CS}_{\mathcal{H}}(\gamma):=\mathcal{S}_{\mathcal{H}}\left(\varphi;1-2\gamma,1\right)\quad(\gamma\in\mathbb{C}\setminus\{0\})$ was defined by Yalçin and Öztürk [23]. In particular, if we put $\gamma:=\dfrac{1-\alpha}{1+e^{i\eta}}$, then we obtain the class $\mathcal{RS}_{\mathcal{H}}(\alpha,\eta):=\mathcal{S}_{\mathcal{H}}\left(\dfrac{2\alpha-1+e^{i\eta}}{1+e^{i\eta}},1\right)$ studied by Yalçin et al. [24]. It is the class of functions $f\in\mathcal{H}_0$ such that

$$\operatorname{Re}\left\{\left(1+e^{i\eta}\right)\frac{D_{\mathcal{H}}f\left(z\right)}{f(z)}-e^{i\eta}\right\}>\alpha,\quad z\in\mathbb{U},\quad\eta\in\mathbb{R}.$$

The classes $\mathcal{S}_{\mathcal{H}}^{n}(k;A,B)$ and $\mathcal{R}_{\mathcal{H}}^{n}(k;A,B)$ are related to harmonic starlike functions, harmonic convex functions and harmonic Janowski functions.

The classes $\mathcal{S}_{\mathcal{H}}(\alpha):=\mathcal{S}_{\mathcal{H}}^{0}(2;2\alpha-1,1)$ and $\mathcal{K}_{\mathcal{H}}(\alpha):=\mathcal{S}_{\mathcal{H}}^{1}(2;2\alpha-1,1)$ were investigated by Jahangiri [10] (see also [2, 18]). They are the classes of starlike and convex functions of order α, respectively. The classes $N_{\mathcal{H}}(\alpha):=\mathcal{R}_{\mathcal{H}}^{1}(2;2\alpha-1,1)$ and $R_{\mathcal{H}}(\alpha):=\mathcal{R}_{\mathcal{H}}^{2}(2;2\alpha-1,1)$ were studied in [1] (see also [14]). Finally, the classes $\mathcal{S}_{\mathcal{H}}:=\mathcal{S}_{\mathcal{H}}(0)$ and $\mathcal{K}_{\mathcal{H}}:=\mathcal{K}_{\mathcal{H}}(0)$ are the classes of functions which are starlike and convex in $\mathbb{U}(r)$, respectively, for all $r\in(0,1\rangle$. We should notice that the classes $\mathcal{S}(A,B):=\mathcal{S}_{\mathcal{H}}(A,B)\cap\mathcal{A}$ and $\mathcal{R}(A,B):=\mathcal{R}_{\mathcal{H}}(A,B)\cap\mathcal{A}$ were introduced by Janowski [13].

Using obtained results to the classes defined above we can obtain new and also well-known results (see, for example, [1 – 9, 10 – 14, 18, 21 – 24]).

***Remark*.** The results obtained in classes of harmonic functions can be transfered to corresponding classes of analytic functions.

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