UDC 519.624.1

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Some remarks on an integral equation arising in applications of numerical-analytic method of solving of boundary value problems

Некоторые замечания к интегральным уравнениям, порожденным численно-аналитическим методом решения граничных задач

In the paper the comparison methob anb the chosing of the appropriate norm of the comparison operator is used to establish the solvability of the integral-functional equation resulted by the application of the numeric-analytic method of solving of boundary value problems for ordinary differential-delay equations of the neutral type.

С помощью метода сравнения и надлежащего подбора нормы оператора сравнения устанавливается разрешимость интегро-функционального уравнения, порожденного численно-аналитическим методом решения граничных задач для обыкновенных дифференциальных уравнений нейтрального типа.

За допомогою методу порівняння і належного добору норми оператора порівняння встановлюється розв'язність інтегро-функціонального рівняння, породженого чисельно-аналітичним методом розв'язування граничних задач для звичайних диференціальних рівнянь нейтрального типу.

The application of the numerical-analytic method (see A. M. Samoilenko and N. I. Ronto [1, 2]) to the solution of the problem:

$$x'(t) = f(t, x(\alpha(t)), x'(\beta(t))), \quad t \in J = [0, T], \quad T > 0, \tag{1}$$

$$Ax(0) + Bx(T) = d \tag{2}$$

by the substitution

$$x(t) = x_0 + \int_0^t z(s)\,ds \tag{3}$$

leads us (see [3]) to the following system of equations

$$z(t) = f\Big(t, x_0 + \alpha(t)\varphi(x_0) + \frac{T - \alpha(t)}{T}\int_0^{\alpha(t)} z(s)\,ds - \frac{\alpha(t)}{T}\int_{\alpha(t)}^T z(s)\,ds,\ z(\beta(t))\Big), \tag{4}$$

$$\varphi(x_0) - \frac{1}{T}\int_0^T z(s)\,ds = 0 \tag{5}$$

with

$$\varphi(x_0) = T^{-1}B^{-1}(d - (A + B)x_0).$$

It is assumed that $f \in C(J \times R^n \times R^n, R^n)$, $\alpha, \beta \in C(J, J)$, $d \in R^n$, A and B are $n \times n$ matrices.

It is quite clear that the solution x^* of (1), (2) is found when a solution (x_0^*, z^*) of the system (4), (5) is given. The formula (3) serves for the relation between the solutions. In the method we are dealing with the system (4), (5) is solved successively. As the first the equation (4) is solved with respect to z at the same time x_0 is considered as a parameter. As a result of this we find a function $z^*(., x_0)$ which together with equation (5) is used to find x_0^*. Finally the solution of (1), (2) is given by the equation $x^*(t) = z^*(t, x_0^*)$.

The aim of the present paper is to discuss the conditions under which the solution $z^*(\cdot, x_0)$ of equation (4) can be obtained by the method of successive approximations

$$z_{k+1}(t, x_0) = (Fz_k)(t, x_0), \quad k = 0, 1, \ldots \tag{6}$$

with z_0 choosen arbitrarilly. Here the operator F is defined by the right hand side of equation (4). The convergence of the sequence $\{z_k\}$ will be obtained by the comparative method (see e. g. [4—6]).

We introduce the following assumptions

A s s u m p t i o n H_1. There are $n \times n$ matrices K and L with positive entries such that

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leqslant K|x - \bar{x}| + L|y - \bar{y}| \tag{7}$$

for all $t \in J$, $x, y, \bar{x}, \bar{y} \in R^n$. Here $|\cdot|$ denotes the absolute value of the vector, so $|x| = (|x_1|, \ldots, |x_n|)^T$.

A s s u m p t i o n H_2. For any positive function $h \in C(J \times R^n, R^n_+)$ there exists the unique solution $u_0 \in C(J \times R^n, R^n_+)$ of the comparison equation

$$u(t, x_0) = (\Omega u)(t, x_0) + h(t, x_0), \tag{8}$$

with the operator Ω defined by the equation

$$(\Omega u)(t, x_0) = \frac{T - \alpha(t)}{T} \int\limits_0^{\alpha(t)} Ku(s, x_0)\, ds + \frac{\alpha(t)}{T} \int\limits_{\alpha(t)}^{T} Ku(s, x_0)\, ds + Lu(\beta(t), x_0).$$

Now for a fixed initial approximation z_0 we define

$$h(t, x_0) = |(Fz_0)(t, x_0) - z_0(t, x_0)| \tag{9}$$

and we construct the sequence $\{u_k\}$ by the formula

$$u_{k+1}(t, x_0) = (\Omega u_k)(t, x_0), \quad k = 1, 2, \ldots, t \in J, \quad x_0 \in R^n, \tag{10}$$

for u_0 defined by equation (8) with h given by equation (9). From Assumption H_2 it follows easily (see for the comparative theory) that the sequence $\{u_k\}$ is nonincreasing and converges almost uniformly (uniformly on any compact subset of $J \times R^n$) to the zero function. Moreover by the mathematical induction rule we get easily the following estimations

$$|z_k(t, x_0) - z_0(t, x_0)| \leqslant u_0(t, x_0), \quad k = 0, 1, \ldots, \tag{11}$$

$$|z_{k+p}(t, x_0) - z_k(t, x_0) \leqslant u_k(t, x_0), \quad k, p = 0, 1, \ldots. \tag{12}$$

From these estimations it follows the following

T h e o r e m. *If the assumptions H_1 and H_2 are satisfied then for every $x_0 \in R^n$ there exists a unique solution $z^*(\cdot, x_0)$ of equation* (4), *moreover*

$$z^*(t, x_0) = \lim z_k(t, x_0), \quad k \to +\infty, \tag{13}$$

$$|z^*(t, x_0) - z_k(t, x_0)| \leqslant u_k(t, x_0), \quad k = 0, 1, \ldots. \tag{14}$$

It follows from the given general discussion that now the crutial point is to find practical sufficient conditions which quarantee that the Assumption H_2 holds true. There is an immediate answer to our problem. It is enough to assume that some norm of the operator Ω is less than one. However there are many different norms of the same operator Ω (they depend on the norm taken in the space of continuous functions where the operator is defined). Гet us consider some special cases.

1. If we take in the space $C(J, R^n)$ the standard supremum norm (the Tchebysheff norm) then we see that the Assumption H_2 holds if the condition

$$m\|k\| + \|L\| < 1, \quad m = 21^{-1} \max[(T - \alpha(t))\alpha(t) : t \in J] \tag{15}$$

holds.

2. If we take in $C(J, R^n)$ the following weighted norm

$$\| u \|_b = \max [| u(t) |_b : t \in J], \quad | u(t) |_b = \max [b_i^{-1} | u_i(t) | : i = 1, 2, \ldots, n]$$

with b the positive eigenvector of the matrix $mk + L$ corresponding to the eigenvalue which is equal to the spectral radius of that matrix (the existence of such eigenvector is quaranteed by the well known Perron theorem) then the Assumption H_2 holds true when the spectral radius of $mk + L$ is less than one (see [3]), i. e.

$$\rho = \rho(mK + L) < 1. \tag{16}$$

In fact in this case we have

$$| u(t) | \leqslant \| u \|_b \cdot b, \quad | (\Omega u)(t) | \leqslant \| u \|_b (mK + L) b = \rho \| u \|_b \cdot b$$

what means that $\| \Omega \|_b \leqslant \rho < 1$.

3. In the case $\alpha(t) = t$, $\beta(t) = t$ the condition (16) can be replaced by the following one

$$\rho' = \rho\left(\frac{T}{3} K + L\right) < 1. \tag{17}$$

To get the result we need to employ the following norm

$$\| u \|_{b,c} = \max [(t(T - t) + c)^{-1} | u(t) |_b : t \in J]$$

for the corresponding eigenvector b and sufficiently small positive number c. From this definition we have

$$| u(t) | \leqslant \| u \|_{b,c} \cdot (t(T - t) + c) b$$

and consequently

$$| (\Omega u)(t) | \leqslant \| u \|_{b,c} \Big[T^{-1} (T - t) \int_0^t (s(T - s) + c) K b ds + T^{-1} t \int_t^T (s(T - s) +$$

$$+ c) K b ds + (t(T - t) + c) L b \Big] \leqslant \| u \|_{b,c} [T^{-1} t (T - t) (6^{-1} (T^2 + 4Tt - 4t^2) +$$

$$+ 2c) K b + (t(T - t) + c) L b].$$

This implies the following inequality

$$(t(T - t) + c)^{-1} | (\Omega u)(t) | \leqslant \| u \|_{b,c} ((3^{-1} T + 2cT^{-1}) K b + L b] =$$

$$= \| u \|_{b,c} (\rho' b + 2cT^{-1} K b) \leqslant \| u \|_{b,c} (\rho' + 2cT^{-1} \| K b \|_b) b$$

because $Kb \leqslant \| Kb \|_b b$. From the last inequality we find

$$\| \Omega u \|_{b,c} \leqslant (\rho' + 2cT^{-1} \| K b \|_b) \| u \|_{b,c}$$

which means that $\| \Omega \|_{b,c} < 1$ if $\rho' < 1$ and c is sufficiently small.

4. Observe that in the case $\alpha(t) = t$, $\beta(t) = t$ one can reason in a different way. Under the condition $\rho(L) < 1$ the equation can be rewritten in the equivalent form

$$u(t) = T^{-1} \left((T - t) \int_0^t K_1 u(s) ds + t \int_t^T K_1 u(s) ds \right) + h_1(t), \tag{18}$$

where

$$K_1 = (I - L)^{-1} K, \quad h_1(t) = (I - L)^{-1} h(t, x_0).$$

Now the Assumption H_i is satisfied if the following conditions hold

$$\rho(L) < 1, \quad \rho(3^{-1} T (I - L)^{-1} K) < 1.$$

In this case it is worthy to observe that the substitution $u(t) - h_1(t) = v(t)$ in the equation (18) reduces this equation to the form which can be considered in the subspace of function of $C(J, R^n)$ having the property

$$\max [(t(T - t))^{-1} | v(t) | : t \in J] < + \infty.$$

Now the norm $\| u \|_{b,c}$ can be replaced by that which corresponds to $c = 0$.

5. Finally we will seek the solution of equation (18) by the Neumann series. For the convenience we will write K_1 and h_1 as simply K and h. Let

$$(Gu)(t) = T^{-1}\left[(T - t)\int_0^t Ku(s)\,ds + t\int_t^T Ku(s)\,ds\right], \tag{19}$$

$$(G^0u)(t) = u(t), \quad (G^{i+1}u)(t) = G(G^iu)(t), \quad i = 0, 1, \ldots, \quad u_0(t) = \sum_{i=0}^{+\infty}(G^ih)(t). \tag{20}$$

It is clear that the series (20) converges uniformly when the norm of the operator G is less than one. So one can use the results of our discussion. Now we will present the aproach based on some pointwise estimations. This will be close to the considerations given in the books [1, 2]. Let the sequence $\{\alpha_i\}$ be defined as follows

$$\alpha_{i+1}(t) = T^{-1}\left((T - t)\int_0^t \alpha_i(s)\,ds + t\int_t^T \alpha_i(s)\,ds\right), \quad \alpha_0(t) = 1, \quad i = 0, 1, \ldots \tag{21}$$

Take $H \in R^n$, $H > 0$ such that $|h(t)| \leqslant H$ for $t \in J$. Now by the mathematical induction rule we infer

$$0 \leqslant (G^ih)(t) \leqslant \alpha_i(t)K^iH, \quad i = 0, 1, \ldots. \tag{22}$$

By the dirrect calculations (see [1]) one can find that

$$\alpha_1(t) \leqslant 2^{-1}T\alpha_0(t), \quad \alpha_2(t) \leqslant 3^{-1}T\alpha_1(t), \quad \alpha_3(t) \leqslant 10^{-1}T^2\alpha_1(t),$$

$$\alpha_{2i}(t) \leqslant \left(\frac{T}{\sqrt{10}}\right)^{2i-2}\alpha_2(t), \quad \alpha_{2i+1}(t) \leqslant \left(\frac{T}{\sqrt{10}}\right)^{2i}\alpha_1(t), \quad i = 1, 2, \ldots.$$

From these inequalities we get the estimation

$$\alpha_i(t) \leqslant \left(\frac{T}{\sqrt{10}}\right)^{i-1}\bar{\alpha}_1(t), \quad t \in J, \quad i = 1, 2, \ldots, \quad \bar{\alpha}_1(t) = \frac{\sqrt{10}}{3}\alpha_1(t). \tag{23}$$

From (22) and (23) we have

$$0 \leqslant (G^i(h)(t) \leqslant \left(\frac{T}{\sqrt{10}}K\right)^{i-1}\bar{\alpha}_1(t)KH, \quad t \in J, \quad i = 1, 2, \ldots.$$

Now we see that the series (20) converges uniformly when

$$\rho\left(\frac{T}{\sqrt{10}}K\right) < 1. \tag{24}$$

Clearly the function u_0 defined by (20) is a solution of (18) and

$$u_0(t) \leqslant h(t) + \bar{\alpha}_1(t)\left(I - \frac{T}{\sqrt{10}}K\right)^{-1}KH.$$

We observe also that in the case considered the corresponding functions u_k (see formula (10)) satisfy the relation

$$u_k(t) = \sum_{i=k}^{+\infty}(G^ih)(t) \leqslant \bar{\alpha}(t)\left(\frac{T}{\sqrt{10}}\right)^{k-1}\left(I - \frac{T}{\sqrt{10}}K\right)^{-1}KH. \tag{25}$$

Now the estimation (14) takes the form

$$|z^*(t, x_0) - z_k(t, x_0)| \leqslant T^{-1}\sqrt{10}\cdot\bar{\alpha}(t)\left(\frac{T}{\sqrt{10}}K\right)^k\left(I - \frac{T}{\sqrt{10}}K\right)^{-1}H,$$

for $k = 1, 2, \ldots$.

We observe that in the book [2, p. 13, 59] in the corresponding estimations the the number 10 does not appear, it is replaced by the number π. In the merit there are no special reasons for the presence of this specific number. The only justification is that these two numbers are very close. Indeed, we have $\pi = 3,14159265 \ldots < \sqrt{10} = 3,16227766 \ldots$.

As it was mentioned at the beginning of the paper we are not intend to discuss farther deteils which are concerned with the solving the corresponding equation which determines $\overset{*}{x}_0$. The readers are advised to consult the publications [1, 2, 7] for the necessary details.

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Received 30.01.91