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NONLINEAR SKEW COMMUTING MAPS ON $*$-RINGS*
НЕЛІНІЙНІ СКІСНІ КОМУТУЮЧІ ВІДОБРАЖЕННЯ НА $*$-КІЛЬЦЯХ

Let $\mathcal{R}$ be a unital $*$-ring with the unit I. Assume that $\mathcal{R}$ contains a symmetric idempotent P which satisfies $A\mathcal{R}P = 0$ implies $A = 0$ and $A\mathcal{R}(I-P) = 0$ implies $A = 0$. In this paper, it is proved that if $\phi : \mathcal{R} \to \mathcal{R}$ is a nonlinear skew commuting map, then there exists an element $Z \in \mathcal{Z}_S(\mathcal{R})$ such that $\phi(X) = ZX$ for all $X \in \mathcal{R}$, where $\mathcal{Z}_S(\mathcal{R})$ is the symmetric center of $\mathcal{R}$. As an application, the form of nonlinear skew commuting maps on factors is obtained.

Нехай $\mathcal{R}$ — унітарне $*$-кільце з одиницею I. Припустимо, що $\mathcal{R}$ має симетричний ідемпотент P, для якого з $A\mathcal{R}P = 0$ випливає $A = 0$, а з $A\mathcal{R}(I-P) = 0$ — $A = 0$. У цій статті доведено, що якщо $\phi : \mathcal{R} \to \mathcal{R}$ є нелінійним скісним комутуючим відображенням, то існує елемент $Z \in \mathcal{Z}_S(\mathcal{R})$ такий, що $\phi(X) = ZX$ для всіх $X \in \mathcal{R}$, де $\mathcal{Z}_S(\mathcal{R})$ — симетричний центр $\mathcal{R}$. Як застосування отримано форму нелінійних скісних комутуючих відображень на факторах.

1. Introduction. Let $\mathcal{R}$ be a ring. A map $\phi\colon \mathcal{R} \to \mathcal{R}$ is called commuting if

$$\phi(X)X = X\phi(X) \tag{1.1}$$

for all $X \in \mathcal{R}$. The usual goal when treating a commuting map is to describe its form. The first important result on commuting maps is Posner's theorem, which proved that the existence of a nonzero commuting derivation on a prime ring $\mathcal{R}$ implies that $\mathcal{R}$ is commutative [12]. For $X, Y \in \mathcal{R}$, denote by $[X,Y] = XY - YX$ the Lie product of X and Y. Accordingly, the commuting maps can be written as $[\phi(X), X] = 0$ for all $X \in \mathcal{R}$. If ϕ is additive, then for any $X, Y \in \mathcal{R}$, replacing X by $X+Y$ in Eq. (1.1) implies that

$$[\phi(X), Y] = [X, \phi(Y)]$$

for all $X, Y \in \mathcal{R}$. Assuming that ϕ is additive, Brešar [3] proved that additive commuting map ϕ on simple unital ring $\mathcal{R}$ must be of the form

$$\phi(X) = ZX + f(X)$$

for some $Z \in \mathcal{Z}(\mathcal{R})$ and additive $f\colon \mathcal{R} \to \mathcal{Z}(\mathcal{R})$, where $\mathcal{Z}(\mathcal{R})$ is the center of $\mathcal{R}$. The problem of describing commuting maps is closely related with the theory of functional identities and many results have been obtained on this subject. The reader is referred to the survey paper [5] and the book [4]. Recently, Bounds [2] described commuting maps over the ring of strictly upper triangular matrices. Brešar and Šemrl [6] gave the form of continuous commuting functions on matrix algebras.

Recall that a ring $\mathcal{R}$ is called a $*$-ring if there is an additive map $*\colon \mathcal{R} \to \mathcal{R}$ satisfying $(XY)^* = Y^*X^*$ and $(X^*)^* = X$ for all $X, Y \in \mathcal{R}$. For $X, Y \in \mathcal{R}$, denote by $[X,Y]_* = XY - YX^*$ the skew Lie product of X and Y. The skew Lie product arose in the problem of representing quadratic

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functionals with sesquilinear functionals and characterizing ideals [11, 14–16]. In the last decade, skew Lie product has attracted attention of several authors [7–10, 13, 17, 18]. Motivated by the above mentioned work, we introduce the concept of nonlinear skew commuting maps. A map $\phi\colon \mathcal{R} \to \mathcal{R}$ (without the additivity assumption) is called a nonlinear skew commuting maps if

$$[\phi(X), Y]_* = [X, \phi(Y)]_*$$

for all $X, Y \in \mathcal{R}$.

Let $\mathcal{Z}(\mathcal{R})$ be the centre of $\mathcal{R}$. An element $X \in \mathcal{R}$ is called symmetric if $X^* = X$, $\mathcal{Z}_S(\mathcal{R}) = \{X \in \mathcal{Z}(\mathcal{R})\colon X^* = X\}$ is called symmetric center of $\mathcal{R}$. In this paper, we describe the form of nonlinear skew commuting maps on $*$-rings. As an application, the form of nonlinear skew commuting maps on factors is obtained.

2. Main result. In this section, we will prove the following theorem.

Theorem 2.1. *Let $\mathcal{R}$ be a unital $*$-ring with the unit I. Assume that $\mathcal{R}$ contains a symmetric idempotent P which satisfies: $(\mathcal{Q}_1)$ $A\mathcal{R}P = 0$ implies $A = 0$, $(\mathcal{Q}_2)$ $A\mathcal{R}(I - P) = 0$ implies $A = 0$. If a map $\phi\colon \mathcal{R} \to \mathcal{R}$ (without the additivity assumption) satisfies*

$$[\phi(X), Y]_* = [X, \phi(Y)]_*$$

for all $X, Y \in \mathcal{R}$, then there exists an element $Z \in \mathcal{Z}_S(\mathcal{R})$ such that $\phi(X) = ZX$ for all $X \in \mathcal{R}$.

It is clear that $P \neq 0, P \neq I$. Write $P_1 = P, P_2 = I - P_1$. Put $\mathcal{R}_{ij} = P_i\mathcal{R}P_j$, $i, j = 1, 2$. Then

$$\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{21} + \mathcal{R}_{22}$$

and so for each $A \in \mathcal{R}$, $A = A_{11} + A_{12} + A_{21} + A_{22}$, $A_{ij} \in \mathcal{R}_{ij}$, $i, j = 1, 2$.

We will complete the proof by several lemmas.

Lemma 2.1 ([1], Lemma 4)**.** *Let $\mathcal{R}$ be a unital ring with the unit I. Assume that $\mathcal{R}$ satisfies $A\mathcal{R}P_1 = 0$ implies $A = 0$ and $A\mathcal{R}P_2 = 0$ implies $A = 0$. For $A_{ii} \in \mathcal{R}$, $i = 1, 2$, if $P_iXA_{ii} = A_{ii}XP_i$ for all $X \in \mathcal{R}$, then there exists an element $Z \in \mathcal{Z}(\mathcal{R})$ such that $A_{ii} = ZP_i$.*

Lemma 2.2. $\phi(I) \in \mathcal{Z}_S(\mathcal{R})$.

Proof. For any $Y \in \mathcal{R}$, we have

$$[\phi(I), Y]_* = [I, \phi(Y)]_* = 0,$$

which implies

$$\phi(I)Y = Y\phi(I)^* \tag{2.1}$$

for all $Y \in \mathcal{R}$. Taking $Y = I$ in Eq. (2.1), we get $\phi(I) = \phi(I)^*$. Hence, $\phi(I) \in \mathcal{Z}_S(\mathcal{R})$.

Lemma 2.3. *For every $X, Y \in \mathcal{R}$, we have*

$$\phi(X + Y) - \phi(X) - \phi(Y) \in \mathcal{Z}_S(\mathcal{R}).$$

Proof. For any $X, Y, T \in \mathcal{R}$, it follows that

$$[\phi(X + Y) - \phi(X) - \phi(Y), T]_* = [\phi(X + Y), T]_* - [\phi(X), T]_* - [\phi(Y), T]_* =$$

$$= [X + Y, \phi(T)]_* - [X, \phi(T)]_* - [Y, \phi(T)]_* = 0.$$

Hence, $\phi(X + Y) - \phi(X) - \phi(Y) \in \mathcal{Z}_S(\mathcal{R})$.

Lemma 2.4. *There exist elements* $Z_1, Z_2 \in \mathcal{Z}_S(\mathcal{R})$ *such that* $\phi(P_1) = Z_1P_1 + Z_2$.

Proof. It follows from Lemma 2.2 that

$$0 = [P_1, \phi(I)]_* = [\phi(P_1), I]_* = \phi(P_1) - \phi(P_1)^*.$$

Hence, $\phi(P_1)^* = \phi(P_1)$.

For any $X \in \mathcal{R}$, it is easy to check that

$$[P_1, [P_1, [P_1, \phi(X)]_*]_*]_* = [P_1, \phi(X)]_*.$$

Hence,

$$[P_1, [P_1, [\phi(P_1), X]_*]_*]_* = [\phi(P_1), X]_*.$$

Write $\phi(P_1) = S_{11} + S_{12} + S_{21} + S_{22}$. Since $\phi(P_1)^* = \phi(P_1)$, the above equation becomes

$$(S_{11} + S_{12})XP_1 - P_1X(S_{11} + S_{21}) + (S_{21} + S_{22})XP_2 - P_2X(S_{12} + S_{22}) = 0. \tag{2.2}$$

Taking $X = X_{12}$ in Eq. (2.2) and multiplying by P_2 from both sides, we get $S_{21}XP_2 = = S_{21}X_{12} = 0$ for all $X \in \mathcal{R}$. It follows from the condition $(\mathcal{Q}_2)$ of Theorem 2.1 that $S_{21} = 0$. Taking $X = X_{21}$ in Eq. (2.2) and multiplying by P_1 from both sides, we get $S_{12}XP_1 = S_{12}X_{21} = 0$ for all $X \in \mathcal{R}$. It follows from the condition $(\mathcal{Q}_1)$ of Theorem 2.1 that $S_{12} = 0$.

Taking $X = X_{11}$ in Eq. (2.2), we get $S_{11}X_{11} = X_{11}S_{11}$. It follows from Lemma 2.1 that $S_{11} = P_1\phi(P_1)P_1 = ZP_1$ for some $Z \in \mathcal{Z}(\mathcal{R})$. By $\phi(P_1)^* = \phi(P_1)$, we have $Z^*P_1 = ZP_1$, and so $Z^*XP_1 = ZXP_1$ for all $X \in \mathcal{R}$. It follows from the condition $(\mathcal{Q}_1)$ of Theorem 2.1 that $Z^* = Z$, that is, $Z \in \mathcal{Z}_S(\mathcal{R})$. Similarly, taking $X = X_{22}$ in Eq. (2.2), we get $S_{22} = Z_2P_2$ for some $Z_2 \in \mathcal{Z}_S(\mathcal{R})$. Hence,

$$\phi(P_1) = S_{11} + S_{22} = ZP_1 + Z_2P_2 = Z_1P_1 + Z_2,$$

where $Z_1 = Z - Z_2 \in \mathcal{Z}_S(\mathcal{R})$.

Lemma 2.5. *For every* $X_{ij} \in \mathcal{R}_{ij}$, $1 \le i \ne j \le 2$, *we have*

$$\phi(X_{ij}) = Z_1X_{ij}.$$

Proof. Take any $X_{12} \in \mathcal{R}_{12}$ and let $\phi(X_{12}) = A_{11}+A_{12}+A_{21}+A_{22}$. It follows from Lemma 2.4 that

$$P_1\phi(X_{12}) - \phi(X_{12})P_1 = [P_1, \phi(X_{12})]_* = [\phi(P_1), X_{12}]_* = Z_1X_{12},$$

which implies that $A_{12} = Z_1X_{12}$ and $A_{21} = 0$.

Take any $B \in \mathcal{R}$ and let $\phi(B) = Y_{11} + Y_{12} + Y_{21} + Y_{22}$. Since

$$[B, \phi(X_{12})]_* = [\phi(B), X_{12}]_*,$$

we obtain

$$BA_{11} + BA_{12} + BA_{22} - A_{11}B^* - A_{12}B^* - A_{22}B^* =$$

$$= Y_{11}X_{12} + Y_{21}X_{12} - X_{12}Y_{12}^* - X_{12}Y_{22}^*. \tag{2.3}$$

Multiplying Eq. (2.3) by P_1 from the right, we get

$$BA_{11} - A_{11}B^*P_1 - A_{12}B^*P_1 - A_{22}B^*P_1 = -X_{12}Y_{12}^*. \tag{2.4}$$

Replacing B by P_2BP_1 in Eq. (2.4), we have

$$P_2BA_{11} = -X_{12}Y_{12}^*,$$

which implies $P_2BA_{11} = 0$ for all $B \in \mathcal{R}$, and then $A_{11}^*B^*P_2 = 0$ for all $B \in \mathcal{R}$. It follows from the condition $(\mathcal{Q}_2)$ of Theorem 2.1 that $A_{11} = 0$. Similarly, multiplying Eq. (2.3) by P_2 from the left, and then replacing B by P_1BP_2, we can get $A_{22} = 0$. Hence,

$$\phi(X_{12}) = A_{11} + A_{12} + A_{21} + A_{22} = Z_1X_{12}.$$

The proof of $\phi(X_{21}) = Z_1X_{21}$ is similar.

Lemma 2.6. *For every $X_{ii} \in \mathcal{R}_{ii}$, $i = 1, 2$, we have*

$$\phi(X_{ii}) = Z_1X_{ii}.$$

Proof. Take any $X_{11} \in \mathcal{R}_{11}$ and let $\phi(X_{11}) = S_{11} + S_{12} + S_{21} + S_{22}$. For any $X_{12} \in \mathcal{R}_{12}$, it follows from Lemma 2.5 that

$$0 = [\phi(X_{12}), X_{11}]_* = [X_{12}, \phi(X_{11})]_*,$$

which implies that

$$X_{12}S_{21} + X_{12}S_{22} - S_{12}X_{12}^* - S_{22}X_{12}^* = 0. \tag{2.5}$$

Multiplying Eq. (2.5) by P_2 from the left and P_1 from the right, we have $S_{22}X^*P_1 = S_{22}X_{12}^* = 0$ for all $X \in \mathcal{R}$, and so $S_{22} = 0$. Since

$$[X_{11}, \phi(X_{12})]_* = [\phi(X_{11}), X_{12}]_*,$$

it follows that

$$X_{11}\phi(X_{12}) = S_{11}X_{12} + S_{21}X_{12} - X_{12}S_{12}^*. \tag{2.6}$$

Multiplying Eq. (2.6) by P_1 from both sides, we can get $X_{12}S_{12}^* = 0$, and so $S_{12} = 0$. Multiplying Eq. (2.6) by P_2 from both sides, we can get $S_{21}X_{12} = 0$, and so $S_{21} = 0$. Multiplying Eq. (2.6) by P_1 from the left and P_2 from the right, we have

$$X_{11}\phi(X_{12})P_2 = S_{11}X_{12}.$$

It follows from Lemma 2.5 that $(S_{11} - Z_1X_{11})X_{12} = 0$ and so $S_{11} = Z_1X_{11}$. Hence,

$$\phi(X_{11}) = S_{11} + S_{12} + S_{21} + S_{22} = Z_1X_{11}.$$

The proof of $\phi(X_{22}) = Z_1X_{22}$ is similar.

Now we are in a position to prove the main theorem.

***Proof of Theorem* 2.1.** It follows from Lemmas 2.5 and 2.6 that $\phi(X_{ij}) = Z_1X_{ij}$, $i, j = 1, 2$. For any $X = \sum_{i,j=1}^{2} X_{ij} \in \mathcal{R}$, it follows from Lemma 2.3 that

$$\phi(X) - Z_1X = \phi(X) - Z_1 \sum_{i,j=1}^{2} X_{ij} = \phi(X) - \sum_{i,j=1}^{2} \phi(X_{ij}) \in \mathcal{Z}_S(\mathcal{R}).$$

Define a map $f\colon \mathcal{R} \to \mathcal{Z}_S(\mathcal{R})$ by $f(X) = \phi(X) - Z_1X$. Then we have

$$\phi(X) = Z_1X + f(X)$$

for all $X \in \mathcal{R}$.

Since

$$[\phi(X), Y]_* = [X, \phi(Y)]_*$$

for all $X, Y \in \mathcal{R}$, we obtain

$$[Z_1X + f(X), Y]_* = [X, Z_1Y + f(Y)]_*.$$

Hence,

$$f(Y)\,(X - X^*) = 0 \tag{2.7}$$

for all $X, Y \in \mathcal{R}$.

For any $X_{12} \in \mathcal{R}_{12}$, replacing X by X_{12} in Eq. (2.7), we get

$$f(Y)X_{12} - f(Y)X_{12}^* = 0$$

for all $Y \in \mathcal{R}$. Multiplying the above equation by P_2 from the right, we have $f(Y)X_{12} = 0$, and so $f(Y)P_1 = 0$.

For any $X_{21} \in \mathcal{R}_{21}$, replacing X by X_{21} in Eq. (2.7), we obtain

$$f(Y)X_{21} - f(Y)X_{21}^* = 0$$

for all $Y \in \mathcal{R}$. Multiplying the above equation by P_1 from the right, we get $f(Y)X_{21} = 0$, and so $f(Y)P_2 = 0$. Hence,

$$f(Y) = f(Y)P_1 + f(Y)P_2 = 0$$

for all $Y \in \mathcal{R}$, and, thus, $\phi(X) = Z_1X$.

Let $\mathbb{R}$ be the real number field. We denote by H the complex Hilbert space and by $B(H)$ the algebra of all bounded linear operators on H. Let $\mathcal{A} \subseteq B(H)$ be a von Neumann algebra. Recall that $\mathcal{A}$ is a factor if its center contains only the scalar operators.

***Corollary* 2.1.** *Let $\mathcal{A}$ be a factor acting on a complex Hilbert space H. If a map $\phi\colon \mathcal{A} \to \mathcal{A}$ satisfies*

$$[\phi(X), Y]_* = [X, \phi(Y)]_*$$

for all $X, Y \in \mathcal{A}$, then $\phi(X) = \alpha X$ for all $X \in \mathcal{A}$, where $\alpha \in \mathbb{R}$.

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