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FATOU AND JULIA LIKE SETS

МНОЖИНИ, ПОДІБНІ ДО МНОЖИН ФАТУ ТА ЖУЛІА

For a family of holomorphic functions on an arbitrary domain, we introduce Fatou and Julia like sets, and establish some of their interesting properties.

Для сім'ї голоморфних функцій на довільних областях визначено множини, що є подібними до множин Фату та Жуліа, і встановлено деякі цікаві властивості цих множин.

1. Introduction and main results. Throughout, we shall denote by $\mathcal{H}(D)$ the class of all holomorphic functions on a domain $D \subseteq \mathbb{C}$. A subfamily $\mathcal{F}$ of $\mathcal{H}(D)$ is said to be normal if every sequence in $\mathcal{F}$ contains a subsequence that converges locally uniformly on D. $\mathcal{F}$ is said to be normal at a point $z_0 \in D$ if it is normal in some neighborhood of z_0 in D (see [12, 15]).

Let f be an entire function and let $f^n := \underbrace{f \circ f \circ \ldots \circ f}_{n-\text{times}}$, $n \geq 1$, be the nth iterate of f. The *Fatou set* of f, denoted by $F(f)$, is defined as

$$F(f) = \{z \in \mathbb{C} \colon \{f^n\} \text{ is a normal family in some neighborhood of } z\}$$

and the complement $\mathbb{C} \setminus F(f)$ of $F(f)$ is called the *Julia set* of f and is denoted by $J(f)$. $F(f)$ is an open subset of $\mathbb{C}$ and $J(f)$ is a closed subset of $\mathbb{C}$, and both are completely invariant sets under f. The study of Fatou and Julia sets of holomorphic functions is a subject matter of Complex Dynamics for which one can refer to [2, 4, 14].

For a given domain D and a subfamily $\mathcal{F}$ of $\mathcal{H}(D)$, we denote by $F(\mathcal{F})$, a subset of D on which $\mathcal{F}$ is normal and $J(\mathcal{F}) := D \setminus F(\mathcal{F})$. If $\mathcal{F}$ happens to be a family of iterates of an entire function f, then $F(\mathcal{F})$ and $J(\mathcal{F})$ reduce to the Fatou set of f and the Julia set of f, respectively, therefore, it is reasonable to call $F(\mathcal{F})$ and $J(\mathcal{F})$ as *Fatou and Julia like sets*. Note that Julia set of an entire function is always nonempty (see [2]) whereas Julia like set $J(\mathcal{F})$ can be empty. For example, consider the family

$$\mathcal{F} := \{f(az+b) \colon a, b \in \mathbb{C}, a \neq 0\},$$

where f is a normal function on $\mathbb{C}$ (see [12, p. 179]). Then since f is a normal function on $\mathbb{C}$, $\mathcal{F}$ is a normal family on $\mathbb{C}$, that is, $F(\mathcal{F}) = \mathbb{C}$ and, hence, $J(\mathcal{F}) = \phi$.

Also, it is interesting to note that Julia set of any meromorphic function is an uncountable set (see [2]) but Julia like set is not so, for example, $J(\mathcal{F}) = \{0\}$, where $\mathcal{F} := \{nz : n \in \mathbb{N}\} \subset \mathcal{H}(\mathbb{D})$, where $\mathbb{D}$ is the open unit disk.

If $\mathcal{F}$ and $\mathcal{G}$ are two subfamilies of $\mathcal{H}(D)$, then $J(\mathcal{F} \cap \mathcal{G}) \subset J(\mathcal{F}) \cap J(\mathcal{G})$, however $J(\mathcal{F} \cap \mathcal{G}) = J(\mathcal{F}) \cap J(\mathcal{G})$ may not hold in general. For example, let

$$\mathcal{F} = \{nz \colon n \in \mathbb{N}\} \cup \{n(z-1) \colon n \in \mathbb{N}\}$$

and

$$\mathcal{G} = \{n(z-1) : n \in \mathbb{N}\} \cup \{e^{nz} : n \in \mathbb{N}\}$$

be the families of entire functions. Then $J(\mathcal{F} \cap \mathcal{G}) = \{1\}$ and $J(\mathcal{F}) \cap J(\mathcal{G}) = \{0, 1\}$.

This paper is devoted to the problem of normality of families of mappings that have been actively studied recently (see [5, 8–11]). In particular, we give some interesting properties of Fatou and Julia like sets.

Theorem 1.1. (a) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two subfamilies of $\mathcal{H}(D)$, then $J(\mathcal{F}_1 \cup \mathcal{F}_2) = J(\mathcal{F}_1) \cup \cup J(\mathcal{F}_2)$.*

(b) *If $z_0 \in J(\mathcal{F})$ and N is any neighborhood of z_0, then $\mathbb{C} \setminus U$ contains at most one point, where $U = \bigcup_{f \in \mathcal{F}} f(N)$.*

***Example* 1.1.** For $\alpha \in \mathbb{C}$, consider one-parameter family of entire functions $\mathcal{F}_\alpha := \{n(z-\alpha) : n \in \mathbb{N}\}$. Then $\mathcal{F}_\alpha$ is not normal at $z = \alpha$, that is, $\mathcal{F}_\alpha$ is not normal in any open set containing $z = \alpha$. Consider the family of entire functions $\mathcal{F} = \cup_{|\alpha| \leq 1} \mathcal{F}_\alpha$. Then we show that $J(\mathcal{F}) = \{z : |z| \leq 1\}$ and, hence, $\mathrm{Int}(J(\mathcal{F})) \neq \phi$ and $J(\mathcal{F}) \neq \mathbb{C}$. The inclusion $\{z : |z| \leq 1\} \subset J(\mathcal{F})$ holds trivially. To show the other way inclusion, let $z_0 \in \mathbb{C}$ such that $|z_0| > 1$ and let $\{f_n\}$ be a sequence in $\mathcal{F}$. Then we have two cases:

Case I: When $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which is locally bounded at z_0.

In this case by Montel's theorem $\{f_{n_k}\}$ further has a subsequence which converges uniformly in some neighborhood of z_0. Thus, $\{f_n\}$ has a subsequence which converges uniformly in some neighborhood of z_0, that is, $z_0 \in F(\mathcal{F})$.

Case II: When $\{f_n\}$ has no subsequence which is locally bounded at z_0.

Since $f_n(z) = m_n(z - \alpha_n)$, where $m_n \in \mathbb{N}$ and $|\alpha_n| \leq 1$ for each $n \in \mathbb{N}$, it follows that $\{m_n\}$ has an increasing subsequence $\{m_{n_k}\}$ which converges to ∞. Let $N \subset \{z : |z| > 1\}$ be a small neighborhood of z_0 . Then $\{f_{n_k}\}$ converges uniformly to ∞ in N. Thus $\{f_n\}$ has a subsequence which converges uniformly in some neighborhood of z_0, that is, $z_0 \in F(\mathcal{F})$.

Thus in both the cases we find that $J(\mathcal{F}) \subset \{z : |z| \leq 1\}$. Hence, $J(\mathcal{F}) = \{z : |z| \leq 1\}$.

Note that for a family of iterates of an entire function f, $J(f) = \mathbb{C}$ or $J(f)$ has empty interior [2] (Lemma 3).

A set $A \subset D$ is said to be *forward invariant (backward invariant)* under the family $\mathcal{F}$ if, for each $f \in \mathcal{F}$, $f(A) \subset A \left(f^{-1}(A) \subset A\right)$.

If $\mathcal{F}_0$ is a semigroup of entire functions, then $F(\mathcal{F}_0)$ is forward invariant and $J(\mathcal{F}_0)$ is backward invariant under the family $\mathcal{F}_0$ (see [7]), whereas for an arbitrary subfamily $\mathcal{G}$ of $\mathcal{H}(D)$, $F(\mathcal{G})$ and $J(\mathcal{G})$ may not be forward invariant or backward invariant. For example, $J(\mathcal{G})$ is not forward invariant as well as backward invariant for $\mathcal{G} = \{nz : n \in \mathbb{N}\} \cup \{z^n : n \in \mathbb{N}\}$. Forward invariance of $J(\mathcal{F})$ and $F(\mathcal{F})$ for the family $\mathcal{F}$, implies the following theorem.

Theorem 1.2. *Let $\mathcal{F}$ be a subfamily of $\mathcal{H}(D)$. Then the following statements hold*:

(a) *If $J(\mathcal{F})$ is forward invariant, then $J(\mathcal{F}) = D$ or $\mathrm{Int}(J(\mathcal{F})) = \phi$. In particular, if $\mathbb{C} \setminus D$ contains at least two points, then $\mathrm{Int}(J(\mathcal{F})) = \phi$.*

(b) *If $J(\mathcal{F})$ contains at least two points and $F(\mathcal{F})$ is forward invariant, then $J(\mathcal{F})$ is a perfect set.*

***Example* 1.2.** Let $\mathcal{F}_1 = \{nz : n \in \mathbb{N}\}$, $\mathcal{F}_2 = \{z^n : n \in \mathbb{N}\}$. Then $J(\mathcal{F}_1 \cup \mathcal{F}_2) = \{z : |z| = 1\} \cup \cup \{0\}$. Clearly, $J(\mathcal{F}_1 \cup \mathcal{F}_2)$ is not perfect and $F(\mathcal{F}_1 \cup \mathcal{F}_2)$ is not forward invariant.

Example 1.2 shows that the condition, "$F(\mathcal{F})$ is forward invariant" in Theorem 1.2 can not be dropped.

Recall that a point $z_0 \in D$ is said to be a periodic point of an entire function f, of order k, if $f^k(z_0) = z_0$. In the dynamics of transcendental entire functions, it is well-known that Julia set is the closure of the repelling periodic points (see [13]). This can be extended for Julia like set too. In this context, we need some basic notations from the Nevanlinna value distribution theory of meromorphic functions (see [6]).

Let f be a meromorphic function on $\mathbb{C}$. The proximity function $m(r, a, f)$ of f and the counting function $N(r, a, f)$ of a-points of $f (a \neq \infty)$ are given by

$$m(r, a, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\phi}) - a|} d\phi.$$

For $a = \infty$, we write

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi,$$

$$N(r, a, f) := \int_0^r \frac{n(t, 1/f - a)}{t} dt$$

and

$$N(r, f) := \int_0^r \frac{n(t, f)}{t} dt,$$

where $n(t, 1/f - a)$ is the number of a-points of f in $|z| \leq t$ and, in particular, $n(t, f)$ is the number of poles of f in $|z| \leq t$. The characteristic function of f, denoted by $T(r, f)$, is given by

$$T(r, f) = m(r, f) + N(r, f)$$

and it behaves like $\log^+ M(r, f)$, whenever f happens to be an entire function, where $M(r, f) = = \max_{|z|=r} |f(z)|$. Further, we define

$$\delta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, a, f)}{T(r, f)}$$

and is called the Nevanlinna deficiency of f at a, and the truncated defect is given by

$$\Theta(a, f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, a, f)}{T(r, f)},$$

where $\overline{N}(r, a, f)$ is the counting function of f corresponding to the distinct a-points of f, that is, by ignoring the multiplicities of a-points of f.

Theorem 1.3. *Let $\mathcal{F}$ be a family of transcendental entire functions and $J(\mathcal{F})$ contains at least three points. Then for any $w_0 \in J(\mathcal{F})$ and $f \in \mathcal{F}$ with $\Theta(w_0, f) < \dfrac{1}{2}$, there exist a sequence $\{w_n\}$ such that $w_n \to w_0$ and a sequence $\{f_n\} \subset \mathcal{F}$ such that w_n is a repelling fixed point of $f \circ f_n$.*

The polynomial analogue of Theorem 1.3 also holds as follows theorem.

Theorem 1.4. *If $\mathcal{F}$ is a family of nonconstant polynomials in which for each $w_0 \in J(\mathcal{F})$, there is $P_0 \in \mathcal{F}$ such that $P_0 - w_0$ has at least three distinct simple roots. Then $J(\mathcal{F})$ is contained in the closure of repelling fixed points of the polynomials of the form $P \circ Q$, where P, $Q \in \mathcal{F}$.*

Definition 1.1. *For a subfamily $\mathcal{F}$ of $\mathcal{H}(D)$ and $z \in \mathbb{C}$, define*

$$\mathcal{O}^{-}_{\mathcal{F}}(z) := \{w \in D \colon f(w) = z \text{ for some } f \in \mathcal{F}\} = \bigcup_{f \in \mathcal{F}} f^{-1}\{z\}$$

and

$$E(\mathcal{F}) := \{z \in \mathbb{C} \colon \mathcal{O}^{-}_{\mathcal{F}}(z) \text{ is finite}\}.$$

For a family $\mathcal{F}$ of nonconstant entire functions and $z_0 \in \mathbb{C}$, $\mathcal{O}^{-}_{\mathcal{F}}(z_0)$ is finite implies that $f^{-1}\{z_0\}$ is finite for each $f \in \mathcal{F}$. In this case $N(r, z_0, f) = O(1)$ and, hence, $\delta(z_0, f) = 1$ for all $f \in \mathcal{F}$. While $\delta(z_0, f) = 1$ for all $f \in \mathcal{F}$ may not always imply that $\mathcal{O}^{-}_{\mathcal{F}}(z_0)$ is finite as shown by the following example.

***Example* 1.3.** Let $\mathcal{F} = \{(z-n)\, e^z : n \in \mathbb{N}\}$. Then $\mathcal{F}$ is a family of transcendental entire functions and $\mathcal{O}^{-}_{\mathcal{F}}(0) = \{n \colon n \in \mathbb{N}\}$ is infinite and $N(r, 0, f) = O(\log(r))$ as $r \to \infty$ and, hence, $\delta(0, f) = 1$ for all $f \in \mathcal{F}$.

By an extension of Montel's theorem [3, p. 203], it follows that if $\mathcal{O}^{-}_{\mathcal{F}}(z_0)$ is omitted by $\mathcal{F}$ on some deleted neighborhood of some $w \in J(\mathcal{F})$, then $\mathcal{O}^{-}_{\mathcal{F}}(z_0)$ contains at most one point and, hence, $z_0 \in E(\mathcal{F})$.

Let $\mathcal{F}$ be a uniformly bounded family of holomorphic functions on a domain D. Then by Montel's theorem $J(\mathcal{F}) = \phi$. Note that $E(\mathcal{F})$ is an infinite set. Indeed, there exists $M > 0$ such that $|f(z)| \le M$ for all $f \in \mathcal{F}$ and so $\{w \colon |w| > M\} \subset E(\mathcal{F})$ showing that $E(\mathcal{F})$ is uncountable. Let $\mathcal{F} = \{f \in \mathcal{H}(D) \colon f \text{ omits two distinct fixed values } a \text{ and } b \text{ on } D\}$. Then by Montel's theorem, $J(\mathcal{F}) = \phi$ and $E(\mathcal{F}) = \{a, b\}$. The size of $E(\mathcal{F})$ has a definite relation with $J(\mathcal{F})$. In fact, we have the following result.

Theorem 1.5. *Let $\mathcal{F}$ be a subfamily of $\mathcal{H}(\mathcal{D})$.*

(a) *If $E(\mathcal{F}) \ne \phi$, then, for $z \notin E(\mathcal{F})$, $J(\mathcal{F}) \subseteq \overline{\mathcal{O}^{-}_{\mathcal{F}}(z)}$.*

(b) *If $J(\mathcal{F}) \ne \phi$, then $\#E(\mathcal{F}) \le 1$.*

Following example shows that $E(\mathcal{F})$ may contain exactly one point.

***Example* 1.4.** Let $\mathcal{F} = \{nz \colon n \in \mathbb{N}\}$ be the family of entire functions. Then $\mathcal{O}^{-1}_{\mathcal{F}}(0) = \{0\}$, it follows that $0 \in E(\mathcal{F})$. Note that $J(\mathcal{F}) = \{0\}$ and, by Theorem 1.5, $E(\mathcal{F}) = \{0\}$.

For a family $\mathcal{F}$ of entire functions with $F(\mathcal{F}) \ne \phi$, the set

$$F_\infty(\mathcal{F}) := \{z \in F(\mathcal{F}) \colon \text{ there is a sequence } \{f_n\} \subset \mathcal{F} \text{ such that } f_n(z) \to \infty\}$$

is an open as well as closed subset of $F(\mathcal{F})$. Indeed, let $z_0 \in F_\infty(\mathcal{F})$. Then there is a sequence $\{f_n\}$ such that $f_n(z_0) \to \infty$. By normality of $\mathcal{F}$ at z_0, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges uniformly to ∞ in some neighborhood U of z_0 and, hence, $U \subset F_\infty(\mathcal{F})$. This proves that $F_\infty(\mathcal{F})$ is an open subset of $F(\mathcal{F})$. Similarly, $F_\infty(\mathcal{F})$ is closed also.

We say that $f \in \partial(\mathcal{F})$ if and only if there is an open disk $D(z_0, r) \subset F(\mathcal{F})$ and a sequence $\{f_n\}$ in $\mathcal{F}$ such that $\{f_n\}$ converges uniformly to f on $D(z_0, r)$ and $f \notin \mathcal{F}$. By using Vitali's theorem [1, p. 56], for a family $\mathcal{F}$ of entire functions, $f \in \partial(\mathcal{F})$ if and only if there is a sequence $\{f_n\} \subset \mathcal{F}$ which converges locally uniformly to f on a component of $F(\mathcal{F})$ and $f \notin \mathcal{F}$.

It is observed that $F_\infty(\mathcal{F}) \neq \phi$ if and only if $\infty \in \partial(\mathcal{F})$. Further, if $F_\infty(\mathcal{F})$ is a nonempty proper subset of $F(\mathcal{F})$, then $F(\mathcal{F})$ is disconnected. Following example shows that the converse of this statement is not true.

***Example* 1.5.** Let $\mathcal{F}_1 = \{\sin kz : k \in \mathbb{N}\}$. Then we show that $J(\mathcal{F}_1) = \mathbb{R}$.

Let $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Then choose a disk $D(z_0, r)$ about z_0 such that $D(z_0, r) \cap \mathbb{R} = \phi$. Note that for every $z \in D(z_0, r)$ and $k \in \mathbb{N}$, $kz \notin \mathbb{R}$ and

$$|\sin kz| = \sqrt{\sin^2 kx \cosh^2 ky + \cos^2 kx \sinh^2 ky} =$$

$$= \sqrt{(1 - \cos^2 kx) \cosh^2 ky + \cos^2 kx \sinh^2 ky} =$$

$$= \sqrt{\cosh^2 ky - \cos^2 kx \left(\cosh^2 ky - \sinh^2 ky\right)} =$$

$$= \sqrt{\cosh^2 ky - \cos^2 kx}.$$

Thus $|\sin kz| \to \infty$ as $k \to \infty$ uniformly on $D(z_0, r)$. Therefore, $\mathbb{C} \setminus \mathbb{R} \subset F(\mathcal{F}_1)$. Next, if $z_0 \in \mathbb{R}$, then any disk $D(z_0, s)$ about z_0 contains a segment of $\mathbb{R}$ which is mapped into $[-1, 1]$ by $\sin kz$ for every $k \in \mathbb{N}$, whereas, for any other point $z \in D(z_0, s) \setminus \mathbb{R}$, $|\sin kz| \to \infty$ as $k \to \infty$. So the family $\mathcal{F}_1 = \{\sin kz : k \in \mathbb{N}\}$ can not be normal on $z_0 \in \mathbb{R}$. Thus, $\mathbb{R} \subset J(\mathcal{F}_1)$. But $\mathbb{C} \setminus \mathbb{R} \subset F(\mathcal{F}_1)$, hence, $J(\mathcal{F}_1) = \mathbb{R}$.

For $\mathcal{F}_2 = \{z^n : n \in \mathbb{N}\}$, $J(\mathcal{F}_2) = \{z : |z| = 1\}$. Let $\mathcal{F}_3 = \mathcal{F}_2 \cup \mathcal{F}_1$. Then, by Theorem 1.1, $J(\mathcal{F}_3) = \mathbb{R} \cup \{z : |z| = 1\}$. Clearly, $F(\mathcal{F}_3)$ is disconnected and consists of four components. But $F_\infty(\mathcal{F}_3)$ is not proper subset of $F(\mathcal{F}_3)$, since it can be easily shown that $F_\infty(\mathcal{F}_3) = F(\mathcal{F}_3)$.

2. Proof of main results. ***Proof of Theorem* 1.1.** (a) Clearly, $J(\mathcal{F}_1) \cup J(\mathcal{F}_2) \subset J(\mathcal{F}_1 \cup \mathcal{F}_2)$. To show that the other way inclusion, let $z_0 \in J(\mathcal{F}_1 \cup \mathcal{F}_2)$. Then, by Zalcman lemma [15], there is a sequence $\{f_n\} \subset \mathcal{F}_1 \cup \mathcal{F}_2$, a sequence of positive real numbers $r_n \to 0$ and a sequence $\{z_n\}$: $z_n \to z_0$ as $n \to \infty$ such that $f_n(z_n + r_n z)$ converges locally uniformly on $\mathbb{C}$ to a nonconstant entire function $f(z)$. There is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which lies entirely either in $\mathcal{F}_1$ or $\mathcal{F}_2$ and $f_{n_k}(z_{n_k} + r_{n_k} z)$ converges locally uniformly on $\mathbb{C}$ to the nonconstant entire function $f(z)$. Hence, by the converse to Zalcman lemma, $z_0 \in J(\mathcal{F}_1) \cup J(\mathcal{F}_2)$. Therefore, $J(\mathcal{F}_1 \cup \mathcal{F}_2) \subset J(\mathcal{F}_1) \cup J(\mathcal{F}_2)$.

(b) Suppose that $\mathbb{C} \setminus U$ contains at least two points. Since $U = \bigcup_{f \in \mathcal{F}} f(N)$, each $f \in \mathcal{F}$ omits at least two distinct values on N and, hence, by Montel's theorem, $\mathcal{F}$ is normal in N, which is a contradiction as $z_o \in J(\mathcal{F}) \cap N$. Hence, $\mathbb{C} \setminus U$ contains at most one point.

***Proof of Theorem* 1.2.** (a) If $J(\mathcal{F}) = D$, then there is nothing to prove. Suppose that $J(\mathcal{F}) \neq D$. Assume on the contrary that $\mathrm{Int}(J(\mathcal{F})) \neq \phi$. Let N be a neighborhood of some $z \in J(\mathcal{F})$ such that $N \subset J(\mathcal{F})$. Since $J(\mathcal{F})$ is forward invariant, $U = \bigcup_{f \in \mathcal{F}} f(N) \subset J(\mathcal{F})$. By Theorem 1.1, it follows that $\mathbb{C} \setminus J(\mathcal{F})$ contains at most one point. Since $J(\mathcal{F})$ is properly contained in D, it follows that $D = \mathbb{C}$ and $U = J(\mathcal{F})$. Since $J(\mathcal{F})$ is closed in $D = \mathbb{C}$, we have $J(\mathcal{F}) = \mathbb{C} = D$, a contradiction. Hence, $\mathrm{Int}(J(\mathcal{F})) = \phi$.

To prove (b), suppose $z_0 \in J(\mathcal{F})$ is an isolated point. Then there exists a neighborhood V of z_0 such that $V \setminus \{z_0\} \cap J(\mathcal{F}) = \phi$. Since $f\,(F(\mathcal{F})) \subset F(\mathcal{F})$ for all $f \in \mathcal{F}$, $f\,(V \setminus \{z_0\}) \subset F(\mathcal{F})$ for all $f \in \mathcal{F}$. So the family $\mathcal{F}$ omits $J(\mathcal{F})$ on $V \setminus \{z_0\}$. Therefore, by an extension of Montel's theorem [3, p. 203], $\mathcal{F}$ is normal in V, which is a contradiction.

***Proof of Theorem* 1.3.** We use the method of Schwick [13] to carry out the proof. Let $f \in \mathcal{F}$. By an application of the second fundamental theorem of Nevanlinna [6, p. 44], the set

$$A = \left\{ w : \Theta(w, f) \geq \frac{1}{2} \right\}$$

contains at most two points. Since $J(\mathcal{F})$ contains at least three elements, therefore, for $w_0 \in$ $\in J(\mathcal{F}) \setminus A$, $\Theta(w_0, f) < \dfrac{1}{2}$. This implies that the equation $f(z) = w_0$ has infinitely many simple roots $a_1, a_2, \ldots$, say. Now by Zalcman lemma, there is a sequence $f_n \in \mathcal{F}$, a sequence $z_n \to w_0$ and a sequence of positive real numbers $r_n \to 0$, such that $f_n(z_n + r_n z) \to h(z)$, where $h(z)$ is nonconstant entire function. Continuity of f implies that $f \circ f_n(z_n + r_n z) \to f \circ h(z)$. If h is transcendental, then for each a_n except for two values $\Theta(a_n, h) < \dfrac{1}{2}$ and hence there exists $b \in \mathbb{C}$ such that $h(b) = a_n$ and $h'(b) \neq 0$. Further, if h is a polynomial, then for each a_n, except for one value, $h(z) = a_n$ has simple roots. We pick up one value a_1, say, such that there exists $b \in \mathbb{C}$ with $h(b) = a_1$, and $h'(b) \neq 0$ and, hence, $f(h(b)) = w_0$, $f'(h(b))h'(b) \neq 0$, that is, $(f \circ h)'(b) \neq 0$. Next, $f \circ f_n(z_n + r_n z) - (z_n + r_n z) \to f \circ h(z) - w_0$. Since $f \circ h - w_0$ has zero at $z = b$ and $f \circ h - w_0$ is not constant, by Hurwitz theorem, $f \circ f_n(z_n + r_n z) - (z_n + r_n z)$ has zeros at c_n with $c_n \to b$ for all sufficiently large n. Thus $w_n = z_n + r_n c_n$ is a fixed point of $f \circ f_n$. Since, for large $n, r_n\,(f \circ f_n)'\,(z_n + r_n c_n) = (f \circ f_n(z_n + r_n z))'\,(c_n) \to (f \circ h)'(b) \neq 0$ so that $\left|(f \circ f_n)'\,(z_n + r_n c_n)\right| > 1$.

The proof of Theorem 1.4 is on the similar lines as that of Theorem 1.3.

***Proof of Theorem* 1.5.** (a) If $J(\mathcal{F}) = \phi$, then there is nothing to prove. Suppose that $J(\mathcal{F}) \neq \phi$. Assume the contrary that there is $z_0 \in J(\mathcal{F})$ such that $z_0 \notin \overline{\mathcal{O}^-_{\mathcal{F}}(z_1)}$ for some $z_1 \in \mathbb{C} \setminus E(\mathcal{F})$, that is, there is a neighborhood N of z_0 such that $N \cap \mathcal{O}^-_{\mathcal{F}}(z_1) = \phi$. We choose a neighborhood $N_1 \subset N$ of z_0 such that $(N_1 \setminus \{z_0\}) \cap \mathcal{O}^-_{\mathcal{F}}(z_2) = \phi$ for some $z_2 \in E(\mathcal{F})$ since $\mathcal{O}^-_{\mathcal{F}}(z_2)$ is a finite set. Then $\cup_{f \in \mathcal{F}} f(N_1 \setminus \{z_0\})$ omits z_1 and z_2. Therefore, by an extension of Montel's theorem [3, p. 203], $\mathcal{F}$ is normal in N_1, which is a contradiction as $z_0 \in J(\mathcal{F}) \cap N_1$.

(b) Suppose that $E(\mathcal{F}) \geq 2$ and let $z_1, z_2 \in E(\mathcal{F})$. Let $z_0 \in J(\mathcal{F})$. Since $\mathcal{O}^-_{\mathcal{F}}(z_1) \cup \mathcal{O}^-_{\mathcal{F}}(z_2)$ is a finite set, we choose a neighborhood N of z_0 such that $N \setminus \{z_0\} \cap \left(\mathcal{O}^-_{\mathcal{F}}(z_1) \cup \mathcal{O}^-_{\mathcal{F}}(z_2)\right) = \phi$ and, hence, by an extension of Montel's theorem, $\mathcal{F}$ is normal in N, which is a contradiction as $z_0 \in J(\mathcal{F}) \cap N$.

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