UDC 519.41/47

M. J. Tomkinson, Prof. (Univ. Glasgow, Scotland)

Major subgroups of nilpotent-by-finite groups

Великі підгрупи майже нільпотентних груп

The main result is theorem which states that any major subgroup M of nilpotent-by-finite group G contains derived subgroups of all normal nilpotent subgroups of finite index in G and that G/M_G is a Chernikov group.

Основний результат — теорема, яка стверджує, що довільна велика підгрупа M майже нільпотентної групи G містить комутанти всіх її нормальних нільпотентних підгруп скінченного індексу і G/M_G — черніковська група.

The major subgroups of a group G and their intersection $\mu(G)$ were introduced in [1] as a variation on maximal subgroups and the Frattini subgroup of G. Ie particular, we have proved in [1—4] a number of results of the form: if $G/\mu(G)$ has property X then G has property X. The property X may be that of being hypercentral or hypercyclic or of having some finiteness condition. These results, of course, require some restriction on the group G.

We recall the definitions. Let U be a subgroup of the group G and consider the properly ascending chains

$$U = U_0 < U_1 < \ldots < U_\alpha = G$$

from U to G. We define $\mathfrak{m}(U)$ to be the least upper bound of the types α of all such chains. Thus $\mathfrak{m}(U)$ is a measure of how far from G the subgroup U is; $\mathfrak{m}(U) = 1$ if and only if U is a maximal subgroup of G. A proper subgroup M of G is said to be a *major subgroup* if $\mathfrak{m}(U) = \mathfrak{m}(M)$ whenever $M \leqslant U < G$. Then we define $\mu(G)$ to be the intersection of all major subgroups of G.

The main classes of groups considered in [1—3] were nilpotent-by-finite groups and soluble groups with rank restrictions. In [4], different methods were used to extend the main results on nilpotent-by-finite groups to soluble FC-nilpotent groups. Here we show that the situation in nilpotent-by-finite groups is much less complicated that in the other classes. In [3] it was observed that if G is a Chernikov group then $\mu(G)$ is finite so the results for Chernikov groups are not surprising. Here we see that the major subgroups in a nilpotent-by-finite group all occur in Chernikov factor groups. If X is a subgroup of G, then we write $X_G = \text{core}(X) = \bigcap_{g \in G} g^{-1}Xg$.

Theorem. *Let A be a nilpotent normal subgroup having finite index in the group G. If M is a major subgroup of G then $M \geqslant A'$ and G/M_G is a Chernikov group.*

This result does not extend to any of the other classes we have considered. Let $A \cong C_{p^\infty}$ and let α be an automorphism of A having infinite order and such that $C_A(\alpha) = 1$. Form the split extension G of A by $X = \langle a \rangle$. Then $\mathfrak{m}(X) = \omega$ and X is a major subgroup of G, but $X_G = 1$ and G/X_G is not a Chernikov group. It is clear that G is a soluble minimax group and is also FC-nilpotent, A being the FC-centre of G.

We split the theorem into two parts, first considering abelian-by-finite groups.

Proposition 1. *Let M be a major subgroup of the abelian-by-finite group G. Then G/M_G is a Chernikov group.*

Moreover, if A is an abelian normal subgroup of finite index in G and M is not maximal in G, then A/M_G is a divisible Chernikov p-group, for some prime p.

Proof. We may clearly assume that $M_G = 1$.

If $AM < G$, then $\mathfrak{m}(M) = \mathfrak{m}(AM) = 1$ and so $M = AM$. Thus $A \leqslant M_G = 1$ and so G is finite.

So we may assume that $AM = G$ and $A \cap M \lhd AM = G$ so that $A \cap M = M_G = 1$. If A has a nontrivial finite factor group A/H then G/H_G is finite. Since $H_G < A$, we have $M \leqslant MH_G < G$ and so $\mathfrak{m}(M) = \mathfrak{m}(MH_G) = 1$. Thus $H_G \leqslant M_G = 1$ and so G is finite. Therefore we may assume that A has no nontrivial finite factor groups and so is divisible.

Let $A/K \cong C_{p^\infty}$; then A/K_G is a divisible Chernikov p-group and there is a normal subgroup L/K_G of G/K_G such that A/L is a divisible Chernikov p-group and every normal subgroup of G/L properly contained in A/L is finite. Therefore $\mathfrak{m}(ML) = \omega$ and hence $\mathfrak{m}(M) = \omega$. It follows that every properly ascending chain from M to ML is finite and so L has maximal condition on M-admissible subgroups. Since M is finite it follows that L is a finitely generated abelian group. If L were infinite then, for any prime $q \neq p$, $L^q < L$ and hence $A^q < A$, contrary to A having no finite factor group. Thus L is finite and A is a divisible Chernikov p-group.

Proposition 2. *Let M be a major subgroup of the nilpotent-by-finite group G and let A be a nilpotent normal subgroup of finite index in G. Then $M \geqslant A'$. Hence $\mu(G) \geqslant A'$.*

Proof. Again we assume that $M_G = 1$ and have to show that A is abelian. We use induction on the nilpotency class of A.

If $AM < G$, then $\mathfrak{m}(M) = \mathfrak{m}(AM) = 1$ and $M \geqslant A$. Therefore we may assume that $AM = G$.

Let Z be the centre of A; then $M \cap Z \lhd AM = G$ and so $M \cap Z = M_G = 1$.

Suppose that $MZ < G$; then MZ/Z is a major subgroup of G/Z and, by induction, $MZ \geqslant A'$. If MZ is maximal in G, then $M = MZ \geqslant A'$, as required. So, using Proposition 1, $m(MZ) = \omega$, $A/A \cap MZ$ is a divisible Chernikov p-group and every properly ascending chain from M to MZ is finite. Therefore Z satisfies the maximal condition on M-admissible subgroups and, since $M/C_M(Z)$ is finite, it follows that Z is a finitely generated abelian group.

If Z is infinite then, for any prime $q \neq p$, $Z^q < Z$ and so $(A \cap M) Z^q = A \cap MZ^q < A \cap MZ$. Now $A \cap MZ/A \cap MZ^q$ is an elementary abelian q-group and so also is $A \cap MZ/(A \cap MZ^q)_G$. Since A is nilpotent, $A/(A \cap MZ^q)_G$ is the direct product of $A \cap MZ/(A \cap MZ^q)_G$ and a divisible Chernikov p-group. In particular, $MZ^q \geqslant A'$. So MZ^q/A' is a major subgroup of G/A' and, by Proposition 1, $A/A \cap MZ^q$ is divisible. This is contrary to $A/A \cap MZ^q$ having an elementary abelian direct factor and so Z must be finite. It follows that A has finite exponent [5, Theorem 2.23], contrary to $A/A \cap MZ$ being divisible.

This shows that $MZ = G$. Therefore $A = A \cap MZ = (A \cap M) Z$ and $A \cap M \lhd AM = G$. So $A \cap M = M_G = 1$ and $A = (A \cap M) Z = Z$ is abelian.

For finite groups the usual way to prove that $\varphi(G) \geqslant N'$ if N is a normal nilpotent subgroup would be to first prove that $\varphi(N) \leqslant \varphi(G)$ whenever $N \lhd \lhd G$. This result is clearly false for the subgroup $\mu(G)$ as is shown by taking $G = C_{p^\infty}$ and N the subgroup of order p^2. Even for normal subgroups of finite index we can prove very little in this direction.

Proposition 3. *Let N be a normal subgroup having finite index in the nilpotent group G. Then $\mu(N) \leqslant \mu(G)$.*

Proof. Let M be a major subgroup of G; then $M \geqslant G'$ [1, Lemma 4.1] and either $G/M \cong C_p$ or $G/M \cong C_{p^\infty}$ [1, Theorem 3.1]. If $N \leqslant M$ then clearly $\mu(N) \leqslant M$. If N/M then in both cases $NM = G$, $N/N \cap M \cong G/M$ and $N \cap M$ is a major subgroup of N. Thus we again have $\mu(N) \leqslant M$.

Example. Let $V = A \times B \times C$ be the direct product of three groups each isomorphic to C_{2^∞}, $A = \langle a_1, a_2, \ldots; a_1^2 = 1, \quad a_{n+1}^2 = a_n \rangle$, etc. and let $H \cong \mathrm{Alt}(4)$ act on V as follows: the three elements x, y, z of order two act according to

$$x: a_n \to b_n, \quad c_n \to a_n^{-1} b_n^{-1} c_n^{-1},$$

$$y: a_n \to a_n^{-1} b_n^{-1} c_n^{-1}, \quad b_n \to c_n,$$

$$z: a_n \to c_n, \quad b_n \to a_n^{-1} b_n^{-1} c_n^{-1};$$

an element t of order three acts according to

$$t: a_n \to b_n \to c_n \to a_n.$$

Then the split extension G of V by H can be thought of as the wreath product of C_{2^∞} by Alt (4) with the centre factored out.

The only H-admissible subgroups of V are $V_n = \langle a_n, b_n, c_n \rangle$ and so H is a major subgroup of G and $\mu(G) = H_G = 1$.

Let $N = V \langle x, y, z \rangle \lhd G$. We show that $\mu(N) \neq 1$ and so $\mu(N)/\mu(G)$. Write $X = \langle x, y, z \rangle$. If $\mu(N) = 1$ then there would be a major subgroup M of N not containing the element $a_1 b_1$. It is clear that $VM = N$ and $V \cap M = U$ is an X-admissible subgroup of V such that the X-admissible subgroups of V/U are all finite.

Now $B = \langle a_n b_n; n = 1, 2, \ldots \rangle$ is an X-admissible subgroup of V and clearly $V = B \times U$. If $Z = C_V(X) = \langle a_n b_n, b_n c_n; n = 1, 2, \ldots \rangle$ then $Z = B \times (U \cap Z)$. Without loss of generality, we may assume that

$$U \cap Z = \langle b_1 c_1, b_2 c_2 x_1, b_3 c_3 x_2, \ldots \rangle,$$

where $x_i \in \langle a_i b_i, b_i c_i \rangle$. (The other possibility is that we have $c_n a_n$ in place of the $b_n c_n$.)

We now have $V/U \cap Z = (Z/U \cap Z) \times (U/U \cap Z)$, a direct product of

X-admissible subgroups. But Soc $(V/U \cap Z) = \langle \bar{a}_1, \bar{b}_1 \rangle$ and X acts as follows:

$$x: \bar{a}_1 \to \bar{b}_1, \quad y: \bar{a}_1 \to \bar{b}_1, \quad z \text{ fixes } \bar{a}_1, \bar{b}_1.$$

We see that $\langle \bar{a}_1 \bar{b}_1 \rangle$ is the unique minimal X-admissible subgroup of $V/U \cap Z$, contrary to $V/U \cap Z$ having a direct decomposition.

1. *Tomkinson M. J.* A Frattini-like subgroup // Math. Proc. Cambridge Phil. Soc.— 1975.— 77, N 2.— P. 247—257.
2. *Tomkinson M. J.* Supersolubility conditions and a Frattini-like subgroup // Quart. J. Math.— 1978.— **29**, N 1.— P. 93—99.
3. *Tomkinson M. J.* Finiteness conditions and a Frattini-like subgroup // Rend Circ. mat. Palermo.— 1990.— **23**.— P. 321—335.
4. *Tomkinson M. J.* *FC*-nilpotent groups and a Frattini-like subgroup.— To appear.
5. *Robinson D. J. S.* Finiteness conditions and generalized soluble groups: In 2 pt.— Berlin etc.: Springer, 1972.— Pt. 1.— 210 p.

Received 21.10.91