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# DIFFERENT TYPE PARAMETERIZED INEQUALITIES FOR PREINVEX FUNCTIONS WITH RESPECT TO ANOTHER FUNCTION VIA GENERALIZED FRACTIONAL INTEGRAL OPERATORS AND THEIR APPLICATIONS

# РІЗНІ ТИПИ ПАРАМЕТРИЗОВАНИХ НЕРІВНОСТЕЙ ДЛЯ ПРЕІНВЕКСНИХ ФУНКЦІЙ ВІДНОСНО ІНШОЇ ФУНКЦІЇ З ВИКОРИСТАННЯМ УЗАГАЛЬНЕНИХ ДРОБОВИХ ІНТЕГРАЛЬНИХ ОПЕРАТОРІВ ТА ЇХ ЗАСТОСУВАННЯ

The authors have proved an identity with two parameters for differentiable function with respect to another function via generalized integral operator. By applying the established identity, the generalized trapezium, midpoint and Simpson type integral inequalities have been discovered. It is pointed out that the results of this research provide integral inequalities for almost all fractional integrals discovered in recent past decades. Various special cases have been identified. Some applications of presented results to special means and new error estimates for the trapezium and midpoint quadrature formula have been analyzed. The ideas and techniques of this paper may stimulate further research in the field of integral inequalities.

Доведено тотожність з двома параметрами для диференційовних функцій відносно іншої функції з використанням узагальненого інтегрального оператора. За допомогою цієї тотожності отримано інтегральні нерівності типу трапеції, середньої точки та типу Сімпсона. Зазначено, що результати цього дослідження охоплюють майже всі дробові інтеграли, які були відкриті упродовж кількох останніх десятиліть. Розглянуто різні спеціальні випадки. Також наведено деякі застосування цих результатів у спеціальних випадках і нові оцінки похибок для квадратурних формул типу трапеції та середньої точки. Ідеї та методи цієї роботи мають стимулювати подальші дослідження в галузі інтегральних нерівностей.

**1. Introduction.** The following inequality, named Hermite – Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1.1.** *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $p_1, p_2 \in I$ with $p_1 < p_2$. Then the following inequality holds:*

$$f\left(\frac{p_1+p_2}{2}\right) \le \frac{1}{p_2-p_1}\int\limits_{p_1}^{p_2} f(x)dx \le \frac{f(p_1)+f(p_2)}{2}. \tag{1.1}$$

*This inequality* (1.1) *is also known as trapezium inequality.*

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1.1) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [1 – 6, 8, 10, 11, 13, 14, 18, 20 – 25, 27 – 33].

The following inequality is well-known in the literature as Simpson's inequality.

**Theorem 1.2.** *Let $f : [p_1, p_2] \longrightarrow \mathbb{R}$ be four time differentiable on the interval $(p_1, p_2)$ and having the fourth derivative bounded on $(p_1, p_2)$ that is $\|f^{(4)}\|_\infty = \sup_{x\in(p_1,p_2)} |f^{(4)}| < \infty$. Then we have*

$$\left|\int_{p_1}^{p_2} f(x)dx - \frac{p_2-p_1}{3}\left[\frac{f(p_1)+f(p_2)}{2} + 2f\left(\frac{p_1+p_2}{2}\right)\right]\right| \le \frac{1}{2880}\|f^{(4)}\|_\infty (p_2-p_1)^5. \quad (1.2)$$

Inequality (1.2) gives an error bound for the classical Simpson quadrature formula, which is one of the most used quadrature formulae in practical applications. In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Simpson type inequalities, see [19, 26].

The aim of this paper is to establish trapezium, midpoint and Simpson type generalized integral inequalities for preinvex functions with respect to another function, some applications to special means and new error bounds for midpoint and trapezium quadrature formula. Interestingly, the special cases of presented results, are fractional integral inequalities. Therefore, it is important to summarize the study of fractional integrals.

At start, let us recall some mathematical preliminaries and definitions which will be helpful for further study.

**Definition 1.1** [23]. *Let $f \in L[p_1, p_2]$. Then $k$-fractional integrals of order $\alpha$, $k > 0$ with $p_1 \ge 0$ are defined by*

$$I^{\alpha,k}_{p_1^+} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{p_1}^{x} (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > p_1,$$

*and*

$$I^{\alpha,k}_{p_2^-} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{p_2} (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad p_2 > x,$$

*where $\Gamma_k(\cdot)$ is $k$-gamma function.*

For $k = 1$, $k$-fractional integrals give Riemann – Liouville integrals. For $\alpha = k = 1$, $k$-fractional integrals give classical integrals.

**Definition 1.2** [15, 16]. *Let $g\colon [p_1, p_2] \to \mathbb{R}$ be an increasing and positive monotone function on $[p_1, p_2]$, having a continuous derivative on $(p_1, p_2)$. The left-hand side fractional integral of $f$ with respect to $g$ on $[p_1, p_2]$ of order $\alpha > 0$ is defined by*

$$I^{\alpha,g}_{p_1+} f(x) = \frac{1}{\Gamma(\alpha)} \int_{p_1}^{x} \frac{g'(u)f(u)}{[g(x)-g(u)]^{1-\alpha}} du, \quad x > p_1,$$

*provided that the integral exists. The right-hand side fractional integral of $f$ with respect to $g$ on $[p_1, p_2]$ of order $\alpha > 0$ is defined by*

$$I^{\alpha,g}_{p_2-} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{p_2} \frac{g'(u)f(u)}{[g(u)-g(x)]^{1-\alpha}} du, \quad x < p_2,$$

*provided that the integral exists.*

Jleli and Samet in [10] proved the Hadamard type inequality for Riemann – Liouville fractional integral of a convex function $f$ with respect to another function $g$. Also in [27], Sarikaya and Ertuğral defined a function $\varphi\colon [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

$$\int\limits_0^1 \frac{\varphi(t)}{t}dt < \infty, \tag{1.3}$$

$$\frac{1}{A} \le \frac{\varphi(s)}{\varphi(r)} \le A \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{1.4}$$

$$\frac{\varphi(r)}{r^2} \le B\frac{\varphi(s)}{s^2} \quad \text{for} \quad s \le r, \tag{1.5}$$

$$\left|\frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2}\right| \le C|r-s|\frac{\varphi(r)}{r^2} \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{1.6}$$

where $A,\ B,\ C > 0$ are independent of $r,\ s > 0$. If $\varphi(r)r^{\alpha}$ is increasing for some $\alpha \ge 0$ and $\dfrac{\varphi(r)}{r^{\beta}}$ is decreasing for some $\beta \ge 0$, then $\varphi$ satisfies (1.3) – (1.6) (see [28]). Therefore, the left- and right-hand sided generalized integral operators are defined as follows:

$$_{p_1^+}I_{\varphi}f(x) = \int\limits_{p_1}^{x} \frac{\varphi(x-t)}{x-t}f(t)dt, \quad x > p_1,$$

$$_{p_2^-}I_{\varphi}f(x) = \int\limits_{x}^{p_2} \frac{\varphi(t-x)}{t-x}f(t)dt, \quad x < p_2.$$

The most important feature of generalized integrals is that they produce Riemann – Liouville fractional integrals, $k$-Riemann – Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals etc. (see [9, 12, 27]).

Recently, Farid in [7] generalized the above integral by introducing an increasing and positive monotone function $g$ on $[p_1, p_2]$, having continuous derivative on $(p_1, p_2)$. The generalized fractional integral operator defined by Farid may be given as follows.

**Definition 1.3.** *The left- and right-hand sided generalized fractional integral of a function $f$ with respect to another function $g$ may be given as follows, respectively*:

$$G^{\varphi,g}_{p_1+}f(x) = \int\limits_{p_1}^{x} \frac{\varphi(g(x)-g(u))}{g(x)-g(u)}g'(u)f(u)du, \quad x > p_1, \tag{1.7}$$

$$G^{\varphi,g}_{p_2-}f(x) = \int\limits_{x}^{p_2} \frac{\varphi\left(g(u)-g(x)\right)}{g(u)-g(x)}g'(u)f(u)du, \quad x < p_2. \tag{1.8}$$

This operator generalizes the various fractional integrals of a function $f$ with respect to another function $g$.

The following special cases are focussed in our study.

(i) If we take $\varphi(u) = u,$ then the operator (1.7) and (1.8) reduces to Riemann – Liouville integral of $f$ with respect to function $g$:

$$I^{g}_{p_1+}f(x) = \int\limits_{p_1}^{x} g'(u)f(u)du, \quad x > p_1, \tag{1.9}$$

$$I^{g}_{p_2-}f(x) = \int\limits_{x}^{p_2} g'(u)f(u)du, \quad x < p_2. \tag{1.10}$$

If $g(u) = u,$ then (1.9) and (1.10) will reduce to Riemann integral of $f.$

(ii) If we take $\varphi(u) = \dfrac{u^{\alpha}}{\Gamma(\alpha)},$ then the operator (1.7) and (1.8) reduces to Riemann – Liouville fractional integral of $f$ with respect to function $g$:

$$I^{\varphi,g}_{p_1+}f(x) = \frac{1}{\Gamma(\alpha)}\int\limits_{p_1}^{x} [g(x) - g(u)]^{\alpha-1} g'(u)f(u)du, \quad x > p_1, \tag{1.11}$$

$$I^{\varphi,g}_{p_2-}f(x) = \frac{1}{\Gamma(\alpha)}\int\limits_{x}^{p_2} [g(u) - g(x)]^{\alpha-1} g'(u)f(u)du, \quad x < p_2. \tag{1.12}$$

If $g(u) = u,$ then (1.11) and (1.12) will reduce to left- and right-hand sided Riemann – Liouville fractional integrals of $f$, respectively.

(iii) If we take $\varphi(u) = \dfrac{u^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},$ then the operator (1.7) and (1.8) reduces to $k$-Riemann – Liouville fractional integral of $f$ with respect to function $g$:

$$I^{\varphi,g}_{p_1+,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int\limits_{p_1}^{x} [g(x) - g(u)]^{\frac{\alpha}{k}-1} g'(u)f(u)du, \quad x > p_1, \tag{1.13}$$

$$I^{\varphi,g}_{p_2-,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int\limits_{x}^{p_2} [g(u) - g(x)]^{\alpha-1} g'(u)f(u)du, \quad x < p_2. \tag{1.14}$$

If $g(u) = u,$ then these operators in (1.13) and (1.14) reduces to $k$-fractional integral operators given in [23].

(iv) If we take $\varphi_g(u) = u(g(p_2) - u)^{\alpha-1}$ for $\alpha \in (0,1),$ then the operator given in (1.7) and (1.8) reduces to conformable fractional integral operator of $f$ with respect to a function $g$:

$$K^{\alpha,g}_{p_1}f(x) = \int\limits_{p_1}^{x} [g(u)]^{\alpha-1} g'(u)f(u)du, \quad x > p_1. \tag{1.15}$$

This operator (1.15) generalizes conformable fractional integral operator which was given by Khalil et al. in [17].

(v) If we take $\varphi(u) = \dfrac{u}{\alpha}\exp(-Au)$, where $A = \dfrac{1-\alpha}{\alpha}$ and $\alpha \in (0,1)$, then the operator given in (1.7) and (1.8) reduces to fractional integral operator of $f$ with respect to function $g$ with exponential kernel:

$$J_{p_1+}^{\alpha,g}f(x) = \frac{1}{\alpha}\int\limits_{p_1}^{x} \exp\big(-A(g(x)-g(u))\big)g'(u)f(u)du, \quad x > p_1, \tag{1.16}$$

$$J_{p_2-}^{\alpha,g}f(x) = \frac{1}{\alpha}\int\limits_{x}^{p_2} \exp\big(-A(g(x)-g(u))\big)g'(u)f(u)du, \quad x < p_2. \tag{1.17}$$

Operators in (1.16) and (1.17) generalizes fractional integral operator with exponential kernel which was introduced by Kirane and Torebek in [18].

Motivated by the above literatures, the main objective of this paper is to discover in Section 2, an interesting identity with two parameters in order to study some new bounds regarding trapezium, midpoint and Simpson type integral inequalities. By using the established identity as an auxiliary result, some new estimates for trapezium, midpoint and Simpson type integral inequalities via generalized integrals are obtained. It is pointed out that some new fractional integral inequalities have been deduced from main results. In Section 3, some applications to special means and new error estimates for the midpoint and trapezium quadrature formula are given. The ideas and techniques of this paper may stimulate further research in the field of integral inequalities.

**2. Main results.** Throughout this study, let $P = [mp_1, mp_1 + \eta(p_2, mp_1)]$ be an invex subset with respect to $\eta : P \times P \longrightarrow \mathbb{R}$, where $p_1 < p_2$ and $m \in (0,1]$. Also, for all $t \in [0,1]$, for brevity, we define

$$\Lambda_m^{\varphi,g}(t) := \int\limits_0^t \frac{\varphi\left(g\left(mp_1 + u\eta(p_2,mp_1)\right) - g(mp_1)\right)}{g\left(mp_1 + u\eta(p_2,mp_1)\right) - g(mp_1)} g'\left(mp_1 + u\eta(p_2,mp_1)\right)du < \infty$$

and

$$\Delta_m^{\varphi,g}(t) := \int\limits_t^1 \frac{\varphi\left(g\left(mp_1 + \eta(p_2,mp_1)\right) - g\left(mp_1 + u\eta(p_2,mp_1)\right)\right)}{g\left(mp_1 + \eta(p_2,mp_1)\right) - g\left(mp_1 + u\eta(p_2,mp_1)\right)} \times$$

$$\times\, g'\left(mp_1 + u\eta(p_2,mp_1)\right)du < \infty,$$

where $g$ is an increasing and positive monotone function on $P$, having continuous derivative on $P^\circ = (mp_1, mp_1 + \eta(p_2, mp_1))$.

For establishing some new results regarding general fractional integrals we need to prove the following lemma.

**Lemma 2.1.** *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on $P^\circ$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. If $f' \in L(P)$, then the following identity for generalized fractional integrals holds*:

$$\frac{\gamma_1 f(mp_1) + \gamma_2 f(mp_1 + \eta(p_2,mp_1))}{2} + \left[\frac{\Lambda_m^{\varphi,g}\left(\frac{1}{2}\right) + \Delta_m^{\varphi,g}\left(\frac{1}{2}\right)}{2} - \frac{\gamma_1+\gamma_2}{2}\right] \times$$

$$\times f\left(mp_1+\frac{\eta(p_2,mp_1)}{2}\right)-\frac{1}{2\eta(p_2,mp_1)}\times$$

$$\times\left[G^{\varphi,g}_{\left(mp_1+\frac{\eta(p_2,mp_1)}{2}\right)^+}f\left(mp_1+\eta(p_2,mp_1)\right)+G^{\varphi,g}_{\left(mp_1+\frac{\eta(p_2,mp_1)}{2}\right)^-}f(mp_1)\right]=$$

$$=\frac{\eta(p_2,mp_1)}{2}\left\{\int\limits_0^{\frac{1}{2}}\left[\Lambda^{\varphi,g}_m(t)-\gamma_1\right]f'\left(mp_1+t\eta(p_2,mp_1)dt-\right.\right.$$

$$\left.-\int\limits_{\frac{1}{2}}^{1}[\Delta^{\varphi,g}_m(t)-\gamma_2]f'(mp_1+t\eta(p_2,mp_1))dt\right\}.$$

*We denote*

$$T_{f,\Lambda^{\varphi,g}_m,\Delta^{\varphi,g}_m}(\gamma_1,\gamma_2;p_1,p_2):=\frac{\eta(p_2,mp_1)}{2}\times$$

$$\times\left\{\int\limits_0^{\frac{1}{2}}\left[\Lambda^{\varphi,g}_m(t)-\gamma_1\right]f'(mp_1+t\eta(p_2,mp_1))dt-\right.$$

$$\left.-\int\limits_{\frac{1}{2}}^{1}\left[\Delta^{\varphi,g}_m(t)-\gamma_2\right]f'(mp_1+t\eta(p_2,mp_1))dt\right\}. \tag{2.1}$$

***Proof.*** Integrating by parts equation (2.1) and changing the variable of integration, we have

$$T_{f,\Lambda^{\varphi,g}_m,\Delta^{\varphi,g}_m}(\gamma_1,\gamma_2;p_1,p_2)=$$

$$=\frac{\eta(p_2,mp_1)}{2}\left\{\int\limits_0^{\frac{1}{2}}\Lambda^{\varphi,g}_m(t)f'\left(mp_1+t\eta(p_2,mp_1)\right)dt-\gamma_1\int\limits_0^{\frac{1}{2}}f'\left(mp_1+t\eta(p_2,mp_1)\right)dt-\right.$$

$$\left.-\int\limits_{\frac{1}{2}}^{1}\Delta^{\varphi,g}_m(t)f'\left(mp_1+t\eta(p_2,mp_1)\right)dt+\gamma_2\int\limits_{\frac{1}{2}}^{1}f'\left(mp_1+t\eta(p_2,mp_1)\right)dt\right\}=$$

$$=\frac{\eta(p_2,mp_1)}{2}\left\{\frac{\Lambda^{\varphi,g}_m(t)f\left(mp_1+t\eta(p_2,mp_1)\right)}{\eta(p_2,mp_1)}\Bigg|_0^{\frac{1}{2}}-\frac{1}{\eta(p_2,mp_1)}\times\right.$$

$$\times\int\limits_0^{\frac{1}{2}}\frac{\varphi\left(g\left(mp_1+t\eta(p_2,mp_1)\right)-g(mp_1)\right)}{g\left(mp_1+t\eta(p_2,mp_1)\right)-g(mp_1)}g'\left(mp_1+t\eta(p_2,mp_1)\right)f\left(mp_1+t\eta(p_2,mp_1)\right)dt-$$

$$-\frac{\gamma_1}{\eta(p_2,mp_1)} f\left(mp_1+t\eta(p_2,mp_1)\right)\Big|_0^{\frac{1}{2}} - \left.\frac{\Delta_m^{\varphi,g}(t) f\left(mp_1+t\eta(p_2,mp_1)\right)}{\eta(p_2,mp_1)}\right|_{\frac{1}{2}}^{1} -$$

$$-\frac{1}{\eta(p_2,mp_1)} \int\limits_{\frac{1}{2}}^{1} \frac{\varphi\left(g\left(mp_1+\eta(p_2,mp_1)\right)-g\left(mp_1+t\eta(p_2,mp_1)\right)\right)}{g\left(mp_1+\eta(p_2,mp_1)\right)-g\left(mp_1+t\eta(p_2,mp_1)\right)} \times$$

$$\left.\times g'\left(mp_1+t\eta(p_2,mp_1)\right) f\left(mp_1+t\eta(p_2,mp_1)\right) dt + \frac{\gamma_2}{\eta(p_2,mp_1)} f\left(mp_1+t\eta(p_2,mp_1)\right)\Big|_{\frac{1}{2}}^{1}\right\} =$$

$$= \frac{\gamma_1 f(mp_1)+\gamma_2 f(mp_1+\eta(p_2,mp_1))}{2} + \left[\frac{\Lambda_m^{\varphi,g}\left(\frac{1}{2}\right)+\Delta_m^{\varphi,g}\left(\frac{1}{2}\right)}{2} - \frac{\gamma_1+\gamma_2}{2}\right] \times$$

$$\times f\left(mp_1+\frac{\eta(p_2,mp_1)}{2}\right) - \frac{1}{2\eta(p_2,mp_1)} \times$$

$$\times \left[G^{\varphi,g}_{\left(mp_1+\frac{\eta(p_2,mp_1)}{2}\right)^+} f\left(mp_1+\eta(p_2,mp_1)\right) + G^{\varphi,g}_{\left(mp_1+\frac{\eta(p_2,mp_1)}{2}\right)^-} f(mp_1)\right].$$

Lemma 2.1 is proved.

***Remark* 2.1.** 1. Taking $m=1$, $\gamma_1=\gamma_2=0$, $\eta(p_2,mp_1)=p_2-mp_1$ and $g(t)=\varphi(t)=t$ in Lemma 2.1, we get the classical midpoint type identity.

2. Taking $m=1$, $\gamma_1=\gamma_2=\dfrac{1}{2}$, $\eta(p_2,mp_1)=p_2-mp_1$ and $g(t)=\varphi(t)=t$ in Lemma 2.1, we get the classical Hermite – Hadamard type identity.

3. Taking $m=1$, $\gamma_1=\gamma_2=1$, $\eta(p_2,mp_1)=p_2-mp_1$ and $g(t)=\varphi(t)=t$ in Lemma 2.1, we get the new Simpson type identity.

**Theorem 2.1.** *Let $f\colon P\longrightarrow\mathbb{R}$ be a differentiable mapping on $P^\circ$ and $0\le\gamma_1,\gamma_2\le 1$. If $|f'|^q$ is preinvex on $P$ for $q>1$ and $p^{-1}+q^{-1}=1$, then the following inequality for generalized fractional integrals holds*:

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(\gamma_1,\gamma_2;p_1,p_2)\right| \le \frac{\eta(p_2,mp_1)}{2\sqrt[q]{8}} \times$$

$$\times \left\{\sqrt[p]{B^{\varphi,g}_{\Lambda_m}(\gamma_1;p)}\,\sqrt[q]{3|f'(mp_1)|^q+|f'(p_2)|^q} + \sqrt[p]{B^{\varphi,g}_{\Delta_m}(\gamma_2;p)}\,\sqrt[q]{|f'(mp_1)|^q+3|f'(p_2)|^q}\right\},$$

*where*

$$B^{\varphi,g}_{\Lambda_m}(\gamma_1;p) := \int\limits_0^{\frac{1}{2}} \left|\Lambda_m^{\varphi,g}(t)-\gamma_1\right|^p dt, \qquad B^{\varphi,g}_{\Delta_m}(\gamma_2;p) := \int\limits_{\frac{1}{2}}^{1} \left|\Delta_m^{\varphi,g}(t)-\gamma_2\right|^p dt.$$

***Proof.*** From Lemma 2.1, preinvexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(\gamma_1,\gamma_2;p_1,p_2)\right| \leq \frac{\eta(p_2,mp_1)}{2}\times$$

$$\times\left\{\int\limits_0^{\frac{1}{2}}\left|\Lambda_m^{\varphi,g}(t)-\gamma_1\right|\left|f'\left(mp_1+t\eta(p_2,mp_1)\right)\right|dt+\right.$$

$$\left.+\int\limits_{\frac{1}{2}}^1\left|\Delta_m^{\varphi,g}(t)-\gamma_2\right|\left|f'\left(mp_1+t\eta(p_2,mp_1)\right)\right|dt\right\}\leq$$

$$\leq\frac{\eta(p_2,mp_1)}{2}\left\{\left(\int\limits_0^{\frac{1}{2}}\left|\Lambda_m^{\varphi,g}(t)-\gamma_1\right|^p dt\right)^{\frac{1}{p}}\left(\int\limits_0^{\frac{1}{2}}\left|f'\left(mp_1+t\eta(p_2,mp_1)\right)\right|^q dt\right)^{\frac{1}{q}}+\right.$$

$$\left.+\left(\int\limits_{\frac{1}{2}}^1\left|\Delta_m^{\varphi,g}(t)-\gamma_2\right|^p dt\right)^{\frac{1}{p}}\left(\int\limits_{\frac{1}{2}}^1\left|f'\left(mp_1+t\eta(p_2,mp_1)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\leq$$

$$\leq\frac{\eta(p_2,mp_1)}{2}\left\{\sqrt[p]{B_{\Lambda_m}^{\varphi,g}(\gamma_1;p)}\left(\int\limits_0^{\frac{1}{2}}\left[(1-t)|f'(mp_1)|^q+t|f'(p_2)|^q\right]dt\right)^{\frac{1}{q}}+\right.$$

$$\left.+\sqrt[p]{B_{\Delta_m}^{\varphi,g}(\gamma_2;p)}\left(\int\limits_{\frac{1}{2}}^1\left[(1-t)|f'(mp_1)|^q+t|f'(p_2)|^q\right]dt\right)^{\frac{1}{q}}\right\}=$$

$$=\frac{\eta(p_2,mp_1)}{2\sqrt[q]{8}}\times$$

$$\times\left\{\sqrt[p]{B_{\Lambda_m}^{\varphi,g}(\gamma_1;p)}\,\sqrt[q]{3|f'(mp_1)|^q+|f'(p_2)|^q}+\sqrt[p]{B_{\Delta_m}^{\varphi,g}(\gamma_2;p)}\,\sqrt[q]{|f'(mp_1)|^q+3|f'(p_2)|^q}\right\}.$$

Theorem 2.1 is proved.

We point out some special cases of Theorem 2.1.

***Corollary* 2.1.** *Taking* $p=q=2$ *in Theorem* 2.1, *we get*

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(\gamma_1,\gamma_2;p_1,p_2)\right| \leq \frac{\eta(p_2,mp_1)}{4\sqrt{2}}\times$$

$$\times\left\{\sqrt{B_{\Lambda_m}^{\varphi,g}(\gamma_1;2)}\,\sqrt{3|f'(mp_1)|^2+|f'(p_2)|^2}+\sqrt{B_{\Delta_m}^{\varphi,g}(\gamma_2;2)}\,\sqrt{|f'(mp_1)|^2+3|f'(p_2)|^2}\right\}.$$

***Corollary* 2.2.** *Taking* $|f'|\leq K$ *in Theorem* 2.1, *we get*

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(\gamma_1,\gamma_2;p_1,p_2)\right| \leq \frac{K\eta(p_2,mp_1)}{2\sqrt[q]{2}}\left\{\sqrt[p]{B_{\Lambda_m}^{\varphi,g}(\gamma_1;p)}+\sqrt[p]{B_{\Delta_m}^{\varphi,g}(\gamma_2;p)}\right\}.$$

***Corollary* 2.3.** *Taking* $m=1$, $\gamma_1=\gamma_2=0$, $\eta(p_2,mp_1)=p_2-mp_1$ *and* $g(t)=\varphi(t)=t$ *in Theorem* 2.1, *we get the following midpoint type inequality*:

$$\left|T_{f,\Lambda_1,\Delta_1}(0,0;p_1,p_2)\right| \leq$$

$$\leq \frac{p_2-p_1}{8\sqrt[q]{4}\sqrt[p]{p+1}}\left\{\sqrt[q]{|f'(p_1)|^q+3|f'(p_2)|^q}+\sqrt[q]{3|f'(p_1)|^q+|f'(p_2)|^q}\right\}.$$

***Corollary* 2.4.** *Taking* $m=1$, $\gamma_1=\gamma_2=\dfrac{1}{2}$, $\eta(p_2,mp_1)=p_2-mp_1$ *and* $g(t)=\varphi(t)=t$ *in Theorem* 2.1, *we obtain the following trapezium type inequality:*

$$\left|T_{f,\Lambda_1,\Delta_1}\left(\frac{1}{2},\frac{1}{2};p_1,p_2\right)\right| \leq$$

$$\leq \frac{p_2-p_1}{2\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}}\left\{\sqrt[q]{|f'(p_1)|^q+3|f'(p_2)|^q}+\sqrt[q]{3|f'(p_1)|^q+|f'(p_2)|^q}\right\}.$$

***Corollary* 2.5.** *Taking* $m=1$, $\gamma_1=\gamma_2=1$, $\eta(p_2,mp_1)=p_2-mp_1$ *and* $g(t)=\varphi(t)=t$ *in Theorem* 2.1, *we have the following new Simpson type inequality*:

$$\left|T_{f,\Lambda_1,\Delta_1}(1,1;p_1,p_2)\right| \leq$$

$$\leq \frac{p_2-p_1}{8\sqrt[q]{4}}\sqrt[p]{\frac{2^{p+1}-1}{p+1}}\left\{\sqrt[q]{|f'(p_1)|^q+3|f'(p_2)|^q}+\sqrt[q]{3|f'(p_1)|^q+|f'(p_2)|^q}\right\}.$$

***Corollary* 2.6.** *Taking* $m=1$, $\gamma_1=\dfrac{1}{6}$, $\gamma_2=\dfrac{5}{6}$, $\eta(p_2,mp_1)=p_2-mp_1$ *and* $g(t)=\varphi(t)=t$ *in Theorem* 2.1, *we get*

$$\left|T_{f,\Lambda_1,\Delta_1}\left(\frac{1}{6},\frac{5}{6};p_1,p_2\right)\right| \leq$$

$$\leq \frac{p_2-p_1}{24\sqrt[q]{4}}\sqrt[p]{\frac{3\left(2^{p+1}+1\right)}{p+1}}\left\{\sqrt[q]{|f'(p_1)|^q+3|f'(p_2)|^q}+\sqrt[q]{3|f'(p_1)|^q+|f'(p_2)|^q}\right\}.$$

***Corollary* 2.7.** *Taking* $\gamma_1=\gamma_2=0$ *and* $\varphi(t)=t$ *in Theorem* 2.1, *we have*

$$\left|T_{f,\Lambda_m^g,\Delta_m^g}(0,0,p_1,p_2)\right| \leq \frac{1}{2\sqrt[q]{8}\sqrt[p]{\eta(p_2,mp_1)}}\times$$

$$\times\left\{\sqrt[p]{B_1^g(p)}\sqrt[q]{3|f'(mp_1)|^q+|f'(p_2)|^q}+\sqrt[p]{B_2^g(p)}\sqrt[q]{|f'(mp_1)|^q+3|f'(p_2)|^q}\right\},$$

*where*

$$B_1^g(p) := \int\limits_{mp_1}^{mp_1+\frac{\eta(p_2,mp_1)}{2}} \big[g(t) - g(mp_1)\big]^p dt$$

*and*

$$B_2^g(p) := \int\limits_{mp_1+\frac{\eta(p_2,mp_1)}{2}}^{mp_1+\eta(p_2,mp_1)} \big[g(mp_1 + \eta(p_2, mp_1)) - g(t)\big]^p dt.$$

***Corollary* 2.8.** *Taking* $\gamma_1 = \gamma_2 = 0$ *and* $\varphi(t) = \dfrac{t^\alpha}{\Gamma(\alpha)}$ *in Theorem* 2.1, *we obtain*

$$\big|T_{f,\Lambda_m^g,\Delta_m^g}(0,0,p_1,p_2)\big| \le \frac{1}{2\sqrt[q]{8}\sqrt[p]{\eta(p_2,mp_1)}} \times$$

$$\times \left\{ \sqrt[p]{B_3^g(p,\alpha)}\, \sqrt[q]{3|f'(mp_1)|^q + |f'(p_2)|^q} + \sqrt[p]{B_4^g(p,\alpha)}\, \sqrt[q]{|f'(mp_1)|^q + 3|f'(p_2)|^q} \right\},$$

*where*

$$B_3^g(p,\alpha) := \int\limits_{mp_1}^{mp_1+\frac{\eta(p_2,mp_1)}{2}} \big[g(t) - g(mp_1)\big]^{p\alpha} dt$$

*and*

$$B_4^g(p,\alpha) := \int\limits_{mp_1+\frac{\eta(p_2,mp_1)}{2}}^{mp_1+\eta(p_2,mp_1)} \big[g(mp_1 + \eta(p_2, mp_1)) - g(t)\big]^{p\alpha} dt.$$

***Corollary* 2.9.** *Taking* $\gamma_1 = \gamma_2 = 0$ *and* $\varphi(t) = \dfrac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem* 2.1, *we get*

$$\big|T_{f,\Lambda_m^g,\Delta_m^g}(0,0,p_1,p_2)\big| \le \frac{1}{2\sqrt[q]{8}\sqrt[p]{\eta(p_2,mp_1)}} \times$$

$$\times \left\{ \sqrt[p]{B_5^g(p,\alpha,k)}\, \sqrt[q]{3|f'(mp_1)|^q + |f'(p_2)|^q} + \sqrt[p]{B_6^g(p,\alpha,k)}\, \sqrt[q]{|f'(mp_1)|^q + 3|f'(p_2)|^q} \right\},$$

*where*

$$B_5^g(p,\alpha,k) := \int\limits_{mp_1}^{mp_1+\frac{\eta(p_2,mp_1)}{2}} \big[g(t) - g(mp_1)\big]^{\frac{p\alpha}{k}} dt$$

*and*

$$B_6^g(p,\alpha,k) := \int\limits_{mp_1+\frac{\eta(p_2,mp_1)}{2}}^{mp_1+\eta(p_2,mp_1)} \big[g(mp_1 + \eta(p_2, mp_1)) - g(t)\big]^{\frac{p\alpha}{k}} dt.$$

***Corollary* 2.10.** *Taking* $\gamma_1 = \gamma_2 = 0$ *and* $\varphi_g(t) = t(g(mp_1+\eta(p_2, mp_1))-t)^{\alpha-1}$ *in Theorem* 2.1, *we have*

$$\left|T_{f,\Lambda_m^g,\Delta_m^g}(0,0,p_1,p_2)\right| \leq \frac{1}{2\sqrt[q]{8}\sqrt[p]{\eta(p_2,mp_1)}} \times$$

$$\times\left\{\sqrt[p]{B_7^g(p)}\sqrt[q]{3|f'(mp_1)|^q+|f'(p_2)|^q}+\sqrt[p]{B_8^g(p,\alpha)}\sqrt[q]{|f'(mp_1)|^q+3|f'(p_2)|^q}\right\},$$

*where*

$$B_7^g(p) = \int\limits_{mp_1}^{mp_1+\frac{\eta(p_2,mp_1)}{2}} \left\{g^\alpha(mp_1+\eta(p_2,mp_1))-\Big[g(mp_1)+g(mp_1+\eta(p_2,mp_1))-g(t)\Big]^\alpha\right\}^p dt$$

*and*

$$B_8^g(p,\alpha) := \int\limits_{mp_1+\frac{\eta(p_2,mp_1)}{2}}^{mp_1+\eta(p_2,mp_1)} \Big[g^\alpha(mp_1+\eta(p_2,mp_1))-g^\alpha(t)\Big]^p dt.$$

***Corollary* 2.11.** *Taking* $\gamma_1 = \gamma_2 = 0$ *and* $\varphi(t) = \dfrac{t}{\alpha}\exp(-At)$, *where* $A = \dfrac{1-\alpha}{\alpha}$, *in Theorem* 2.1, *we obtain*

$$\left|T_{f,\Lambda_m^g,\Delta_m^g}(0,0,p_1,p_2)\right| \leq \frac{1}{2\sqrt[q]{8}\sqrt[p]{\eta(p_2,mp_1)}} \times$$

$$\times\left\{\sqrt[p]{B_9^g(p,A)}\sqrt[q]{3|f'(mp_1)|^q+|f'(p_2)|^q}+\sqrt[p]{B_{10}^g(p,A)}\sqrt[q]{|f'(mp_1)|^q+3|f'(p_2)|^q}\right\},$$

*where*

$$B_9^g(p,A) := \int\limits_{mp_1}^{mp_1+\frac{\eta(p_2,mp_1)}{2}} \Big\{1-\exp\big[A\left(g(mp_1)-g(t)\right)\big]\Big\}^p dt$$

*and*

$$B_{10}^g(p,A) := \int\limits_{mp_1+\frac{\eta(p_2,mp_1)}{2}}^{mp_1+\eta(p_2,mp_1)} \Big\{1-\exp\big[A\left(g(t)-g(mp_1+\eta(p_2,mp_1))\right)\big]\Big\}^p dt.$$

**Theorem 2.2.** *Let* $f\colon P\longrightarrow\mathbb{R}$ *be a differentiable mapping on* $P^\circ$ *and* $0\leq\gamma_1,\gamma_2\leq 1$. *If* $|f'|^q$ *is preinvex on* $P$ *for* $q\geq 1$, *then the following inequality for generalized fractional integrals holds*:

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(\gamma_1,\gamma_2;p_1,p_2)\right| \leq$$

$$\leq \frac{\eta(p_2, mp_1)}{2}\left\{\left[B^{\varphi,g}_{\Lambda_m}(\gamma_1;1)\right]^{1-\frac{1}{q}}\sqrt[q]{D^{\varphi,g}_{\Lambda_m}(\gamma_1)|f'(mp_1)|^q + E^{\varphi,g}_{\Lambda_m}(\gamma_1)|f'(p_2)|^q}\, +\right.$$

$$\left. + \left[B^{\varphi,g}_{\Delta_m}(\gamma_2;1)\right]^{1-\frac{1}{q}}\sqrt[q]{F^{\varphi,g}_{\Delta_m}(\gamma_2)|f'(mp_1)|^q + H^{\varphi,g}_{\Delta_m}(\gamma_2)|f'(p_2)|^q}\right\},$$

*where*

$$D^{\varphi,g}_{\Lambda_m}(\gamma_1) := \int_0^{\frac{1}{2}}(1-t)\left|\Lambda^{\varphi,g}_m(t)-\gamma_1\right|dt, \qquad E^{\varphi,g}_{\Lambda_m}(\gamma_1) := \int_0^{\frac{1}{2}} t\left|\Lambda^{\varphi,g}_m(t)-\gamma_1\right|dt,$$

$$F^{\varphi,g}_{\Delta_m}(\gamma_2) := \int_{\frac{1}{2}}^1(1-t)\left|\Delta^{\varphi,g}_m(t)-\gamma_2\right|dt, \qquad H^{\varphi,g}_{\Delta_m}(\gamma_2) := \int_{\frac{1}{2}}^1 t\left|\Delta^{\varphi,g}_m(t)-\gamma_2\right|dt,$$

*and* $B^{\varphi,g}_{\Lambda_m}(\gamma_1;1)$, $B^{\varphi,g}_{\Delta_m}(\gamma_2;1)$ *are defined as in Theorem* 2.1.

***Proof.*** From Lemma 2.1, preinvexity of $|f'|^q$, power mean inequality and properties of the modulus, we have

$$\left|T_{f,\Lambda^{\varphi,g}_m,\Delta^{\varphi,g}_m}(\gamma_1,\gamma_2;p_1,p_2)\right| \leq \frac{\eta(p_2,mp_1)}{2}\times$$

$$\times\left\{\int_0^{\frac{1}{2}}\left|\Lambda^{\varphi,g}_m(t)-\gamma_1\right|\left|f'\left(mp_1+t\eta(p_2,mp_1)\right)\right|dt+\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|\Delta^{\varphi,g}_m(t)-\gamma_2\right|\left|f'\left(mp_1+t\eta(p_2,mp_1)\right)\right|dt\right\}\leq$$

$$\leq \frac{\eta(p_2,mp_1)}{2}\times$$

$$\times\left\{\left(\int_0^{\frac{1}{2}}\left|\Lambda^{\varphi,g}_m(t)-\gamma_1\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\Lambda^{\varphi,g}_m(t)-\gamma_1\right|\left|f'\left(mp_1+t\eta(p_2,mp_1)\right)\right|^q dt\right)^{\frac{1}{q}}+\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^1\left|\Delta^{\varphi,g}_m(t)-\gamma_2\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|\Delta^{\varphi,g}_m(t)-\gamma_2\right|\left|f'\left(mp_1+t\eta(p_2,mp_1)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\leq$$

$$\leq \frac{\eta(p_2,mp_1)}{2}\left\{\left[B^{\varphi,g}_{\Lambda_m}(\gamma_1;1)\right]^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\Lambda^{\varphi,g}_m(t)-\gamma_1\right|\left[(1-t)|f'(mp_1)|^q+t|f'(p_2)|^q\right]dt\right)^{\frac{1}{q}}+\right.$$

$$+\left[B^{\varphi,g}_{\Delta_m}(\gamma_2;1)\right]^{1-\frac{1}{q}}\left(\int\limits_{\frac{1}{2}}^{1}\left|\Delta^{\varphi,g}_m(t)-\gamma_2\right|\left[(1-t)|f'(mp_1)|^q+t|f'(p_2)|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}=$$

$$=\frac{\eta(p_2,mp_1)}{2}\Bigg\{\left[B^{\varphi,g}_{\Lambda_m}(\gamma_1;1)\right]^{1-\frac{1}{q}}\sqrt[q]{D^{\varphi,g}_{\Lambda_m}(\gamma_1)|f'(mp_1)|^q+E^{\varphi,g}_{\Lambda_m}(\gamma_1)|f'(p_2)|^q}+$$

$$+\left[B^{\varphi,g}_{\Delta_m}(\gamma_2;1)\right]^{1-\frac{1}{q}}\sqrt[q]{F^{\varphi,g}_{\Delta_m}(\gamma_2)|f'(mp_1)|^q+H^{\varphi,g}_{\Delta_m}(\gamma_2)|f'(p_2)|^q}\Bigg\}.$$

Theorem 2.2 is proved.

We point out some special cases of Theorem 2.2.

***Corollary* 2.12.** *Taking $q=1$ in Theorem* 2.2, *we get*

$$\left|T_{f,\Lambda^{\varphi,g}_m,\Delta^{\varphi,g}_m}(\gamma_1,\gamma_2;p_1,p_2)\right|\leq$$

$$\leq\frac{\eta(p_2,mp_1)}{2}\Bigg\{\left[D^{\varphi,g}_{\Lambda_m}(\gamma_1)+F^{\varphi,g}_{\Delta_m}(\gamma_2)\right]|f'(mp_1)|+\left[E^{\varphi,g}_{\Lambda_m}(\gamma_1)+H^{\varphi,g}_{\Delta_m}(\gamma_2)\right]|f'(p_2)|\Bigg\}.$$

***Corollary* 2.13.** *Taking $|f'|\leq K$ in Theorem* 2.2, *we have*

$$\left|T_{f,\Lambda^{\varphi,g}_m,\Delta^{\varphi,g}_m}(\gamma_1,\gamma_2;p_1,p_2)\right|\leq\frac{K\eta(p_2,mp_1)}{2}\times$$

$$\times\left\{\left[B^{\varphi,g}_{\Lambda_m}(\gamma_1;1)\right]^{1-\frac{1}{q}}\sqrt[q]{D^{\varphi,g}_{\Lambda_m}(\gamma_1)+E^{\varphi,g}_{\Lambda_m}(\gamma_1)}+\left[B^{\varphi,g}_{\Delta_m}(\gamma_2;1)\right]^{1-\frac{1}{q}}\sqrt[q]{F^{\varphi,g}_{\Delta_m}(\gamma_2)+H^{\varphi,g}_{\Delta_m}(\gamma_2)}\right\}.$$

***Corollary* 2.14.** *Taking $m=1$, $\gamma_1=\gamma_2=0$, $\eta(p_2,mp_1)=p_2-mp_1$ and $g(t)=\varphi(t)=t$ in Theorem* 2.2, *we obtain the following midpoint type inequality*:

$$\left|T_{f,\Lambda_1,\Delta_1}(0,0;p_1,p_2)\right|\leq$$

$$\leq\frac{p_2-p_1}{16\sqrt[q]{3}}\left\{\sqrt[q]{|f'(p_1)|^q+2|f'(p_2)|^q}+\sqrt[q]{2|f'(p_1)|^q+|f'(p_2)|^q}\right\}.$$

***Corollary* 2.15.** *Taking $m=1$, $\gamma_1=\gamma_2=\dfrac{1}{2}$, $\eta(p_2,mp_1)=p_2-mp_1$ and $g(t)=\varphi(t)=t$ in Theorem* 2.2, *we get the following trapezium type inequality*:

$$\left|T_{f,\Lambda_1,\Delta_1}\left(\frac{1}{2},\frac{1}{2};p_1,p_2\right)\right|\leq$$

$$\leq\frac{p_2-p_1}{144\sqrt[q]{6}}\left\{\sqrt[q]{|f'(p_1)|^q+5|f'(p_2)|^q}+\sqrt[q]{5|f'(p_1)|^q+|f'(p_2)|^q}\right\}.$$

***Corollary* 2.16.** *Taking $m=1$, $\gamma_1=\gamma_2=1$, $\eta(p_2,mp_1)=p_2-mp_1$ and $g(t)=\varphi(t)=t$ in Theorem* 2.2, *we have the following new Simpson type inequality*:

$$\left|T_{f,\Lambda_1,\Delta_1}(1,1;p_1,p_2)\right|\leq$$

$$\leq\frac{3(p_2-p_1)}{16\sqrt[q]{9}}\left\{\sqrt[q]{2|f'(p_1)|^q+7|f'(p_2)|^q}+\sqrt[q]{7|f'(p_1)|^q+2|f'(p_2)|^q}\right\}.$$

***Corollary* 2.17.** *Taking* $m = 1$, $\gamma_1 = \dfrac{1}{6}$, $\gamma_2 = \dfrac{5}{6}$, $\eta(p_2, mp_1) = p_2 - mp_1$ *and* $g(t) = \varphi(t) = t$ *in Theorem* 2.2, *we get*

$$\left| T_{f,\Lambda_1,\Delta_1}\left(\frac{1}{6}, \frac{5}{6}; p_1, p_2\right) \right| \leq$$

$$\leq \left(\frac{5}{72}\right)^{1-\frac{1}{q}} \frac{p_2 - p_1}{2} \left\{ \sqrt[q]{\theta_1 |f'(p_1)|^q + \theta_2 |f'(p_2)|^q} + \sqrt[q]{\theta_3 |f'(p_1)|^q + \theta_4 |f'(p_2)|^q} \right\},$$

*where* $\theta_1 = \dfrac{51}{1944} + \dfrac{1}{48}$, $\theta_2 = \dfrac{29}{1296}$, $\theta_3 = \dfrac{325}{648}$ *and* $\theta_4 = \dfrac{125}{648} - \dfrac{7}{48}$.

***Corollary* 2.18.** *Taking* $\gamma_1 = \gamma_2 = 0$ *and* $\varphi(t) = t$ *in Theorem* 2.2, *we have*

$$\left| T_{f,\Lambda_m^g,\Delta_m^g}(0, 0, p_1, p_2) \right| \leq$$

$$\leq \frac{1}{2\eta^{\frac{q+1}{q}}(p_2, mp_1)} \left\{ \left[B_1^g(1)\right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_1^g(1)\eta(p_2, mp_1) - C_1^g\right] |f'(mp_1)|^q + C_1^g |f'(p_2)|^q} + \right.$$

$$\left. + \left[B_2^g(1)\right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_2^g(1)\eta(p_2, mp_1) - E_1^g\right] |f'(mp_1)|^q + E_1^g |f'(p_2)|^q} \right\},$$

*where*

$$C_1^g := \int\limits_{mp_1}^{mp_1 + \frac{\eta(p_2, mp_1)}{2}} (t - mp_1)(g(t) - g(mp_1))dt,$$

$$E_1^g := \int\limits_{mp_1 + \frac{\eta(p_2, mp_1)}{2}}^{mp_1 + \eta(p_2, mp_1)} (t - mp_1)(g(mp_1 + \eta(p_2, mp_1)) - g(t))dt,$$

*and* $B_1^g(1)$, $B_2^g(1)$ *are defined as in Corollary* 2.7 *for value* $p = 1$.

***Corollary* 2.19.** *Taking* $\gamma_1 = \gamma_2 = 0$ *and* $\varphi(t) = \dfrac{t^\alpha}{\Gamma(\alpha)}$ *in Theorem* 2.2, *we obtain*

$$\left| T_{f,\Lambda_m^g,\Delta_m^g}(0, 0, p_1, p_2) \right| \leq \frac{1}{2\eta^{\frac{q+1}{q}}(p_2, mp_1)} \times$$

$$\times \left\{ \left[B_3^g(1, \alpha)\right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_3^g(1, \alpha)\eta(p_2, mp_1) - C_1^g(\alpha)\right] |f'(mp_1)|^q + C_1^g(\alpha) |f'(p_2)|^q} + \right.$$

$$\left. + \left[B_4^g(1, \alpha)\right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_4^g(1, \alpha)\eta(p_2, mp_1) - E_1^g(\alpha)\right] |f'(mp_1)|^q + E_1^g(\alpha) |f'(p_2)|^q} \right\},$$

*where*

$$C_1^g(\alpha) := \int\limits_{mp_1}^{mp_1+\frac{\eta(p_2,mp_1)}{2}} (t-mp_1)(g(t)-g(mp_1))^{\alpha} dt,$$

$$E_1^g(\alpha) := \int\limits_{mp_1+\frac{\eta(p_2,mp_1)}{2}}^{mp_1+\eta(p_2,mp_1)} (t-mp_1)(g(mp_1+\eta(p_2,mp_1))-g(t))^{\alpha} dt,$$

*and* $B_3^g(1,\alpha)$, $B_4^g(1,\alpha)$ *are defined as in Corollary* 2.8 *for value* $p=1$.

***Corollary* 2.20.** *Taking* $\gamma_1=\gamma_2=0$ *and* $\varphi(t)=\dfrac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem* 2.2, *we get*

$$\left|T_{f,\Lambda_m^g,\Delta_m^g}(0,0,p_1,p_2)\right| \le \frac{1}{2\eta^{\frac{q+1}{q}}(p_2,mp_1)} \times$$

$$\times \left\{ \left[B_5^g(1,\alpha,k)\right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_5^g(1,\alpha,k)\eta(p_2,mp_1)-C_1^g(\alpha,k)\right]|f'(mp_1)|^q + C_1^g(\alpha,k)|f'(p_2)|^q} + \right.$$

$$\left. + \left[B_6^g(1,\alpha,k)\right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_6^g(1,\alpha,k)\eta(p_2,mp_1)-E_1^g(\alpha,k)\right]|f'(mp_1)|^q + E_1^g(\alpha,k)|f'(p_2)|^q} \right\},$$

*where*

$$C_1^g(\alpha,k) := \int\limits_{mp_1}^{mp_1+\frac{\eta(p_2,mp_1)}{2}} (t-mp_1)(g(t)-g(mp_1))^{\frac{\alpha}{k}} dt,$$

$$E_1^g(\alpha,k) := \int\limits_{mp_1+\frac{\eta(p_2,mp_1)}{2}}^{mp_1+\eta(p_2,mp_1)} (t-mp_1)(g(mp_1+\eta(p_2,mp_1))-g(t))^{\frac{\alpha}{k}} dt,$$

*and* $B_5^g(1,\alpha,k)$, $B_6^g(1,\alpha,k)$ *are defined as in Corollary* 2.9 *for value* $p=1$.

***Corollary* 2.21.** *Taking* $\varphi_g(t)=t(g(mp_1+\eta(p_2,mp_1))-t)^{\alpha-1}$ *in Theorem* 2.2, *we have*

$$\left|T_{f,\Lambda_m^g,\Delta_m^g}(0,0,p_1,p_2)\right| \le$$

$$\le \frac{1}{2\eta^{\frac{q+1}{q}}(p_2,mp_1)} \left\{ \left[B_7^g(1)\right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_7^g(1)\eta(p_2,mp_1)-L_1^g\right]|f'(mp_1)|^q + L_1^g|f'(p_2)|^q} + \right.$$

$$\left. + \left[B_8^g(1,\alpha)\right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_8^g(1,\alpha)\eta(p_2,mp_1)-L_2^g(\alpha)\right]|f'(mp_1)|^q + L_2^g(\alpha)|f'(p_2)|^q} \right\},$$

*where*

$$L_1^g(\alpha) = \int\limits_{mp_1}^{mp_1+\frac{\eta(p_2,mp_1)}{2}} (t-mp_1) \times$$

$$\times \left[g^{\alpha}(mp_1+\eta(p_2,mp_1))-(g(mp_1)+g(mp_1+\eta(p_2,mp_1))-g(t))^{\alpha}\right]dt,$$

$$L_2^g(\alpha)=\int\limits_{mp_1+\frac{\eta(p_2,mp_1)}{2}}^{mp_1+\eta(p_2,mp_1)}(t-mp_1)\left[g^{\alpha}(mp_1+\eta(p_2,mp_1))-g^{\alpha}(t)\right]dt,$$

*and* $B_7^g(1)$, $B_8^g(1,\alpha)$ *are defined as in Corollary* 2.10 *for value* $p=1$.

***Corollary* 2.22.** *Taking* $\gamma_1=\gamma_2=0$ *and* $\varphi(t)=\dfrac{t}{\alpha}\exp(-At)$, *where* $A=\dfrac{1-\alpha}{\alpha}$, *in Theorem* 2.2, *we obtain*

$$\left|T_{f,\Lambda_m^g,\Delta_m^g}(0,0,p_1,p_2)\right|\le\frac{1}{2(1-\alpha)\eta^{\frac{q+1}{q}}(p_2,mp_1)}\times$$

$$\times\Bigg\{\left[B_9^g(1,A)\right]^{1-\frac{1}{q}}\sqrt[q]{L_3^g(A)|f'(mp_1)|^q+L_4^g(A)|f'(p_2)|^q}+$$

$$+\left[B_{10}^g(1,A)\right]^{1-\frac{1}{q}}\sqrt[q]{L_5^g(A)|f'(mp_1)|^q+L_6^g(A)|f'(p_2)|^q}\Bigg\},$$

*where*

$$L_3^g(A):=\int\limits_{mp_1}^{mp_1+\frac{\eta(p_2,mp_1)}{2}}(mp_1+\eta(p_2,mp_1)-t)\left\{1-\exp\left[A\left(g(mp_1)-g(t)\right)\right]\right\}dt,$$

$$L_4^g(A):=\int\limits_{mp_1}^{mp_1+\frac{\eta(p_2,mp_1)}{2}}(t-mp_1)\left\{1-\exp\left[A\left(g(mp_1)-g(t)\right)\right]\right\}dt,$$

$$L_5^g(A):=\int\limits_{mp_1+\frac{\eta(p_2,mp_1)}{2}}^{mp_1+\eta(p_2,mp_1)}(mp_1+\eta(p_2,mp_1)-t)\times$$

$$\times\left\{1-\exp\left[A\left(g(t)-g(mp_1+\eta(p_2,mp_1))\right)\right]\right\}dt,$$

$$L_6^g(A):=\int\limits_{mp_1+\frac{\eta(p_2,mp_1)}{2}}^{mp_1+\eta(p_2,mp_1)}(t-mp_1)\left\{1-\exp\left[A\left(g(t)-g(mp_1+\eta(p_2,mp_1))\right)\right]\right\}dt,$$

*and* $B_9^g(1,A)$, $B_{10}^g(1,A)$ *are defined as in Corollary* 2.11 *for value* $p=1$.

***Remark* 2.2.** Applying Theorems 2.1 and 2.2 for special values of parameters $\gamma_1$ and $\gamma_2$, for appropriate choices of function $g(t)=t$; $g(t)=\ln t\ \forall t>0$; $g(t)=e^t$ etc., where

$$\varphi(t)=t,\ \frac{t^{\alpha}}{\Gamma(\alpha)},\ \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},$$

$$\varphi_g(t) = t(g(p_2) - t)^{\alpha-1} \quad \text{for} \quad \alpha \in (0,1),$$

$$\varphi(t) = \frac{t}{\alpha} \exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right] \quad \text{for} \quad \alpha \in (0,1),$$

such that $|f'|^q$ to be preinvex (or convex in special case), we can deduce some new general fractional integral inequalities. The details are left to the interested reader.

**3. Applications.** Consider the following special means for different real numbers $p_1, p_2$ and $p_1 p_2 \neq 0$:

1) the arithmetic mean:

$$A := A(p_1, p_2) = \frac{p_1 + p_2}{2},$$

2) the harmonic mean:

$$H := H(p_1, p_2) = \frac{2}{\frac{1}{p_1} + \frac{1}{p_2}},$$

3) the logarithmic mean:

$$L := L(p_1, p_2) = \frac{p_2 - p_1}{\ln|p_2| - \ln|p_1|},$$

4) the generalized log-mean:

$$L_r := L_r(p_1, p_2) = \left[\frac{p_2^{r+1} - p_1^{r+1}}{(r+1)(p_2 - p_1)}\right]^{\frac{1}{r}}, \qquad r \in \mathbb{Z} \setminus \{-1, 0\}.$$

It is well-known that $L_r$ is monotonic nondecreasing over $r \in \mathbb{Z}$ with $L_{-1} := L$. In particular, we have the inequality $H \leq L \leq A$. Now, using the theory results in Section 2, we give some applications to special means for different real numbers.

**Proposition 3.1.** *Let $m \in (0,1]$ be a fixed number and $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, where $p_1 < p_2$ and $\eta(p_2, mp_1) > 0$. Then, for $r \in \mathbb{N}$ and $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds*:

$$\Big|A^r(mp_1, mp_1 + \eta(p_2, mp_1)) - L_r^r(mp_1, mp_1 + \eta(p_2, mp_1))\Big| \leq$$

$$\leq \frac{r\eta(p_2, mp_1)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \left\{ \sqrt[q]{A\left(|mp_1|^{q(r-1)}, 3|p_2|^{q(r-1)}\right)} + \sqrt[q]{A\left(3|mp_1|^{q(r-1)}, |p_2|^{q(r-1)}\right)} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = 0$, $f(t) = t^r$ and $g(t) = \varphi(t) = t$ in Theorem 2.1, one can obtain the result immediately.

**Proposition 3.2.** *Let $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, where $p_1 < p_2$ and $\eta(p_2, mp_1) > 0$. Then, for $r \in \mathbb{N}$ and $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds*:

$$\Big|A((mp_1)^r, (mp_1 + \eta(p_2, mp_1))^r) - L_r^r(mp_1, mp_1 + \eta(p_2, mp_1))\Big| \leq$$

$$\leq \frac{r\eta(p_2, mp_1)}{\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \left\{ \sqrt[q]{A\left(|mp_1|^{q(r-1)}, 3|p_2|^{q(r-1)}\right)} + \sqrt[q]{A\left(3|mp_1|^{q(r-1)}, |p_2|^{q(r-1)}\right)} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = \dfrac{1}{2},\ f(t) = t^r$ and $g(t) = \varphi(t) = t$ in Theorem 2.1, one can obtain the result immediately.

**Proposition 3.3.** *Let $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, where $p_1 < p_2$ and $\eta(p_2, mp_1) > 0$. Then, for $r \in \mathbb{N}$ and $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds*:

$$\left|A\big((mp_1)^r, (mp_1+\eta(p_2, mp_1))^r\big) - \frac{1}{2}\Big[A^r(mp_1, mp_1+\eta(p_2, mp_1)) + L_r^r(mp_1, mp_1+\eta(p_2, mp_1))\Big]\right| \leq$$

$$\leq \frac{r\eta(p_2, mp_1)}{8\sqrt[q]{2}} \sqrt[p]{\frac{2^{p+1}-1}{p+1}} \left\{ \sqrt[q]{A\left(|mp_1|^{q(r-1)}, 3|p_2|^{q(r-1)}\right)} + \sqrt[q]{A\left(3|mp_1|^{q(r-1)}, |p_2|^{q(r-1)}\right)} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = 1,\ f(t) = t^r$ and $g(t) = \varphi(t) = t$ in Theorem 2.1, one can obtain the result immediately.

**Proposition 3.4.** *Let $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, where $p_1 < p_2$ and $\eta(p_2, mp_1) > 0$. Then, for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds*:

$$\left|\frac{1}{A(mp_1, mp_1+\eta(p_2, mp_1))} - \frac{1}{L(mp_1, mp_1+\eta(p_2, mp_1))}\right| \leq$$

$$\leq \sqrt[q]{\frac{3}{4}} \frac{\eta(p_2, mp_1)}{4\sqrt[p]{p+1}} \left\{ \frac{1}{\sqrt[q]{H\left(|mp_1|^{2q}, 3|p_2|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(3|mp_1|^{2q}, |p_2|^{2q}\right)}} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = 0,\ f(t) = \dfrac{1}{t}$ and $g(t) = \varphi(t) = t$ in Theorem 2.1, one can obtain the result immediately.

**Proposition 3.5.** *Let $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, where $p_1 < p_2$ and $\eta(p_2, mp_1) > 0$. Then, for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds*:

$$\left|\frac{1}{H(mp_1, mp_1+\eta(p_2, mp_1))} - \frac{1}{L(mp_1, mp_1+\eta(p_2, mp_1))}\right| \leq$$

$$\leq \frac{\eta(p_2, mp_1)}{\sqrt[q]{4}\sqrt[p]{2^{p+1}(p+1)}} \left\{ \frac{1}{\sqrt[q]{H\left(|mp_1|^{2q}, 3|p_2|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(3|mp_1|^{2q}, |p_2|^{2q}\right)}} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = \dfrac{1}{2},\ f(t) = \dfrac{1}{t}$ and $g(t) = \varphi(t) = t$ in Theorem 2.1, one can obtain the result immediately.

**Proposition 3.6.** *Let $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, where $p_1 < p_2$ and $\eta(p_2, mp_1) > 0$. Then, for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds*:

$$\left|\frac{1}{H(mp_1, mp_1+\eta(p_2, mp_1))} - \frac{1}{2}\left[\frac{1}{A(mp_1, mp_1+\eta(p_2, mp_1))} - \frac{1}{L(mp_1, mp_1+\eta(p_2, mp_1))}\right]\right| \leq$$

$$\leq \frac{\eta(p_2, mp_1)}{8\sqrt[q]{2}} \sqrt[p]{\frac{2^{p+1}-1}{p+1}} \left\{ \frac{1}{\sqrt[q]{H\left(|mp_1|^{2q}, 3|p_2|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(3|mp_1|^{2q}, |p_2|^{2q}\right)}} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = 1,\ f(t) = \dfrac{1}{t}$ and $g(t) = \varphi(t) = t$ in Theorem 2.1, one can obtain the result immediately.

**Proposition 3.7.** *Let* $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, *where* $p_1 < p_2$ *and* $\eta(p_2, mp_1) > 0$. *Then, for* $r \in \mathbb{N}$ *and* $r \geq 2$, *where* $q \geq 1$, *the following inequality holds*:

$$\Big|A^r(mp_1, mp_1 + \eta(p_2, mp_1)) - L_r^r(mp_1, mp_1 + \eta(p_2, mp_1))\Big| \leq$$

$$\leq \sqrt[q]{\frac{2}{3}} \frac{r\eta(p_2, mp_1)}{8} \left\{ \sqrt[q]{A\left(|mp_1|^{q(r-1)}, 2|p_2|^{q(r-1)}\right)} + \sqrt[q]{A\left(2|mp_1|^{q(r-1)}, |p_2|^{q(r-1)}\right)} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = 0,\ f(t) = t^r$ and $g(t) = \varphi(t) = t$ in Theorem 2.2, one can obtain the result immediately.

**Proposition 3.8.** *Let* $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, *where* $p_1 < p_2$ *and* $\eta(p_2, mp_1) > 0$. *Then, for* $r \in \mathbb{N}$ *and* $r \geq 2$, *where* $q \geq 1$, *the following inequality holds*:

$$\Big|A((mp_1)^r, (mp_1 + \eta(p_2, mp_1))^r) - L_r^r(mp_1, mp_1 + \eta(p_2, mp_1))\Big| \leq$$

$$\leq \frac{r\eta(p_2, mp_1)}{72\sqrt[q]{3}} \left\{ \sqrt[q]{A\left(|mp_1|^{q(r-1)}, 5|p_2|^{q(r-1)}\right)} + \sqrt[q]{A\left(5|mp_1|^{q(r-1)}, |p_2|^{q(r-1)}\right)} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = \dfrac{1}{2},\ f(t) = t^r$ and $g(t) = \varphi(t) = t$ in Theorem 2.2, one can obtain the result immediately.

**Proposition 3.9.** *Let* $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, *where* $p_1 < p_2$ *and* $\eta(p_2, mp_1) > 0$. *Then, for* $r \in \mathbb{N}$ *and* $r \geq 2$, *where* $q \geq 1$, *the following inequality holds*:

$$\Bigg|A((mp_1)^r, (mp_1 + \eta(p_2, mp_1))^r) -$$

$$- \frac{1}{2}\left[A^r(mp_1, mp_1 + \eta(p_2, mp_1)) + L_r^r(mp_1, mp_1 + \eta(p_2, mp_1))\right]\Bigg| \leq$$

$$\leq \sqrt[q]{\frac{2}{9}} \frac{3r\eta(p_2, mp_1)}{16} \left\{ \sqrt[q]{A\left(2|mp_1|^{q(r-1)}, 7|p_2|^{q(r-1)}\right)} + \sqrt[q]{A\left(7|mp_1|^{q(r-1)}, 2|p_2|^{q(r-1)}\right)} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = 1,\ f(t) = t^r$ and $g(t) = \varphi(t) = t$ in Theorem 2.2, one can obtain the result immediately.

**Proposition 3.10.** *Let* $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, *where* $p_1 < p_2$ *and* $\eta(p_2, mp_1) > 0$. *Then, for* $q \geq 1$, *the following inequality holds*:

$$\left|\frac{1}{A(mp_1, mp_1 + \eta(p_2, mp_1))} - \frac{1}{L(mp_1, mp_1 + \eta(p_2, mp_1))}\right| \leq$$

$$\leq \sqrt[q]{\frac{2}{3}} \frac{\eta(p_2, mp_1)}{8} \left\{ \frac{1}{\sqrt[q]{H\left(|mp_1|^{2q}, 2|p_2|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(2|mp_1|^{2q}, |p_2|^{2q}\right)}} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = 0,\ f(t) = \dfrac{1}{t}$ and $g(t) = \varphi(t) = t$ in Theorem 2.2, one can obtain the result immediately.

**Proposition 3.11.** *Let* $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, *where* $p_1 < p_2$ *and* $\eta(p_2, mp_1) > 0$. *Then, for* $q \geq 1$, *the following inequality holds*:

$$\left| \frac{1}{H(mp_1, mp_1 + \eta(p_2, mp_1))} - \frac{1}{L(mp_1, mp_1 + \eta(p_2, mp_1))} \right| \leq$$

$$\leq \frac{\eta(p_2, mp_1)}{72\sqrt[q]{3}} \left\{ \frac{1}{\sqrt[q]{H\left(|mp_1|^{2q}, 5|p_2|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(5|mp_1|^{2q}, |p_2|^{2q}\right)}} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = \dfrac{1}{2},\ f(t) = \dfrac{1}{t}$ and $g(t) = \varphi(t) = t$ in Theorem 2.2, one can obtain the result immediately.

**Proposition 3.12.** *Let* $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, *where* $p_1 < p_2$ *and* $\eta(p_2, mp_1) > 0$. *Then, for* $q \geq 1$, *the following inequality holds*:

$$\left| \frac{1}{H(mp_1, mp_1 + \eta(p_2, mp_1))} - \frac{1}{2} \left[ \frac{1}{A(mp_1, mp_1 + \eta(p_2, mp_1))} - \frac{1}{L(mp_1, mp_1 + \eta(p_2, mp_1))} \right] \right| \leq$$

$$\leq \sqrt[q]{\frac{2}{9}} \frac{3\eta(p_2, mp_1)}{16} \left\{ \frac{1}{\sqrt[q]{H\left(2|mp_1|^{2q}, 7|p_2|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(7|mp_1|^{2q}, 2|p_2|^{2q}\right)}} \right\}.$$

***Proof.*** Taking $\gamma_1 = \gamma_2 = 1,\ f(t) = \dfrac{1}{t}$ and $g(t) = \varphi(t) = t$ in Theorem 2.2, one can obtain the result immediately.

***Remark*** **3.1.** Applying our Theorems 2.1 and 2.2 for special values of parameters $\gamma_1$ and $\gamma_2$, for appropriate choices of function $g(t) = t$; $g(t) = \ln t\ \forall t > 0$; $g(t) = e^t$ etc., where

$$\varphi(t) = t,\ \frac{t^{\alpha}}{\Gamma(\alpha)},\ \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},$$

$$\varphi_g(t) = t(g(p_2) - t)^{\alpha - 1} \quad \text{for} \quad \alpha \in (0, 1),$$

$$\varphi(t) = \frac{t}{\alpha} \exp\left[\left(-\frac{1-\alpha}{\alpha}\right) t\right] \quad \text{for} \quad \alpha \in (0, 1),$$

such that $|f'|^q$ to be preinvex (or convex in the special case), we can deduce some new general fractional integral inequalities using above special means. The details are left to the interested reader.

Next, we provide some new error estimates for the midpoint and trapezium quadrature formula. Let $Q$ be the partition of the points $p_1 = x_0 < x_1 < \ldots < x_k = p_2$ of the interval $[p_1, p_2]$. Let consider the quadrature formula

$$\int_{p_1}^{p_2} f(x)dx = M(f, Q) + E(f, Q), \qquad \int_{p_1}^{p_2} f(x)dx = T(f, Q) + E^*(f, Q),$$

where

$$M(f,Q)=\sum_{i=0}^{k-1} f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i), \qquad T(f,Q)=\sum_{i=0}^{k-1}\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)$$

are the midpoint and trapezium version and $E(f,Q)$, $E^*(f,Q)$ are denote their associated approximation errors.

**Proposition 3.13.** *Let* $f:[p_1,p_2]\longrightarrow\mathbb{R}$ *be a differentiable function on* $(p_1,p_2)$, *where* $p_1<p_2$. *If* $|f'|^q$ *is convex on* $[p_1,p_2]$ *for* $q>1$ *and* $p^{-1}+q^{-1}=1$, *then the following inequality holds*:

$$\big|E(f,Q)\big|\le\frac{1}{4\sqrt[q]{4}\sqrt[p]{p+1}}\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2\Big\{\sqrt[q]{|f'(x_i)|^q+3|f'(x_{i+1})|^q}+\sqrt[q]{3|f'(x_i)|^q+|f'(x_{i+1})|^q}\Big\}.$$

***Proof.*** Applying Theorem 2.1 for $m=1$, $\gamma_1=\gamma_2=0$, $\eta(p_2,mp_1)=p_2-mp_1$ and $g(t)=\varphi(t)=t$ on the subintervals $[x_i,x_{i+1}]$, $i=0,\ldots,k-1$, of the partition $Q$, we have

$$\left|f\left(\frac{x_i+x_{i+1}}{2}\right)-\frac{1}{x_{i+1}-x_i}\int\limits_{x_i}^{x_{i+1}} f(x)dx\right|\le$$

$$\le\frac{x_{i+1}-x_i}{4\sqrt[q]{4}\sqrt[p]{p+1}}\Big\{\sqrt[q]{|f'(x_i)|^q+3|f'(x_{i+1})|^q}+\sqrt[q]{3|f'(x_i)|^q+|f'(x_{i+1})|^q}\Big\}. \tag{3.1}$$

Hence, from (3.1), we get

$$\big|E(f,Q)\big|=\left|\int\limits_{p_1}^{p_2} f(x)dx-M(f,Q)\right|\le$$

$$\le\left|\sum_{i=0}^{k-1}\left\{\int\limits_{x_i}^{x_{i+1}} f(x)dx-f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right|\le$$

$$\le\sum_{i=0}^{k-1}\left|\left\{\int\limits_{x_i}^{x_{i+1}} f(x)dx-f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right|\le$$

$$\le\frac{1}{4\sqrt[q]{4}\sqrt[p]{p+1}}\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2\times$$

$$\times\Big\{\sqrt[q]{|f'(x_i)|^q+3|f'(x_{i+1})|^q}+\sqrt[q]{3|f'(x_i)|^q+|f'(x_{i+1})|^q}\Big\}.$$

Proposition 3.13 is proved.

**Proposition 3.14.** *Let* $f:[p_1,p_2]\longrightarrow\mathbb{R}$ *be a differentiable function on* $(p_1,p_2)$, *where* $p_1<p_2$. *If* $|f'|^q$ *is convex on* $[p_1,p_2]$ *for* $q\ge1$, *then the following inequality holds:*

$$\big|E(f,Q)\big|\le\frac{1}{8\sqrt[q]{3}}\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2\Big\{\sqrt[q]{|f'(x_i)|^q+2|f'(x_{i+1})|^q}+\sqrt[q]{2|f'(x_i)|^q+|f'(x_{i+1})|^q}\Big\}.$$

***Proof.*** The proof is analogous as to that of Proposition 3.13 taking $m = 1$, $\gamma_1 = \gamma_2 = 0$, $\eta(p_2, mp_1) = p_2 - mp_1$ and $g(t) = \varphi(t) = t$ using Theorem 2.2.

**Proposition 3.15.** *Let $f : [p_1, p_2] \longrightarrow \mathbb{R}$ be a differentiable function on $(p_1, p_2)$, where $p_1 < p_2$. If $|f'|^q$ is convex on $[p_1, p_2]$ for $q > 1$ and $p^{-1} + q^{-1} = 1$, then the following inequality holds*:

$$\big|E^*(f,Q)\big| \leq \frac{1}{\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \times$$

$$\times \Big\{ \sqrt[q]{|f'(x_i)|^q + 3|f'(x_{i+1})|^q} + \sqrt[q]{3|f'(x_i)|^q + |f'(x_{i+1})|^q} \Big\}.$$

***Proof.*** Applying Theorem 2.1 for $m = 1$, $\gamma_1 = \gamma_2 = \dfrac{1}{2}$, $\eta(p_2, mp_1) = p_2 - mp_1$ and $g(t) = \varphi(t) = t$ on the subintervals $[x_i, x_{i+1}]$, $i = 0, \ldots, k-1$, of the partition $Q$, we have

$$\left| \frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)dx \right| \leq$$

$$\leq \frac{x_{i+1} - x_i}{\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \Big\{ \sqrt[q]{|f'(x_i)|^q + 3|f'(x_{i+1})|^q} + \sqrt[q]{3|f'(x_i)|^q + |f'(x_{i+1})|^q} \Big\}. \tag{3.2}$$

Hence, from (3.2), we get

$$\big|E^*(f,Q)\big| = \left| \int_{p_1}^{p_2} f(x)dx - T(f,Q) \right| \leq$$

$$\leq \left| \sum_{i=0}^{k-1} \left\{ \int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) \right\} \right| \leq$$

$$\leq \sum_{i=0}^{k-1} \left| \left\{ \int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) \right\} \right| \leq$$

$$\leq \frac{1}{\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \Big\{ \sqrt[q]{|f'(x_i)|^q + 3|f'(x_{i+1})|^q} + \sqrt[q]{3|f'(x_i)|^q + |f'(x_{i+1})|^q} \Big\}.$$

Proposition 3.15 is proved.

**Proposition 3.16.** *Let $f : [p_1, p_2] \longrightarrow \mathbb{R}$ be a differentiable function on $(p_1, p_2)$, where $p_1 < p_2$. If $|f'|^q$ is convex on $[p_1, p_2]$ for $q \geq 1$, then the following inequality holds*:

$$\big|E^*(f,Q)\big| \leq \frac{1}{72\sqrt[q]{6}} \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \Big\{ \sqrt[q]{|f'(x_i)|^q + 5|f'(x_{i+1})|^q} + \sqrt[q]{5|f'(x_i)|^q + |f'(x_{i+1})|^q} \Big\}.$$

***Proof.*** The proof is analogous as to that of Proposition 3.15 taking $m = 1$, $\gamma_1 = \gamma_2 = \dfrac{1}{2}$, $\eta(p_2, mp_1) = p_2 - mp_1$ and $g(t) = \varphi(t) = t$ using Theorem 2.2.

***Remark* 3.2.** Applying our Theorems 2.1 and 2.2, where $m = 1$, for special values of parameter $\gamma_1$ and $\gamma_2$, for appropriate choices of function $g(t) = t$; $g(t) = \ln t \ \forall t > 0$; $g(t) = e^t$ etc., where

$$\varphi(t) = t, \ \frac{t^\alpha}{\Gamma(\alpha)}, \ \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},$$

$$\varphi_g(t) = t(g(p_2) - t)^{\alpha - 1} \quad \text{for} \quad \alpha \in (0, 1),$$

$$\varphi(t) = \frac{t}{\alpha} \exp\left[\left(-\frac{1-\alpha}{\alpha}\right) t\right] \quad \text{for} \quad \alpha \in (0, 1),$$

such that $|f'|^q$ to be convex, we can deduce some new bounds for the midpoint and trapezium quadrature formula using above ideas and techniques. The details are left to the interested reader.

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