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LOCAL SPECTRAL THEORY AND SURJECTIVE SPECTRUM OF LINEAR RELATIONS

ТЕОРІЯ ЛОКАЛЬНОГО СПЕКТРА ТА СЮР'ЄКТИВНИЙ СПЕКТР ЛІНІЙНИХ ВІДНОШЕНЬ

This paper initiates a study of local spectral theory for linear relations. At the beginning, we define the local spectrum and study its properties. Then we obtain results related to the correlation analytic core $K'(T)$ and quasinilpotent part $H_0(T)$ of a linear relation T in a Banach space X. As an application, we give a characterization of the surjective spectrum $\sigma_{su}(T)$ in terms of the local spectrum and show that if $X = H_0(\lambda I - T) + K'(\lambda I - T)$, then $\sigma_{su}(T)$ does not cluster at λ.

Ця робота започатковує вивчення теорії локального спектра для лінійних відношень. Спочатку наведено означення та властивості локального спектра. Після цього отримано деякі результати, що відносяться до кореляційного аналітичного ядра $K'(T)$ та квазінільпотентної частини $H_0(T)$ лінійного відношення T у банаховому просторі X. Як застосування наведено характеризацію сюр'єктивного спектра $\sigma_{su}(T)$ у термінах локального спектра та доведено, що якщо $X = H_0(\lambda I - T) + K'(\lambda I - T)$, то $\sigma_{su}(T)$ не кластеризується для λ.

1. Introduction. Notice that throughout this paper $(X, \|\ \|)$ denotes a complex Banach space. A linear relation T is any mapping having domain $D(T)$ a subspace of X, and taking values in $\mathcal{P}(X)\backslash\varnothing$ (the collection of nonempty subsets of X) such that $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all non zero scalars $\alpha, \beta \in \mathbb{K}$ and $x_1, x_2 \in D(T)$. If $x \notin D(T)$ then $Tx = \varnothing$. With this convention we have $D(T) = \{u \in X : T(u) \neq \varnothing\}$. The set of all linear relations in X is denoted by $LR(X)$. A linear relation T is uniquely defined by its graph $G(T) = \{(u, v) \in X \times X : u \in D(T), v \in T(u)\}$. The inverse of T is the relation T^{-1} given by

$$G(T^{-1}) = \{(v, u) \in X \times X : (u, v) \in G(T)\}.$$

We denotes by

$$R(T) := \bigcup_{x \in D(T)} Tx, \quad \ker(T) := \{x \in X : (x, 0) \in G(T)\}, \quad T(0) := \{x \in X : (0, x) \in G(T)\}$$

the range, the kernel and the multivalued part of T, respectively. The generalized range of T is defined by $R^{\infty}(T) = \bigcap_{n \in \mathbb{N}} R(T^n)$. Note that if $T(0) = \{0\}$, then T is said to be an operator or single-valued. If $G(T)$ is closed, then T is said to be closed. Let Q_T denotes the quotient map from X onto $X/\overline{T(0)}$. We can easy show that $Q_T T$ is single valued and so we can define the norm of T by $\|T\| := \|Q_T T\|$. T is said to be continuous if $\|T\| < \infty$, bounded if T is continuous and everywhere defined. We denote by $CR(X)$, $BR(X)$, and $BCR(X)$ the sets of closed, bounded, and bounded closed linear relations, respectively.

Let U and V be two nonempty subsets of X. We define the distance between U and V by the formula

$$\operatorname{dis}(U, V) = \inf\{\|u - v\|, \ u \in U \text{ and } v \in V\}.$$

We shall also write $\operatorname{dis}(x, V)$ for the distance between $\{x\}$ and V.

A linear relation T is called open if T^{-1} is continuous equivalently if $\gamma(T) > 0$ where

$$\gamma(T) = \begin{cases} +\infty, & \text{if} D(T) \subset \overline{\ker(T)}, \\ \inf \left\{ \dfrac{\|Tx\|}{\operatorname{dis}(x, \ker(T))}; x \in D(T) \backslash \overline{\ker(T)} \right\} & \text{otherwise}. \end{cases}$$

A linear operator A is called a selection of T if $T = A + T - T$ and $D(A) = D(T)$. The resolvent set of a closed linear relation T is the set

$$\rho(T) = \{\lambda \in \mathbb{C} \text{ such that } (\lambda I - T)^{-1} \text{ is everywhere defined and single valued}\},$$

and the spectrum of T is defined by

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

For a survey of result related to linear relations, the reader may see [5].

The concept of local spectral theory, for bounded operators in Banach space, was firstly appeared in Dunford [6, 7]. It's studied also by Kjeld B. Laursen and M. M. Neumann in [8] where they mentioned that this theory played a very naturel role in commutative harmonic analysis. More recently, P. Aiena was attached by this notion in [1, 2] and he was developed many results based on the single valued extension property. Our objective is to generalize the concept of local spectral theory to the general setting of linear relations and we give some of their properties. More precisely, let $T \in BR(X)$ and $x \in X$. The local resolvent of T at x, denoted by $\rho_T(x)$, is defined as the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U_λ of λ and an analytic function $f_{\lambda,x} : U_\lambda \to X$ such that the equation $(\mu I - T) f_{\lambda,x}(\mu) = x + T(0)$ holds for all $\mu \in U_\lambda$. The complement of $\rho_T(x)$ in $\mathbb{C}$ is called the local spectrum of T at x and denoted by $\sigma_T(x)$.

This paper is organized as follows. In Section 2, we introduce some auxiliary results which are important for the following sections. We introduce the notions of quasinilpotent part and correlation analytic core of a linear relation and we give some of their basic properties. In Section 3, we introduce the definition of the local spectrum of linear relation and we give some of their properties. Then we give the definitions of the local and glocal spectral subspaces and their applications in Sections 4 and 5, respectively. In particular, we investigate some connections between local spectral subspace and the correlation analytic core on the one hand and between the glocal spectral subspace and the quasinilpotent part on the other hand. We finish by giving, in Section 6 as application of the local spectral theory, an important result concerning the surjective spectrum of a linear relation.

2. Preliminary results. In this section, we will introduce some auxiliary results which are important for the next parts of this paper. We begin by giving the definitions of the algebraic core $C(T)$, the correlation analytic core $K'(T)$ and the quasinilpotent part $H_0(T)$ of a linear relation T defined in a Banach space X.

Definition 2.1. *Let $T \in LR(X)$. The algebraic core $C(T)$ of T is defined to be the greatest subspace M of X for which $T(M) = M$. It's clear that $C(T) \subseteq R^\infty(T)$.*

Definition 2.2. *Let $T \in LR(X)$. The correlation analytic core $K'(T)$ of T is defined by* $K'(T) = \Big\{ x \in X$ *such that there exist* $a > 0$ *and a sequence* $(u_n) \subset X$, *which verify* $x = u_0,\ u_n \in Tu_{n+1},$ *and* $\operatorname{dis}(u_n, T(0) \cap \ker(T)) \leq a^n \operatorname{dis}(x, T(0) \cap \ker(T))\ \forall n \in \mathbb{N} \Big\}$.

Definition 2.3. *Let $T \in LR(X)$. Then, we define its quasinilpotent part by* $H_0(T) = \Big\{x \in \in X$ *such that there exists a sequence* $(x_n) \subset X$, *which verify* $x = x_0$, $x_{n+1} \in Tx_n$ *and* $\lim_{n\mapsto\infty} \|x_n\|^{\frac{1}{n}} = 0\Big\}$.

Proposition 2.1. *Let $T \in BCR(X)$. Then we have:*

1) $T(K'(T)) = K'(T)$;

2) *if F be a closed subspace of X such that $T(F) = F$, then $F \subseteq K'(T)$.*

Proof. 1. Let $x \in K'(T)$. Then there exist $a > 0$ and $(u_n) \subset X$ such that $x = u_0$, $u_n \in Tu_{n+1}$ and $\operatorname{dis}(u_n, T(0) \cap \ker(T)) \leq a^n \operatorname{dis}(x, T(0) \cap \ker(T))$ $\forall n \in \mathbb{N}$. So it suffices to prove that $u_1 \in \in K'(T)$. If $u_1 \in T(0) \cap \ker(T)$, then there is nothing to prove. If not, we consider the sequence (w_n) defined by

$$w_0 := u_1 \qquad \text{and} \qquad w_n := u_{n+1}.$$

Then, for all $n \in \mathbb{N}$, $w_n = u_{n+1} \in Tu_{n+2} = Tw_{n+1}$ and we have

$$\operatorname{dis}(w_n, T(0) \cap \ker(T)) = \operatorname{dis}(u_{n+1}, T(0) \cap \ker(T)) \leq$$
$$\leq b^n \operatorname{dis}(u_1, T(0) \cap \ker(T))$$

with $b > a^2 \dfrac{\operatorname{dis}(x, T(0) \cap \ker(T))}{\operatorname{dis}(u_1, T(0) \cap \ker(T))}$. Hence, $u_1 \in K'(T)$ which permits us to deduce that $K'(T) \subset \subset T(K'(T))$.

Moving to the reverse inclusion. Let $y \in T(K'(T))$. Then $y \in Tx$ for some $x \in K'(T)$. Let $a > 0$ and $(u_n) \subset X$ be such that $x = u_0$, $u_n \in Tu_{n+1}$ and $\operatorname{dis}(u_n, T(0) \cap \ker(T)) \leq \leq a^n \operatorname{dis}(x, T(0) \cap \ker(T))$ $\forall n \in \mathbb{N}$.

We want to show that $y \in K'(T)$. If $y \in T(0) \cap \ker(T)$, then there is nothing to prove. If not, let consider the sequence (w_n) defined by

$$w_0 := y \qquad \text{and} \qquad w_n := u_{n-1}.$$

Then, for all $n \in \mathbb{N}^*$, we have $w_n = Tw_{n+1}$ and

$$\operatorname{dis}(w_n, T(0) \cap \ker(T)) = \operatorname{dis}(u_{n-1}, T(0) \cap \ker(T)) \leq$$
$$\leq b^n \operatorname{dis}(y, T(0) \cap \ker(T))$$

with $b > \max\left(a, \dfrac{\operatorname{dis}(x, T(0) \cap \ker(T))}{\operatorname{dis}(y, T(0) \cap \ker(T))}\right)$. Thus, $y \in K'(T)$ and, therefore, $T(K'(T)) \subseteq K'(T)$.

2. Let F be a closed subspace of X such that $T(F) = F$. Then the restriction $T_0 : F \longrightarrow F$ is an open map (see [5], Theorem III.4.2). Therefore, by [5] (Proposition II.3.2) we see that $\gamma(T_0) > 0$. Now let $x \in F$. So, there exists $u \in F$ such that $x \in Tu$.

On the other hand, we have

$$\operatorname{dis}(u, \ker(T_0)) \leq \frac{1}{\gamma(T_0)} \parallel T_0 u \parallel .$$

Let $\delta > \dfrac{1}{\gamma(T_0)}$. Then there exists $y \in \ker(T) \cap F$ such that

$$\parallel u - y \parallel \leq \delta \operatorname{dis}(x, T(0)) \leq \delta \operatorname{dis}(x, T(0) \cap \ker(T)).$$

Take $u_1 = u - y$. We have $x \in Tu_1$ and

$$\operatorname{dis}(u_1, T(0) \cap \ker(T)) \leq \| u_1 \| \leq \delta \operatorname{dis}(x, T(0) \cap \ker(T)).$$

By repeating this process we find a sequence $(u_n) \subset X$ such that $x = u_0$, $u_n \in Tu_{n+1}$, and $\operatorname{dis}(u_n, T(0) \cap \ker(T)) \leq \delta^n \operatorname{dis}(x, T(0) \cap \ker(T))$ $\forall n \in \mathbb{N}$. This implies that $x \in K'(T)$ and, therefore, $F \subseteq K'(T)$.

Proposition 2.1 is proved.

The next theorem gives the connection between algebraic and correlation analytic core when $C(T)$ is closed.

Theorem 2.1. *Let $T \in BCR(X)$. If $C(T)$ is closed, then we have*

$$K'(T) = C(T).$$

Proof. From the definition of $C(T)$, by using the first assertion of Proposition 2.1, we observe that $K'(T) \subseteq C(T)$. Now, since $C(T)$ is closed and $T(C(T)) = C(T)$, then from the second assertion of Proposition 2.1 we get $C(T) \subseteq K'(T)$. Thus, $K'(T) = C(T)$.

3. The local spectrum of a linear relation. In this section, we will define the local spectrum for linear relations and we want to give some of their properties. It's well-known that if $\mu \in \rho(T)$, then the resolvent function $R(\mu, T) := (\mu I - T)^{-1}$ is everywhere defined and single valued. Moreover, for any $x \in X$, the function $f_x \colon \rho(T) \to X$, defined by $f_x(\mu) := (\mu I - T)^{-1}x$, is an analytic function which verify

$$(\mu I - T) f_x(\mu) = x + T(0) \quad \text{for all } \mu \in \rho(T).$$

Definition 3.1. *Let X be a Banach space, $T \in LR(X)$, and $x \in X$. Let $\rho_T(x)$ denote the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood $\mathcal{U}_\lambda$ and an analytic function $f_{\lambda,x} \colon \mathcal{U}_\lambda \to X$ such that the equation*

$$(\mu I - T) f_{\lambda,x}(\mu) = x + T(0)$$

holds for all $\mu \in \mathcal{U}_\lambda$.

$\rho_T(x)$ is called the local resolvent set of T at the point x. The local spectrum of T at the point x is then defined by

$$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x).$$

Evidently, by Definition 3.1, we have $\rho_T(x) := \bigcup_{\lambda \in \rho_T(x)} \mathcal{U}_\lambda$ and, hence, it is an open subset of $\mathbb{C}$. Moreover, for all $x \in X$,

$$\rho(T) \subseteq \rho_T(x) \text{ and } \sigma_T(x) \subseteq \sigma(T).$$

In the following three propositions we gather some elementary properties of $\sigma_T(x)$.

Proposition 3.1. *Let $T \in LR(X)$. Then:*

1) *for all $x \in T(0)$, we have $\sigma_T(x) = \varnothing$;*

2) *$\sigma_T(\alpha x + \beta y) \subseteq \sigma_T(x) \cup \sigma_T(y)$ for all x, $y \in X$ and $\alpha, \beta \in \mathbb{C}$;*

3) *$\sigma_{-T}(x) = \{-\lambda,\ \lambda \in \sigma_T(x)\}$ for all $x \in X$;*

4) *for every $F \subseteq \mathbb{C}$, $\sigma_{\lambda I+T}(x) \subseteq F$ if and only if $\sigma_T(x) \subseteq F - \{\lambda\}$; in particular, $\sigma_{\lambda I-T}(x) \subseteq \subseteq \{0\}$ if and only if $\sigma_T(x) \subseteq \{\lambda\}$.*

Proof. 1. It suffices to prove that for all $x \in T(0)$ we have $\rho_T(x) = \mathbb{C}$. Let $\lambda \in \mathbb{C}$. The null analytic function $f \equiv 0$ defined on $\mathbb{C}$ verify that $(\mu I - T)f(\mu) = x + T(0)$ for all $x \in T(0)$ and for all $\mu \in \mathbb{C}$. Then $\lambda \in \rho_T(x)$.

2. It is equivalent to prove that $\rho_T(x) \cap \rho_T(y) \subseteq \rho_T(\alpha x + \beta y)$ for all $x,\ y \in X$ and for all $\alpha, \beta \in \mathbb{C}$. If $\alpha = \beta = 0$, then there is nothing to prove. If not, let $\delta \in \rho_T(x) \cap \rho_T(y)$. Then there exist two open neighborhoods U_δ, V_δ and two analytic functions $f_{\delta,x} \colon U_\delta \to X,\ g_{\delta,y} \colon V_\delta \to X$ such that

$$(\mu I - T)f_{\delta,x}(\nu) = x + T(0) \quad \forall \mu \in U_\delta,$$

$$(\mu I - T)g_{\delta,y}(\mu) = y + T(0) \quad \forall \mu \in V_\delta.$$

Now, let $W_\delta = U_\delta \cap V_\delta$ and let $h \colon W_\delta \to X$ be the analytic function defined by $h = \alpha f_{\delta,x} + \beta g_{\delta,y}$. Then we have

$$(\mu I - T)h(\mu) = (\alpha x + \beta y) + T(0) \quad \forall \mu \in W_\delta.$$

So, $\delta \in \rho_T(\alpha x + \beta y)$.

3. Let $\lambda \in \rho_{-T}(x)$. Then there exist an open neighborhood U_λ and an analytic function $f \colon U_\lambda \to X$ such that $(\mu I - (-T))f(\mu) = x + T(0)$ for all $\mu \in U_\lambda$. If we take the change of variables $\nu = -\mu$, then we find $(\nu I - T)g(\nu) = x + T(0)\ \forall \nu \in U_{-\lambda}$, where $g(\nu) = -f(-\nu)$ is an analytic function defined on an open neighborhood $U_{-\lambda}$ of $-\lambda$. Therefore, $-\lambda \in \rho_T(x)$.

4. It is similar to prove that $\mathbb{C} \backslash F \subseteq \rho_{\lambda I + T}(x)$ if and only if $\mathbb{C} \backslash (F - \{\lambda\}) \subseteq \rho_T(x)$. For the only if part, let $\lambda_0 \in \mathbb{C} \backslash (F - \{\lambda\})$. So, $\lambda_0 + \lambda \in \mathbb{C} \backslash F \subseteq \rho_{\lambda I + T}(x)$. Then there exist an open neighborhood $U_{\lambda_0 + \lambda}$ and an analytic function $f_{\lambda_0+\lambda,x} \colon U_{\lambda_0+\lambda} \to X$ such that $(\mu I - (\lambda I + T))f_{\lambda_0+\lambda,x}(\mu) = x + T(0)$ for all $\mu \in U_{\lambda_0+\lambda}$. If we take the change of variables $\nu = \mu - \lambda$, then we get

$$(\nu I - T)g(\nu) = x + T(0) \quad \forall \nu \in U_{\lambda_0},$$

where $g(\nu) = f(\nu + \lambda)$ is an analytic function defined on an open neighborhood U_{λ_0} of λ_0. Thus, $\lambda_0 \in \rho_T(x)$.

In particular, let $\lambda_0 \in \mathbb{C} \setminus F$. Then $\lambda_0 - \lambda \in \mathbb{C} \setminus (F - \{\lambda\}) \subseteq \rho_T(x)$. So, there exist an open neighborhood $U_{\lambda_0-\lambda}$ and an analytic function $f_{\lambda_0-\lambda,x} \colon U_{\lambda_0-\lambda} \to X$ such that

$$((\mu + \lambda)I - (\lambda I + T))f_{\lambda_0-\lambda,x}(\mu) = x + T(0) \quad \forall \mu \in U_{\lambda_0-\lambda}.$$

If we take the change of variables $\nu = \mu + \lambda$, we get

$$(\nu I - (\lambda I + T))g(\nu) = x + T(0) \quad \forall \nu \in U_{\lambda_0},$$

where $g(\nu) = f(\nu - \lambda)$ is an analytic function defined on an open neighborhood U_{λ_0} of λ_0. Thus, $\lambda_0 \in \rho_{\lambda I + T}(x)$.

Now, if we take $F = \{0\}$ and we replace T by $-T$, we can easily find the particular result.

Proposition 3.1 is proved.

Proposition 3.2. *Let $T \in BR(X)$, $x \in X$ and $\mathcal{U}$ an open subset of $\mathbb{C}$. If there exists an analytic function $f_x \colon \mathcal{U} \to X$ such that $(\mu I - T)f(\mu) = x + T(0)$ for all $\mu \in \mathcal{U}$, then $\mathcal{U} \subseteq \rho_T(f(\lambda))$ for all $\lambda \in \mathcal{U}$. If moreover T has a continuous selection A such that $T(0) \subset \ker(A)$, then*

$$\sigma_T(x) = \sigma_T(f(\lambda)) \text{ for all } \lambda \in \mathcal{U}.$$

Proof. Let $\lambda \in \mathcal{U}$. Define

$$h(\mu) = \begin{cases} \dfrac{f(\lambda) - f(\mu)}{\mu - \lambda}, & \text{if } \mu \neq \lambda, \\ -f'(\lambda), & \text{if } \mu = \lambda. \end{cases}$$

Then h is an analytic function on $\mathcal{U}$ and we have, for all $\mu \in \mathcal{U}\backslash\{\lambda\}$,

$$(\mu I - T)h(\mu) = \frac{1}{\mu - \lambda}\big[((\mu - \lambda)I + (\lambda I - T))(f(\lambda)) - x + T(0)\big] = f(\lambda) + T(0).$$

Therefore $(\mu I - T)h(\mu) = f(\lambda) + T(0)$ holds for all $\mu \in \mathcal{U}\backslash\{\lambda\}$. On the other hand, we have $(\mu I - T)f(\mu) = x + T(0)$. Then $\mu Q_T(f(\mu)) - Q_T T f(\mu) = Q_T x$. Now, as Q_T is a bounded operator, by derivation on μ we get $Q_T(f(\mu)) + \mu Q_T(f'(\mu)) - Q_T T f(\mu) = 0$. Thus, for $\mu = \lambda$, we have $-(\lambda I - T)f'(\lambda) = f(\lambda) + T(0)$. Hence, the equality $(\mu I - T)h(\mu) = f(\lambda) + T(0)$ holds for all $\mu \in \mathcal{U}$. Therefore, $\mathcal{U} \subseteq \rho_T(f(\lambda))$.

To show the equality $\sigma_T(x) = \sigma_T(f(\lambda))$ we begin by proving the inclusion $\rho_T(x) \subseteq \rho_T(f(\lambda))$. Let $w \in \rho_T(x)$. If $w \in \mathcal{U}$ then $w \in \rho_T(f(\lambda))$ for all $\lambda \in \mathcal{U}$, by the first part of the proof. If $w \in \rho_T(x)\backslash\mathcal{U}$, then there exist an open neighborhood W which not contains λ and an analytic function $g: W \to X$ such that $(\mu I - T)g(\mu) = x + T(0)$ for all $\mu \in W$. Define

$$K(\mu) = \frac{g(\lambda) - g(\mu)}{\mu - \lambda} \quad \text{for all} \quad \mu \in W.$$

K is an analytic function on W, and we have $(\mu I - T)K(\mu) = f(\lambda) + T(0)$ for all $\mu \in W$. Thus, $w \in \rho_T(f(\lambda))$. So, $\rho_T(x) \subseteq \rho_T(f(\lambda))$.

It remains to prove the reverse inclusion. Let $\nu \in \rho_T(f(\lambda))$. Then there exist an open neighborhood $\mathcal{V}$ of ν and an analytic function $h: \mathcal{V} \to X$ such that $(\mu I - T)h(\mu) = f(\lambda) + T(0)$ is satisfied for all $\mu \in \mathcal{V}$. By hypothesis A is a continuous selection of T then the function h_A defined on $\mathcal{V}$ by $h_A(\mu) = (\lambda I - A)h(\mu)$ is analytic on $\mathcal{V}$. Besides, since $T(0) \subset \ker(A)$, then we have, for all $\mu \in \mathcal{V}$,

$$(\mu I - T)h_A(\mu) = (\lambda I - A)(\mu I - T)h(\mu) =$$

$$= (\lambda I - A + T - T)(f(\lambda)) + T(0) - AT(0) =$$

$$= x + T(0).$$

Thus, $\nu \in \rho_T(x)$ and, hence, $\rho_T(f(\lambda)) \subseteq \rho_T(x)$ which ends the proof.

Proposition 3.3. *Let T, $S \in BR(X)$ be such that $ST = TS$, $S(0) \subset \ker(T)$, and S have a linear continuous selection S_1. Then, for all $y \in Sx$, we have*

$$\sigma_T(y) \subset \sigma_T(x).$$

Proof. We shall prove that $\rho_T(x) \subset \rho_T(y)$ for all $y \in Sx$. Let $y \in Sx$ and $\lambda \in \rho_T(x)$. Then there exist an open neighborhood U_λ and an analytic function $f: U_\lambda \to X$ such that $(\nu I - T)f(\nu) = x + T(0)$ for all $\nu \in U_\lambda$. Then we have

$$S(\nu I - T)f(\nu) = S(x + T(0)).$$

By using the fact that $TS = ST$, $S(0) \subset \ker(T)$ and [3] (Lemma 2.4), we get $(\nu I - T)Sf(\nu) = = Sx + ST(0)$. Thus, $(\nu I - T)(S_1 f(\nu) + S(0)) = Sx + ST(0)$. So, $(\nu I - T)S_1 f(\nu) + ST(0) = = y + S(0) + ST(0)$. Hence, $(\nu I - T)S_1 f(\nu) = y + T(0)$. Then $\lambda \in \rho_T(y)$ which ends the proof.

Let $T \in LR(X)$. We say that T verifies the stabilization criteria if $T(0) = T^2(0)$. We denote by $\mathcal{STR}_1(X)$ the set of all linear relations satisfying the stabilization criteria.

Lemma 3.1. *Let $R, S \in BR(X)$. Then the following equivalence holds:*

$$S(0) \subset \ker(SR) \text{ and } R(0) \subset \ker(RS) \text{ if and only if}$$
$$SR \text{ and } RS \text{ are both in } \mathcal{STR}_1(X).$$

Proof. For the only if part, we have by hypothesis $SRS(0) = SR(0)$ and $RSR(0) = RS(0)$. Then $SRSR(0) = SR(0)$. Thus, $SR \in \mathcal{STR}_1(X)$. In a similar way we can prove that $RS \in \in \mathcal{STR}_1(X)$.

For the if part, we have $SRSR(0) = SR(0)$. So, $SRS(0) \subset SR(0)$. Then, for all $y \in S(0)$, we have $SR(y) \subset SR(0)$. So, $SR(y) = SR(0)$. Thus, $SRS(0) = SR(0)$ and we conclude that $S(0) \subset \ker(SR)$. By the same way we can prove that $R(0) \subset \ker(RS)$.

Lemma 3.1 is proved.

In the next theorem we study the relation between the local spectrums of the linear relations SR and RS.

Theorem 3.1. *Let $S, R \in BR(X)$ be such that SR and RS are both in $\mathcal{STR}_1(X)$. If S and R have continuous linear selection S_1 and R_1, respectively, then, for all $x \in X$ and $y \in Sx$, we have*

$$\sigma_{SR}(y) \subseteq \sigma_{RS}(x) \subseteq \sigma_{SR}(y) \cup \{0\}.$$

Proof. Let beginning by showing the first inclusion. It is equivalent to prove that $\rho_{RS}(x) \subseteq \subseteq \rho_{SR}(y)$. Let $\lambda \in \rho_{RS}(x)$. Then there exist an open neighborhood U_λ and an analytic function $f : U_\lambda \to X$ such that $(\nu I - RS)f(\nu) = x + RS(0)\ \forall \nu \in U_\lambda$. Then we have $S(\nu I - RS)f(\nu) = = Sx + SRS(0)$. By using Lemma 3.1 and [3] (Lemma 2.4), we get $(\nu I - SR)Sf(\nu) = Sx + +SRS(0)$. Thus, $(\nu I - SR)(S_1 f(\nu) + S(0)) = Sx + SRS(0)$. Since $(\nu I - SR)S(0) \subseteq (\nu I - SR)(0)$, then we obtain $(\nu I - SR)S_1 f(\nu) = y + SR(0)$ for all $\nu \in U_\lambda$. So we conclude that $\lambda \in \rho_{SR}(y)$ and, therefore, $\rho_{RS}(x) \subseteq \rho_{SR}(y)$.

In order to show the second inclusion, it suffices to prove that $\rho_{SR}(y) \cap \mathbb{C}^* \subset \rho_{RS}(x)$. Let $\lambda \in \rho_{SR}(y) \cap \mathbb{C}^*$. So, there exist an open neighborhood $V_\lambda \subset \mathbb{C}^*$ and an analytic function $f : V_\lambda \to X$ such that $(\mu I - SR)f(\mu) = y + SR(0)\ \forall \mu \in V_\lambda$. Let h be the function defined by

$$h(\mu) = \frac{1}{\mu}(x + R_1 f(\mu)).$$

So, we have

$$(\mu I - RS)h(\mu) = x - \frac{1}{\mu}RS(x) + \frac{1}{\mu}(\mu R_1 - RSR_1)f(\mu) =$$

$$= x - \frac{1}{\mu}RS(x) + \frac{1}{\mu}R[(\mu I - SR_1)f(\mu)] \subset x - \frac{1}{\mu}R(y) + \frac{1}{\mu}R(y) + RS(0) \subset x + RS(0).$$

Then $(\mu I - RS)h(\mu) = x + RS(0)$ for all $\mu \in V_\lambda$. Therefore, $\lambda \in \rho_{RS}(x)$, and we obtain the desired inclusion.

Theorem 3.1 is proved.

4. The local spectral subspace. For every subset F of $\mathbb{C}$ the local spectral subspace of T associated with F is the set

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}.$$

Obviously, if $F_1 \subseteq F_2$ then $X_T(F_1) \subseteq X_T(F_2)$.

We begin with the following proposition, which gives some properties of the local spectral subspace.

Proposition 4.1. *Let $T \in BR(X)$ and F be a subset of $\mathbb{C}$. Then:*

1) *$X_T(F)$ is a subspace of X;*

2) *$X_T(F)$ is a linear hyperinvariant subspace for T, i.e., for every bounded operator S such that $TS = ST$ we have $S(X_T(F)) \subseteq X_T(F)$;*

3) *if moreover $T \in \mathcal{STR}_1(X)$ and have a continuous selection, then*

$$T(X_T(F)) \subseteq X_T(F).$$

Proof. 1. By the first part of Proposition 3.1 we have $T(0) \subset X_T(F)$. Then $X_T(F) \neq \varnothing$. Let $x, y \in X_T(F)$. We have $\sigma_T(x) \subset F$ and $\sigma_T(y) \subset F$. Then, by using the second part of Proposition 3.1, we get, for all $\alpha, \beta \in \mathbb{C}$,

$$\sigma_T(\alpha x + \beta y) \subseteq \sigma_T(x) \cup \sigma_T(y) \subseteq F.$$

Thus, $\alpha x + \beta y \in X_T(F)$ and, therefore, $X_T(F)$ is a subspace of X.

2. Let $S \in B(X)$ be such that $ST = TS$. Let $x \in X_T(F)$ and let $\lambda \notin \sigma_T(x)$. Then there exist an open neighborhood U_λ of λ and an analytic function $f : U_\lambda \to X$ which satisfies $(\mu I - T)f(\mu) = x + T(0)$ for all $\mu \in U_\lambda$. Thus, $S(\mu I - T)f(\mu) = Sx + ST(0)$ for all $\mu \in U_\lambda$. Since $ST = TS$ and, by using [3] (Lemma 2.4), we get, for all $\mu \in U_\lambda$,

$$(\mu I - T)Sf(\mu) = Sx + T(0).$$

Since S is a bounded operator, then Sf is analytic on U_λ. So, we conclude that $\lambda \notin \sigma_T(Sx)$. Thus, $\sigma_T(Sx) \subseteq \sigma_T(x) \subseteq F$ and, therefore, $Sx \in X_T(F)$ which implies that $S(X_T(F)) \subseteq X_T(F)$.

3. Since T admits a continuous linear selection and $T(0) \subseteq \ker(T)$, then by using Proposition 3.3 the result follows immediately.

We prove the following auxiliary assertion.

Lemma 4.1. *Let T be a bounded linear relation in X and (x_n) be a sequence of X such that $x_n \in Tx_{n+1}$ for all $n \in \mathbb{N}$. Let R denote the convergence radius of the entire series $\sum_{n\geq 1} \lambda^{n-1}x_n$. Then, for all scalar λ such that $|\lambda| < R$, we have*

$$T\left(\sum_{n\geq 1} \lambda^{n-1}x_n\right) = \sum_{n\geq 1} \lambda^{n-1}x_{n-1} + T(0).$$

Proof. Let λ such that $|\lambda| < R$. We get

$$T\left(\sum_{n\geq 1} \lambda^{n-1}x_n\right) = \sum_{n=1}^{N} \lambda^{n-1}Tx_n + T\left(\sum_{n\geq N+1} \lambda^{n-1}x_n\right) =$$

$$= \sum_{n\geq 1} \lambda^{n-1}x_{n-1} - \sum_{n\geq N+1} \lambda^{n-1}x_{n-1} + T\left(\sum_{n\geq N+1} \lambda^{n-1}x_n\right).$$

Now, since the operators Q_T and $Q_T T$ are bounded, we have

$$Q_T\left(T\left(\sum_{n\geq 1}\lambda^{n-1}x_n\right)\right) = Q_T\left[\sum_{n\geq 1}\lambda^{n-1}x_{n-1} - \sum_{n\geq N+1}\lambda^{n-1}x_{n-1} + T\left(\sum_{n\geq N+1}\lambda^{n-1}x_n\right)\right] =$$

$$= Q_T\left(\sum_{n\geq 1}\lambda^{n-1}x_{n-1}\right).$$

Then $Q_T\Big(T\Big(\sum_{n\geq 1}\lambda^{n-1}x_n\Big) - \sum_{n\geq 1}\lambda^{n-1}x_{n-1}\Big) = 0$, which implies that

$$T\left(\sum_{n\geq 1}\lambda^{n-1}x_n\right) - \sum_{n\geq 1}\lambda^{n-1}x_{n-1} \subseteq T(0).$$

Thus, $T\Big(\sum_{n\geq 1}\lambda^{n-1}x_n\Big) = \sum_{n\geq 1}\lambda^{n-1}x_{n-1} + T(0)$.

Lemma 4.1 is proved.

We now establish the relationship between the correlation analytic core and the local spectral subspace of a relation T.

Theorem 4.1. *Let X be a Banach space and $T \in BCR(X)$. We have*

$$K'(T) = X_T(\mathbb{C}\setminus\{0\}) = \{x \in X \textit{ such that } 0 \in \rho_T(x)\}.$$

Proof. Let prove the first inclusion $K'(T) \subseteq X_T(\mathbb{C}\setminus\{0\})$. Let $x \in K'(T)$. So, there exist a sequence $(x_n) \subset X$ and a positive scalar a such that $x = x_0$, $x_n \in Tx_{n+1}$ and

$$\operatorname{dis}(x_n, T(0)\cap\ker(T)) \leq a^n \operatorname{dis}(x, T(0)\cap\ker(T)) \qquad \forall n \in \mathbb{N}.$$

Let $b > a$. From the last inequality, we deduce that for all $n \geq 1$ there exists $\alpha_n \in T(0)\cap\ker(T)$ such that $\|x_n - \alpha_n\| \leq b^n\|x\|$. Let (y_n) be the sequence defined by $y_0 = x$, and, for all $n \geq 1$, $y_n = x_n - \alpha_n$. Then we have $\|y_n\| \leq b^n\|x\|$. Let f be the analytic function $f\colon B\Big(0, \dfrac{1}{b}\Big) \to X$ defined by $f(\lambda) = -\sum_{n\geq 1}\lambda^{n-1}y_n$. We can easily verify that $y_n \in Ty_{n+1}$ for all $n \in \mathbb{N}$. Moreover, by Lemma 4.1, for all $\lambda \in B\Big(0, \dfrac{1}{b}\Big)$, we have

$$(\lambda I - T)f(\lambda) = \sum_{n\geq 1}\lambda^{n-1}y_{n-1} + T(0) - \lambda\sum_{n\geq 1}\lambda^{n-1}y_n =$$

$$= y_0 + T(0) = x + T(0).$$

Thus, $0 \in \rho_T(x)$ and then $x \in X_T(\mathbb{C}\setminus\{0\})$, which provide the desired inclusion.

Moving to the reverse inclusion. Let $x \in X_T(\mathbb{C}\setminus\{0\})$. So, $0 \in \rho_T(x)$ which implies that there exist an open disc $\mathbb{D}(0,\varepsilon)$ and an analytic function $f\colon \mathbb{D}(0,\varepsilon) \to X$ such that the equation

$(\mu I - T)f(\mu) = x + T(0)$ holds for all $\mu \in \mathbb{D}(0,\varepsilon)$. Since f is an analytic function, then there exists a sequence $(u_n)_{n\geq 1} \subset X$ such that

$$f(\lambda) = -\sum_{n\geq 1} \lambda^{n-1} u_n \qquad \text{for all} \quad \lambda \in \mathbb{D}(0,\varepsilon).$$

Clearly, $f(0) = -u_1$. Then $T(u_1) = x + T(0)$ and $x \in Tu_1$. Take $u_0 = x$ and let prove that $u_n \in Tu_{n+1}$ for all $n \in \mathbb{N}$. The proof is given by induction. For $n = 0$, we have $u_0 \in Tu_1$. Suppose that this property is true until the order n, and let prove it for the order $n+1$. For all $\lambda \in \mathbb{D}(0,\varepsilon)$, we get

$$(\lambda I - T)f(\lambda) = -\sum_{k\geq 1} \lambda^k u_k + T\left(\sum_{k\geq 1} \lambda^{k-1} u_k\right) =$$

$$= -\sum_{k=1}^{n+1} \lambda^k u_k - \sum_{k\geq n+2} \lambda^k u_k + T\left(\sum_{k=1}^{n+2} \lambda^{k-1} u_k\right) + T\left(\sum_{k\geq n+3} \lambda^{k-1} u_k\right).$$

So,

$$Tu_1 + \lambda^{n+1}(Tu_{n+2} - u_{n+1}) - \sum_{k\geq n+2} \lambda^k u_k + T\left(\sum_{k\geq n+3} \lambda^{k-1} u_k\right) = Tu_1.$$

Since

$$\lambda^{n+1} T\left(\sum_{k\geq n+3} \lambda^{k-(n+2)} u_k\right) \subset T\left(\lambda^{n+1} \sum_{k\geq n+3} \lambda^{k-(n+2)} u_k\right),$$

then

$$(Tu_{n+2} - u_{n+1}) - \sum_{k\geq n+2} \lambda^{k-(n+1)} u_k + T\left(\sum_{k\geq n+3} \lambda^{k-(n+2)} u_k\right) \subset T(0).$$

If we take $\lambda = 0$, then we find that $Tu_{n+2} - u_{n+1} \subset T(0)$ and, hence, $u_{n+1} \in Tu_{n+2}$. Consequently, $u_n \in Tu_{n+1}$ for all $n \in \mathbb{N}$.

It remains to prove that there exists a positive scalar which verify the second condition of $K'(T)$. If $x \in T(0) \cap \ker(T)$ then there is nothing to prove. If not, since the series $\left(-\sum_{n\geq 1} \lambda^{n-1} u_n\right)$ converges then $|\lambda|^{n-1}\|u_n\| \to 0$ as $n \to \infty$ for all $|\lambda| < \varepsilon$. In particular, $\dfrac{1}{\mu^{n-1}}\|u_n\| \to 0$ as $n \to \infty$ for $\mu > \dfrac{1}{\varepsilon}$. Take $\mu_0 > \dfrac{1}{\varepsilon}$. So, there exists $c > 0$ such that $\|u_n\| \leq c\mu_0^{n-1}$. Hence, we obtain

$$\|u_n\| \leq \left(\mu_0 + \frac{c}{\operatorname{dis}(x, T(0) \cap \ker(T))}\right)^n \operatorname{dis}(x, T(0) \cap \ker(T)).$$

Thus, we have

$$\operatorname{dis}(u_n, T(0) \cap \ker(T)) \leq \|u_n\| \leq b^n \operatorname{dis}(x, T(0) \cap \ker(T))$$

with $b = \mu_0 + \dfrac{c}{\operatorname{dis}(x, T(0) \cap \ker(T))}$. So, $x \in K'(T)$ and, therefore, $X_T(\mathbb{C} \setminus \{0\}) \subseteq K'(T)$.

Theorem 4.1 is proved.

5. The glocal spectral subspace. Let $F \subset \mathbb{C}$ be a closed subset and let $T \in LR(X)$. We define the glocal spectral subspace $\mathcal{X}_T(F)$ as the set of all $x \in X$ such that there exists an analytic function $f : \mathbb{C} \setminus F \to X$ checking

$$(\lambda I - T)f(\lambda) = x + T(0) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus F.$$

Some basic properties of the glocal spectral subspace are gathered in the following proposition.

Proposition 5.1. *Let $T \in BR(X)$ and $F \subset \mathbb{C}$ be a closed subset. Then:*

1) *$\mathcal{X}_T(F)$ is a subspace of X and $\mathcal{X}_T(F) \subset X_T(F)$;*

2) *$\mathcal{X}_T(F)$ is a linear hyperinvariant subspace for T, i.e., for every bounded operator S such that $TS = ST$ we have $S(\mathcal{X}_T(F)) \subseteq \mathcal{X}_T(F)$;*

3) *if moreover $T \in \mathcal{STR}_1(X)$ and have a continuous selection, then*

$$T(\mathcal{X}_T(F)) \subseteq \mathcal{X}_T(F).$$

Proof. Proceeding as in the proof of Proposition 4.1 we obtain the desired results.

In the sequel we need the two following elementary lemmas.

Lemma 5.1. *Let T belongs to $BCR(X)$ and $(x_n) \subset X$ be such that $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}^*$. Let R be the convergence radius of the power series $\sum_{n\geq 1} \lambda^n x_{n-1}$. Then, for all scalar λ such that $|\lambda| > \dfrac{1}{R}$, we have*

$$T\left(\sum_{n\geq 1} \lambda^{-n} x_{n-1}\right) = \sum_{n\geq 1} \lambda^{-n} x_n + T(0).$$

Proof. For a scalar λ such that $|\lambda| > \dfrac{1}{R}$, we obtain

$$T\left(\sum_{n\geq 1} \lambda^{-n} x_{n-1}\right) = \sum_{n=1}^{N} \lambda^{-n} T x_{n-1} + T\left(\sum_{n\geq N+1} \lambda^{-n} x_{n-1}\right) =$$

$$= \sum_{n\geq 1} \lambda^{-n} x_n - \sum_{n\geq N+1} \lambda^{-n} x_n + T\left(\sum_{n\geq N+1} \lambda^{-n} x_{n-1}\right).$$

Now, since the operators Q_T and $Q_T T$ are bounded, then we can easily seen that

$$Q_T\left(T\left(\sum_{n\geq 1} \lambda^{-n} x_{n-1}\right)\right) = Q_T\left(\sum_{n\geq 1} \lambda^{-n} x_n\right).$$

Then

$$T\left(\sum_{n\geq 1} \lambda^{-n} x_{n-1}\right) - \sum_{n\geq 1} \lambda^{-n} x_n \subseteq T(0).$$

Thus,

$$T\left(\sum_{n\geq 1} \lambda^{-n} x_{n-1}\right) = \sum_{n\geq 1} \lambda^{-n} x_n + T(0).$$

Lemma 5.1 is proved.

Lemma 5.2. *Let T belongs to $BCR(X)$ and $(x_n) \subset X$ be such that $\limsup_{n\to\infty} \|x_n\|^{\frac{1}{n}} \le \varepsilon$. Then, for all $n \in \mathbb{N}$, there exists $\alpha_{n+1} \in Tx_n$ such that for all $|\lambda| < \dfrac{1}{\varepsilon}$ we have*

$$T\left(\sum_{n\ge 0} \lambda^n x_n\right) = \sum_{n\ge 0} \lambda^n \alpha_{n+1} + T(0).$$

Proof. Let $\alpha'_{n+1} \in Tx_n$. Then we have $\operatorname{dis}(\alpha'_{n+1}, T(0)) = \|Tx_n\| \le \|T\|\|x_n\|$. Hence, for a fixed $\gamma > 0$, there exists $\beta_{n+1} \in T(0)$ such that

$$\|\alpha'_{n+1} - \beta_{n+1}\| \le (\|T\| + \gamma)\|x_n\|.$$

Take $\alpha_{n+1} = \alpha'_{n+1} - \beta_{n+1}$. Then $\alpha_{n+1} \in Tx_n$ and the series $\sum_{n\ge 0} \lambda^n \alpha_{n+1}$ is absolutely convergent for all λ such that $|\lambda| < \dfrac{1}{\varepsilon}$. Thus, we obtain

$$Q_T T\left(\sum_{n\ge 0} \lambda^n x_n\right) = Q_T\left(\sum_{n\ge 0} \lambda^n \alpha_{n+1}\right).$$

Hence,

$$T\left(\sum_{n\ge 0} \lambda^n x_n\right) = \sum_{n\ge 0} \lambda^n \alpha_{n+1} + T(0).$$

Lemma 5.2 is proved.

Theorem 5.1. *Let $T \in BCR(X)$. Then*

$$\mathcal{X}_T(\mathbb{D}(0,\varepsilon)) \supset H_\varepsilon(T) + T(0),$$

where

$$H_\varepsilon(T) := \left\{ x \in X : \exists (x_n) \subset X \text{ such that } x = x_0,\ x_{n+1} \in Tx_n \text{ and } \limsup_{n\to\infty} \|x_n\|^{\frac{1}{n}} \le \varepsilon \right\}.$$

If moreover $\sigma(T)$ is bounded, then the equality holds.

Proof. Let $x \in H_\varepsilon(T)$. Then there exists $(x_n) \subset X$ such that $x = x_0$, $x_{n+1} \in Tx_n$, and $\limsup_{n\to\infty} \|x_n\|^{1/n} \le \varepsilon$. Thus, the series defined by $f(\lambda) := \sum_{n\ge 1} \lambda^{-n} x_{n-1}$ converges uniformly on $\mathbb{C} \setminus \mathbb{D}(0,\varepsilon)$. So, f is analytic on $\mathbb{C} \setminus \mathbb{D}(0,\varepsilon)$. Besides, using Lemma 5.1, we get, for all λ such that $|\lambda| > \varepsilon$,

$$(\lambda I - T)f(\lambda) = \sum_{n\ge 0} \lambda^{-n} x_n - \sum_{n\ge 1} \lambda^{-n} x_n - T(0) = x + T(0).$$

Then $x \in \mathcal{X}_T(\mathbb{D}(0,\varepsilon))$. On the other hand, since $T(0) \subset \mathcal{X}_T(\mathbb{D}(0,\varepsilon))$ and $\mathcal{X}_T(\mathbb{D}(0,\varepsilon))$ is a subspace of X, then the inclusion $H_\varepsilon(T) + T(0) \subseteq \mathcal{X}_T(\mathbb{D}(0,\varepsilon))$ is satisfied.

Conversely, let $x \in \mathcal{X}_T(\mathbb{D}(0,\varepsilon))$. There exists an analytic function $f : \mathbb{C}\setminus\mathbb{D}(0,\varepsilon) \to X$ such that $(\lambda I - T)f(\lambda) = x + T(0)$ holds for every $\lambda \in \mathbb{C}\setminus\mathbb{D}(0,\varepsilon)$. By assumption, we have $\sigma(T)$ is bounded, then the set $V := \big(\mathbb{C} \setminus \mathbb{D}(0,\varepsilon)\big) \cap \rho(T)$ is not empty and open and the function f is analytic on V. Besides, for all $\lambda \in V$, we have $f(\lambda) = (\lambda I - T)^{-1}x$. Indeed, we get $(\lambda I - T)f(\lambda) = x + T(0)$ so, $f(\lambda) + \ker(\lambda I - T) = (\lambda I - T)^{-1}x + \ker(\lambda I - T)$. Then $f(\lambda) = (\lambda I - T)^{-1}x$, and we get the result. Now, let $\lambda \in V$. By using [5] (Proposition VI.3.2), we have $\lim_{|\lambda| \mapsto \infty} f(\lambda) = 0$. Let consider the analytic function g defined by

$$g(\mu) := \begin{cases} f\left(\dfrac{1}{\mu}\right), & \text{if} \quad 0 \neq \mu \in \mathbb{D}(0, 1/\varepsilon), \\ 0, & \text{if} \quad \mu = 0. \end{cases}$$

Since g is analytic on $\mathbb{D}(0,1/\varepsilon)$ and $g(0) = 0$, then there exists a sequence $(x_n) \subset X$ such that $x_0 = 0$ and $g(\mu) = \sum_{n\geq 0} \mu^n x_n$ holds even for all $\mathbb{D}(0,1/\varepsilon)$. This shows that the radius of convergence of the power series representing $g(\mu)$ is greater then $1/\varepsilon$. Hence, $\limsup\limits_{n\to\infty} \|x_n\|^{1/n} \leq \varepsilon$. Besides, we obtain $f(\lambda) = g\Big(\dfrac{1}{\lambda}\Big) = \sum_{n\geq 0} \lambda^{-n} x_n$. From Lemma 5.2, for all $n \in \mathbb{N}$, there exists $\alpha_{n+1} \in Tx_n$ such that for all λ with $|\lambda| > \varepsilon$ we get

$$(\lambda I - T)f(\lambda) = \sum_{n\geq 0} \lambda^{-n+1} x_n - \sum_{n\geq 0} \lambda^{-n}\alpha_{n+1} + T(0) =$$

$$= \sum_{n\geq 0} \lambda^{-n}(x_{n+1} - \alpha_{n+1}) + T(0).$$

Thus, we have $Q_T(x) = \sum_{n\geq 0} \lambda^{-n} Q_T(x_{n+1} - \alpha_{n+1})$. Hence,

$$Q_T(x_1 - \alpha_1) = Q_T(x),$$

$$Q_T(x_{n+1} - \alpha_{n+1}) = 0, \quad n \geq 1,$$

and, so,

$$x = x_1 + \alpha \quad \text{with} \quad \alpha \in T(0),$$

$$x_{n+1} \in Tx_n, \quad n \geq 1.$$

Now, let prove that $x_1 \in H_\varepsilon(T)$. Let consider the sequence (y_n) defined by $y_n := x_{n+1}$. We have $x_1 = y_0$ and $y_{n+1} = x_{n+2} \in Tx_{n+1} = Ty_n$. Besides, we have

$$\limsup_{n\to\infty} \|y_n\|^{1/n} \leq \limsup_{n\to\infty} \|x_{n+1}\|^{1/n} \leq \varepsilon.$$

So, $x_1 \in H_\varepsilon(T)$ and, therefore, $\mathcal{X}_T(\mathbb{D}(0,\varepsilon)) \subseteq H_\varepsilon(T) + T(0)$.

Theorem 5.1 is proved.

As a consequence of Theorem 5.1, we prove that the quasinilpotent part of a linear relation may be characterized in terms of the glocal spectral subspace as follows.

***Corollary* 5.1.** *For every $T \in BCR(X)$, we have*

$$\mathcal{X}_T(\{0\}) \supset H_0(T) + T(0).$$

If moreover $\sigma(T)$ is bounded, then the equality holds.

6. Some properties of the surjective spectrum of a linear relation. Let T be a linear relation in $BCR(X)$. Recall that the surjective spectrum of T is defined by

$$\sigma_{su}(T) := \big\{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\big\}.$$

The next lemma is a consequence of [3] (Corollary 1.4.3).

Lemma 6.1. *Let $T \in BCR(X)$. Then $\sigma_{su}(T)$ is a closed subset in $\mathbb{C}$.*

The surjective spectrum may be characterized by means of the local spectrum as follows.

Theorem 6.1. *Let $T \in BCR(X)$. Then we have:*

1) $\sigma_{su}(T) = \bigcup_{x\in X} \sigma_T(x)$;

2) *the set $\{x \in X$ such that $\sigma_T(x) = \sigma_{su}(T)\}$ is of the second category in X.*

Proof. 1. First observe that the desired result is equivalent to $\rho_{su}(T) = \bigcap_{x\in X} \rho_T(x)$. For the opposite inclusion, let $\lambda \in \bigcap_{x\in X} \rho_T(x)$. Then $\lambda \in \rho_T(x)$ for all $x \in X$. Let $x \in X$. There exists an open neighborhood U_λ of λ and an analytic function $f_x : U_\lambda \to X$ such that

$$(\mu I - T)f_x(\mu) = x + T(0) \quad \forall \mu \in U_\lambda.$$

This implies that $x \in (\lambda I - T)f_x(\lambda) \subset R(\lambda I - T)$ for all $x \in X$. Then $(\lambda I - T)$ is surjective. Thus, $\lambda \in \rho_{su}(T)$.

For the direct inclusion, let $\lambda \in \rho_{su}(T)$. Then $(\lambda I - T)$ is surjective. So, by Proposition 2.1, we have $K'(\lambda I - T) = X$ and, from Theorem 4.1, we obtain that $0 \in \rho_{\lambda I - T}(x)$. So, there exist an open neighborhood U_0 of 0 and an analytic function $f : U_0 \to X$ such that

$$((\mu - \lambda)I + T)f(\mu) = x + T(0) \quad \forall \mu \in U_0.$$

Therefore,

$$(\delta I - T)g(\delta) = x + T(0) \quad \forall \delta \in U_\lambda,$$

where U_λ be the open neighborhood of λ given by $U_\lambda = \lambda - U_0$ and g be the analytic function defined on U_λ by $g(\delta) = -f(\lambda - \delta)$. Therefore, $\lambda \in \rho_T(x)$ for all $x \in X$, which ends the proof.

2. Let E be a dense countable subset of $\sigma_{su}(T)$. Then, for each $\lambda \in E$, we have $(\lambda I - T)X \neq X$. We claim that $(\lambda I - T)X$ is of the first category in X. Indeed, let us suppose that $(\lambda I - T)X$ is of the second category in X. We show that $(\lambda I - T)X = X$, which is absurd. To do this, it suffices to prove that $(\lambda I - T)$ is an open mapping. Let U be the open ball in X with center 0 and radius $r > 0$. We prove that $(\lambda I - T)(U)$ contains a neighborhood of 0 in X. Define

$$U_n := \{x \in X : \|x\| < 2^{-n} r\}, \quad n = 0, 1, 2, \ldots .$$

We note that $U_1 \supset U_2 - U_2$. So, $(\lambda I - T)U_1 \supset (\lambda I - T)U_2 - (\lambda I - T)U_2$. Hence,

$$\overline{(\lambda I - T)U_1} \supset \overline{(\lambda I - T)U_2 - (\lambda I - T)U_2} \supset \overline{(\lambda I - T)U_2} - \overline{(\lambda I - T)U_2}. \tag{6.1}$$

On the other hand, we have

$$(\lambda I - T)X = \bigcup_{k=1}^{\infty} k(\lambda I - T)(U_2).$$

As, the union of countably many first category sets is first category and as $(\lambda I-T)X$ is of the second category, then at least one $k(\lambda I-T)(U_2)$ is of the second category of X and, so, $(\lambda I-T)(U_2)$ is of the second category. Hence, $\operatorname{int}(\overline{(\lambda I-T)(U_2)})\neq\varnothing$. Thus, by (6.1), there is some neighborhood W of 0 in X such that

$$W\subset\overline{(\lambda I-T)(U_1)}.$$

We shall now show that $\overline{(\lambda I-T)(U_1)}\subset(\lambda I-T)(U)$. Fix $y_1\in\overline{(\lambda I-T)(U_1)}$. As what just proved for U_1 holds by the same way for U_2, so, $\overline{(\lambda I-T)(U_2)}$ contains a neighborhood of 0. Thus,

$$\big(y_1-\overline{(\lambda I-T)(U_2)}\big)\cap(\lambda I-T)(U_1)\neq\varnothing.$$

Hence, there exists $\alpha_1\in(\lambda I-T)(x_1)$ with $x_1\in U_1$ such that $y_2=y_1-\alpha_1\in\overline{(\lambda I-T)(U_2)}$. Proceeding by induction we can construct the sequences $(\alpha_n)_{n\geq1}$, $(x_n)_{n\geq1}$ and $(y_n)_{n\geq1}$ such that for all $n\geq1$, $x_n\in U_n$,

$$\alpha_n\in(\lambda I-T)(x_n)\quad\text{and}\quad y_{n+1}=y_n-\alpha_n\in\overline{(\lambda I-T)(U_{n+1})}.$$

Since, $\|x_n\|<\dfrac{r}{2^n}$, then $\sum_{n\geq1}x_n$ converges. Let $x=\sum_{n=1}^{\infty}x_n$. Then $\|x\|<r$ and, so, $x\in U$. By the construction of the sequences $(y_n)_{n\geq1}$ and $(\alpha_n)_{n\geq1}$, we have

$$\sum_{n=1}^{m}Q_T\alpha_n=\sum_{n=1}^{m}Q_T(y_n-y_{n+1})=Q_Ty_1-Q_Ty_{m+1}.$$

So, $\sum_{n=1}^{m}Q_T(\lambda I-T)x_n=Q_Ty_1-Q_Ty_{m+1}$. Thus,

$$Q_T(\lambda I-T)\left(\sum_{n=1}^{m}x_n\right)=Q_Ty_1-Q_Ty_{m+1}.$$

Now, we claim that $Q_Ty_m\to0$ as $m\to\infty$. Indeed, let $\alpha\in(\lambda I-T)(U_1)$. Then there exists $\beta\in U_1$ such that $\alpha\in(\lambda I-T)(\beta)$. Hence, $\operatorname{dis}(\alpha,T(0))=\|(\lambda I-T)(\beta)\|\leq\|(\lambda I-T)\|\|\beta\|$. Thus, we obtain

$$(\lambda I-T)(U_1)\subset\left\{\alpha\in X:\operatorname{dis}(\alpha,T(0))\leq\|(\lambda I-T)\|\frac{r}{2^n}\right\}.$$

But, the map $\alpha\mapsto\operatorname{dis}(\alpha,T(0))$ is continuous on X. Then the set

$$\left\{\alpha\in X:\operatorname{dis}(\alpha,T(0))\leq\|(\lambda I-T)\|\frac{r}{2^n}\right\}$$

is closed and we get

$$y_n\in\overline{(\lambda I-T)(U_n)}\subset\left\{\alpha\in X:\operatorname{dis}(\alpha,T(0))\leq\|(\lambda I-T)\|\frac{r}{2^n}\right\}.$$

So, $\operatorname{dis}(y_n,T(0))\leq\|(\lambda I-T)\|\dfrac{r}{2^n}$. Therefore $\|Q_Ty_n\|\to0$ as $n\to\infty$. Then we conclude that $Q_T(\lambda I-T)(x)=Q_T(y_1)$ and, hence, $y_1\in(\lambda I-T)(x)$, which confirm our claim.

Now, since, for all $\lambda\in E$, we have $(\lambda I-T)X$ is of the first category, then $F=\bigcup_{\lambda\in E}(\lambda I-T)X$ is also of the first category. Thus, $X\backslash F$ is of the second category. We claim that, for all $x\in X\backslash F$,

$\sigma_{su}(T) \subset \sigma_T(x)$. Indeed, let $x \in X \backslash F$ then $E \subset \sigma_T(x)$. So,

$$\sigma_{su}(T) = \overline{E} \subset \overline{\sigma_T(x)} = \sigma_T(x).$$

Hence, $\sigma_{su}(T) = \sigma_T(x)$. Therefore, $\{x \in X$ such that $\sigma_T(x) = \sigma_{su}(T)\}$ is of the second category in X.

Theorem 6.1 is proved.

We can now state the main result of this section, which is a sufficient condition to ensure that the surjective spectrum $\sigma_{su}(T)$ is not cluster at a point λ.

Theorem 6.2. *Let $T \in BCR(X)$. If $X = H_0(\lambda I - T) + K'(\lambda I - T)$, then $\sigma_{su}(T)$ does not cluster at λ.*

Proof. Without loss of generality we can suppose that $\lambda = 0$. Suppose that $0 \in \sigma_{su}(T)$ and $X = H_0(T) + K'(T)$. Then every $x \in X$ may be written as $x = x_1 + x_2$, where $x_1 \in H_0(T)$ and $x_2 \in K'(T)$. From Corollary 5.1 we have $\sigma_T(x_1) \subseteq \{0\}$. Therefore,

$$\sigma_T(x) \subseteq \sigma_T(x_1) \cup \sigma_T(x_2) \subseteq \{0\} \cup \sigma_T(x_2).$$

Now, by Theorem 4.1 and since $\sigma_T(x_2)$ is closed, we conclude that 0 is isolated in $\sigma_T(x)$ for any $x \in X$. By using Theorem 6.1, we conclude that there exists $x_0 \in X$ such that $\sigma_T(x_0) = \sigma_{su}(T)$. Hence, 0 is isolated in $\sigma_{su}(T)$.

Theorem 6.2 is proved.

We close this section by stating the next corollary which is related to Theorems 6.1 and 6.2.

***Corollary* 6.1.** *Let $T \in BCR(X)$. Then, if $X = H_0(\lambda I - T) + K'(\lambda I - T)$, $\sigma_T(x)$ does not cluster at λ for every $x \in X$.*

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