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Expansion in eigenvectors of multiparameter spectral problems

Розклад за власними векторами багатопараметричних спектральних задач

We prove the expansion theorems for abstract multiparameter spectral problems. These theorems contain expansion results for general operators with continuous spectrum and partial differential elliptic operators.

Встановлені теореми про розклад за власними векторами абстрактної багатопараметричної спектральної задачі. Ці теореми містять результати для загальних операторів з неперервним спектром і еліптичних диференціальних операторів.

1. Introduction. The study of the completeness of eigenfunctions of multiparameter spectral problems started with the works of A. C. Dixon and D. Hilbert at beginning of this century. The motivation of these investigations is connected with the solutions of boundary value problems for partial differential equations by the mean of separation of variables. Systematic investigations of multiparameter problems began 30 years ago and were inspired by F. V. Atkinson [1]. Note here the works [2—5] in which under different assumptions the theorems about such expansions were established for ordinary differential operators with both discrete and continuous spectrum. Simultaneously the abstract multiparameter theory in Hilbert space was developed, see [6—11]. The book *[10] contains* the contemporary state of the question and also the last bibliography.

In the present paper we would like to show that for the investigation of such problems the approach of generalized eigenfunctions [12—14] is very convenient and fruitful. It gives highly general results which comprise the theorems for ordinary differential operators and, on the other hand, gives the theorems about such expansions for general partial differential elliptic operators and abstract operators with continuous spectrum (last classes of operators in this direction were not investigated earlier).

The plan of this paper is the following. In sec. 2 we give some little survey of the results concerning the abstract multiparameter theory, which are necessary for us. In sec. 3, 4 the abstract theorems about expansion in generalized eigenvectors of multiparameter problem for selfadjoint operators in Hilbert space are proved. Sec. 5 is devoted to the investigation of the multiparameter spectral problems for differential operators. We give in this paragraph the particular attention to consideration of elliptic partial differential operators. In sec. 6 we develop for differential operators the approach of sec. 4, connected with Carleman property for corresponding operators. The fundamental technical means of sec. 5, 6 are (as for classical spectral theory) the theorems of rise of smoothness of solutions for elliptic (and ordinary) differential equations [12, 14, 15]. In sec. 5, 6 we formulate the results only for elliptic differential operators. For ordinary differential operators such theorems are only outlined. These theorems contain results of type [3, 4], but with some more hard restrictions on smoothness of coefficients of ordinary differential operators. Note that the notion of generalized eigenvectors was contained in [16], also in this work the problems of constructions of investigated eigenfunction expansions for abstract and partial differential operators are set.

The results of this paper were briefly announced in [17, 18] and were included into the report of authors on International Conference on Differential Equations (Moscow, May 28 — June 1, 1991). The most of these results are contained in [19].

2. Abstract multiparameter theory. In this section a short account of main results of multiparameter spectral theory in Hilbert space is given.

Let $H_1, \dots, H_n$ be separable Hilbert spaces; A_j is a selfadjoint and B_{jk} is a bounded selfadjoint operator acting in the space H_j, $j, k = 1, \dots, n$. Consider in the tensor product $H = \bigotimes_{j=1}^{n} H_j$ the selfadjoint commuting operators

$$\hat{A}_j = 1 \otimes \dots \otimes 1 \otimes A_j \otimes 1 \otimes \dots \otimes 1$$

(A_j stands on j-th place) and analogously defined selfadjoint bounded operators $\hat{B}_{jk}$. Note that the operators $\hat{A}_j$, $j = 1, \dots, n$, are essential selfadjoint on dense in H domain $\mathcal{D} = \bigcap_{j=1}^{n} D(\hat{A}_j)$. Here and below $\mathcal{D}(A)$ is the domain of operator A. We shall say that $0 \neq \varphi = \varphi(\lambda) \in \mathcal{D}$ is the eigenvector corresponding to (mulitparameter) eigenvalue $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ if

$$\hat{A}_j \varphi = \sum_{k=1}^{n} \lambda_k \hat{B}_{jk} \varphi, \qquad j = 1, \dots, n. \tag{1}$$

We shall suppose that the bounded operator $\Delta = \det(\hat{B}_{jk})_{j,k=1}^{n}$ is positive and has a bounded inverse in H (the usual definition of determinant is correct since the operators $\hat{B}_{jk}$ with-different j commute). Introduce by means of operator Δ the new inner product in H:

$$\langle f, g\rangle = (\Delta f, g)_H, \quad f, g \in H.$$

The norms, defined by $\langle \cdot,\cdot \rangle$ and $(\cdot,\cdot)_H$, are equivalent, therefore H with respect to $\langle \cdot, \cdot \rangle$ is also a Hilbert space. The main result of abstract multiparameter spectral theory in Hilbert space in the case of discrete spectrum of operators A_j can be, formulated in the following manner.

Theorem 1 [1, 7, 8, 10]. *Let* $\forall j = 1, \ldots, n$ *resolvents of the operator* A_j *are compact. Then in H orthonormal with respect to* $\langle \cdot, \cdot \rangle$ *basis exists which consists of eigenvectors of problem* (1).

Explain shortly the idea of the proof of Theorem 1. Introduce the set of operators $X_1, \ldots, X_n$ acting in H as formal solution of the system $\sum_{k=1}^{n} \hat{B}_{jk} X_k = \hat{A}_j$, $j = 1, \ldots, n$. More exactly, we shall define X_k by means of the Kramer rule:

$$X_k = \Delta^{-1} \sum_{j=1}^{n} \Delta_{jk} \hat{A}_j, \qquad k = 1, \ldots, n. \tag{2}$$

Here Δ_{jk} is the algebraic completement of operators $\hat{B}_{jk}$ in the matrix $(\hat{B}_{il})_{i,l=1}^{n}$. It is easy to understand that Δ_{jk} is a bounded selfadjoint operator in H. The formula (2) defines the operator X_k on $\mathcal{D} = \bigcap_{j=1}^{n} \mathcal{D}(\hat{A}_j)$ correctly. From commutativity of Δ_{jk} and $\hat{A}_j$ (they act with respect to different variables) it is easy to get the Hermitness of operators X_k on $\mathcal{D}$ concerning the inner product $\langle \cdot, \cdot \rangle$. H. Volkmer has shown that the operators X_k, in reality, are essential selfadjoint on $\mathcal{D}$ with respect to $\langle \cdot, \cdot \rangle$. Below, for convenience of writing, we shall remain the notation X_k for closure of operators X_k, so X_k is now selfadjoint ($k = 1, \ldots, n$). Cite the complete result of work [9].

Theorem 2. *The operators* $X_1, \ldots, X_n$ *commute and are selfadjoint with respect to* $\langle \cdot, \cdot \rangle$; $\bigcap_{k=1}^{n} \mathcal{D}(X_k) = \mathcal{D}$ and $\forall f \in \mathcal{D}$

$$\sum_{k=1}^{n} \hat{B}_{jk} X_k f = \hat{A}_j f, \qquad j = 1, \ldots, n. \tag{3}$$

Note that the result of Theorem 2 is not evident even for the finite-dimensional case (see [1]). Its proof in the general situation has demanded the efforts of series of mathematicians (see works [1, 6, 8] for the case of bounded operator and [7, 9] for unbounded ones.)

Explain, in what manner the Theorem 1 can be deduced from Theorem 2. From the compactness of resolvents of operators A_k $k = 1, \ldots, n$ it is possible to show that the joint spectrum of the family $(X_k)_{k=1}^{n}$ of selfadjoint and commuting operators X_k is discrete (from compactness of operators A_k such conslusion, in general, is wrong [6]). Therefore in the space H there exists the orthonormal concerning $\langle \cdot, \cdot \rangle$ basis which consits of joint eigenvectors of family $(X_k)_{k=1}^{n}$. There remain only to note that the application of (3) to joint eigenvector $f = \varphi$ of this family gives (1).

In the following section we shall show in what manner it is possible by means of Theorem 2 to get the expansion in generalized eigenvectors of problem (1) in the general case, when A_k are arbitrary selfadjoint operators with, in general, nondiscrete spectrum.

3. Expansion in generalized eigenvectors of multiparameter spectral problems. Consider some rigging [12—14] of the space H by positive and negative Hilbert spaces H_+, H_-:

$$H_- \supseteq H \supseteq H_+ \supseteq D, \tag{4}$$

here D is some linear topological space embedded densely and topologically into H_+. Thus, $\forall \alpha \in H_-$, $u \in H_+$ the duality $(\alpha, u)_H$ exists.

Suppose that $D \subseteq D$ and for corresponding restrictions

$$\hat{A}_j \in L(D, H_+), \quad \hat{B}_{jk} \in L(D, D), \qquad j, k = 1, .,. , n,$$
$$\Delta, \Delta^{-1} \in L(H_+, H_+) \tag{5}$$

($L(F, G)$ denotes the class of linear continuous operators acting from F into G). We shall say that $0 \neq \varphi = \varphi(\lambda) \in H_-$ is a generalized eigenvector of problem (1), corresponding to eigenvalue $\lambda \in \mathbb{R}^n$, if $\forall u \in D$

$$(\varphi, \hat{A}_i u)_H = \sum_{k=1}^{n} \lambda_k (\varphi, \hat{B}_{jk} u)_H, \quad j = 1, \ldots, n. \tag{6}$$

When φ belongs to H and, moreover, $\varphi \in D$, then we can throw A_j and B_{jk} in (6) over φ and get (1), since u runs the dense set from H. Thus, above given definition generalizes (1).

The main result of the present section is contained in the following theorem.

Theorem 3. *Let the inbedding operator $H_+ \subseteq H$ is quasinuclear (i. e. of Hilbert—Schmidt type). The finite Borel measure δ in $\mathbb{R}^n$ exists for which it is possible to construct for ρ-almost all $\lambda \in \mathbb{R}^n$ the set $(\varphi_\alpha(\lambda))_{\alpha=1}^{N(\lambda)}$ $(N(\lambda) \leqslant \infty)$ of generalized eigenvectors of problem* (1) *and the Fourier transformation*

$$H_+ \ni u \mapsto \tilde{u}(\lambda) = (\langle u, \varphi_\alpha(\lambda)\rangle)_{\alpha=1}^{N(\lambda)} \in l_2(N(\lambda)) \tag{7}$$

for which the Parseval equality is valid: $\forall u, v, \in H_+$

$$\langle u, v\rangle = \int_{\mathbb{R}^n} (\tilde{u}(\lambda), \tilde{v}(\lambda))_{l_2(N(\lambda))} d\rho(\lambda) = \int_{\mathbb{R}^n} \sum_{\alpha=1}^{N(\lambda)} \langle u, \varphi_\alpha(\lambda)\rangle \overline{\langle v, \varphi_\alpha(\lambda)\rangle} d\rho(\lambda) \tag{8}$$

(*here* $l_2(\infty) = l_2$, $\forall N < \infty$ $l_2(N) = \mathbb{C}^N$).

Explain: since $\Delta \in L(H_+, H_+)$ then bilinear form $\langle \cdot, \cdot \rangle$ is defined and bounded on $H_- \times H_+$, therefore $\forall \alpha \in H_-$, $u \in H_+$ the duality $\langle \alpha, u\rangle$ exists.

Note that ρ (spectral measure) is defined by problem (1) uniquely up to equivalence of measures (independently with respect to rigging (4)).

For our case it is possible to formulate also the «projection» form of Theorem 3. Namely, the following result is true.

Theorem 4. *Let, asabove, the imbedding $H_+ \subseteq H$ be quasinuclear and ρ be spectral measure of problem* (1). *Then for ρ-almost all $\lambda \in \mathbb{R}^n$ the function $P(\lambda)$ exists, which values are Hilbert — Schmidt operators acting from H_+ into H_-, with following properties:* $\forall u \in H_+$ $\langle P(\lambda) u, u\rangle \geqslant 0$,

$$u = \Big(\int_{\mathbb{R}^n} P(\lambda) d\rho(\lambda)\Big) u \tag{9}$$

(*the integral converges with respect to Hilbert—Schmidt norm*) *and range of operators $P(\lambda)$ consists of generalized eigenvectors of problem* (1) *corresponding to eigenvalue* λ.

Note that «generalized projector $P(\lambda)$ on generalized eigensubspace» is expressed by $\varphi_\alpha(\lambda)$ from (7) in the following manner (it is sufficient to compare (8) and (9)):

$$P(\lambda) u = \sum_{\alpha=1}^{N(\lambda)} \langle u, \varphi_\alpha(\lambda)\rangle_\alpha(\lambda), \quad u \in H_+.$$

Before the proof of these two theorems for us is necessary to introduce some notions. Let D' be a conjugate space for D, duality between D' and D is given by $(\cdot, \cdot)_H$. Introduce the extensions $\hat{A}_j^+ \in L(H_-, D')$, $\hat{B}_{jk}^+ \in L(D', D')$ of operators $\hat{A}_j$, $\hat{B}_{jk}$ respectively by

$$(\hat{A}_j^+\varphi, u)_H = (\varphi, \hat{A}_j u)_H, \quad \varphi \in H_-, \ u \in D; \quad j = 1, \dots, n,$$

$$(\hat{B}_{jk}^+\varphi, u)_H = (\varphi, \hat{B}_{jk} u)_H, \quad \varphi \in D', \ u \in D; \quad j, k = 1, \dots, n.$$

For us it is also necessary to introduce some extensions of operators X_k. Note that from (2), (5) and commutativity of Δ_{jk} and $\hat{A}_j$ one can conclude: $X_k \in L(D, H_+)$, $k = 1, \dots, n$. Due to selfadjointness of X_k with respect to $\langle\cdot, \cdot\rangle$ we can define the extensions $X_k^{\#} \in L(H_-, D')$ of operators X_k by

$$\langle X_k^{\#}\varphi, u\rangle = \langle\varphi, X_k u\rangle, \ \varphi \in H_-, \ u \in D; \quad k = 1, \dots, n.$$

We shall say that the vector $0 \neq \varphi = \varphi(\lambda) \in H_-$ is a generalized eigenvector of family of operators $(X_k)_{k=1}^n$, corresponding to eigenvalue $\lambda = (\lambda_1, \dots$ $\dots, \lambda_n) \in \mathbb{R}^n$, if

$$X_k^{\#}\varphi = \lambda_k \varphi, \quad k = 1, \dots, n. \tag{10}$$

P r o o f of Theorems 3, 4. As it was noted the operators X_k act continuously from D into H_+ and, therefore, (4) is suitable for expansion with respect to joint generalized eigenvectors of family $(X_k)_{k=1}^n$ of commuting selfadjoint operators [12—14]. According to projection spectral theorem and its form in terms of Fourier transformation ([13], Theorem 2.8 and subsection 2.11 from Ch. 2; or [14], Theorem 2.4 and § 3, subsections 1, 2, from Ch. 15) it is possible to assert that all results of Theorems 3, 4 are valid with replacing the generalized eigenvectors of problem (1) on such joint vectors of family $(X_k)_{k=1}^n$. Thus, our assertions will be proved if we check that every joint generalized eigenvector of family $(X_k)_{k=1}^\infty$ is the generalized eigenvector of problem (1).

Let $\varphi = \varphi(\lambda) \in H_-$ be joint generalized eigenvector of family $(X_k)_{k=1}^n$, i. e. φ satisfies the equalities (10). It is necessary to show that the equalities (6) it also satisfies. Since $D \subseteq \mathcal{D}$, then $\forall u \in D$

$$\sum_{k=1}^n \hat{B}_{jk}^+ X_k^{\#} u = \sum_{k=1}^n \hat{B}_{jk} X_k u = \hat{A}_j^+ u, \qquad j = 1, \dots, n. \tag{11}$$

(see (3)). According to continuity of operators $\hat{B}_{jk}^+$, $X_k^{\#}$, $\hat{A}_j^+$ and density D in H_- the equality (11) is extended on H_-, i. e. we have

$$\sum_{k=1}^n \hat{B}_{jk}^+ X_k^{\#} = \hat{A}_j^+, \qquad j = 1, \dots, n. \tag{12}$$

Now after application of (12) to φ from (10) it follows:

$$\sum_{k=1}^n \lambda_k \hat{B}_{jk}^+ \varphi = \hat{A}_j^+ \varphi, \qquad j = 1, \dots, n.$$

But last equality is equivalent to definition (6).

R e m a r k 1. Using (12) it is easy to show that (6) and (10) are equivalent, i. e. the notion of generalized eigenvectors of multiparameter problem (1) and notion of joint generalized eigenvectors of family $(X_k)_{k=1}^n$ of operators coincides. The proof of this fact is quite analogous to the proof of such fact for the ordinary eigenvectors [1, 6—8, 10].

4. T h e c o n s t r u c t i o n o f e x p a n s i o n s i n t h e c a s e o f n o n q u a s i n u c l e a r r i g g i n g. It is good known [12—14] that the condition of quasinuclearity of mapping H_+ into H_- is necessary for construction of chain which is suitable for expansions in generalized eigenfunctions of arbitrary selfadjoint operator A (or commuting family of such operators). But for some concrete situations this condition can (and useful) omit. In particular in the applications it is very convenient to use the Carleman property

of corresponding operators. In present section we develope abstract approach which is useful in such cases when it is possible to prove the Carleman property for our operators A_j, $j = 1, \ldots, n$.

Let O be the imbedding operator of space H_- into H_0, $S_2(F, G)$ denote the class of Hilbert-Schmidt operators acting from F into G. We formulate at first one simple general result.

Lemma 1. *The assertions of Theorems* 3, 4 *are valid if instead of quasinuclearity of imbedding* $H_+ \subseteq H$ *we shall suppose that some bounded continous nonzero function* $\mathbb{R}^n \ni (\lambda_1, \ldots, \lambda_n) \mapsto \alpha(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^1$ *exists for which the operator* $\alpha(X_1, \ldots, X_n) O \in S_2(H_+, H)$.

For the case $n = 1$ this fact is contained in [13] (Ch. 2, Theorem 5.2) or in [14] (Ch. 15, Theorem 4.2). The proof for arbitrary n is analogous.

Suppose that we have the rigging $H_{-,k} \supseteq H_k \supseteq H_{+,k}$, $k = 1, \ldots, n$, and the spaces H_+, H_- in (4) are constructed as the tensor products: $H_+ = \bigotimes_{k=1}^{n} H_{+,k}$, $H_- = \bigotimes_{k=1}^{n} H_{-,k}$. In this case $O = \bigotimes_{k=1}^{n} O_k$ where O_k is the imbedding operator $H_{+,k} \subseteq H_k$. Suppose: 1) that for some set $(l_j)_{j=1}^n$, $l_j = 1, 2, \ldots$,

$$(A_j^{l_j} - i1)^{-1} O_j \in S_2(H_{+,j}, H_j), \quad j = 1, \ldots, n; \tag{13}$$

2) the operators B_{jk} transform $\mathcal{D}(A_j^{l_j-1})$ in itself and are continous in the norm of corresponding graph, i. e. $\exists c_{jk} > 0$:

$$\| A_j^{l_j-1} B_{jk} u \|_{H_j} \leqslant c_{jk} (\| A_j^{l_j-1} u \|_{H_j} + \| u \|_{H_j}), \quad u \in \mathcal{D}(A_j^{l_j-1}); \quad j, k = 1, \ldots, n. \tag{14}$$

Denote $X = (X_1^2 + \ldots + X_n^2 + 1)^{\frac{1}{2}}$.

Lemma 2. *Assume that conditions* (13), (14) *are fulfilled*, $l = l_1 + \ldots + l_n$. *Then* $X^{-l} O \in S_2(H_+, H)$.

It is convenient to prove this lemma a little later. Now we formulate some consequence of Lemmas 1 and 2.

Theorem 5. *Suppose that the conditions* (13), (14) *and* (5) *are fulfilled. Then* (4) *is suitable for the construction of expansion on generalized eigenvectors of multiparameter problem* (1). *In particular, the assertions of Theorems* 3, 4 *are valid.*

Proof. According to Lemma 1 for the proof of our theorem it is sufficient to find the function α with corresponding properties. But according to Lemma 2 as such function we can take $\alpha(\lambda_1, \ldots, \lambda_n) = (\lambda_1^2 + \ldots + \lambda_n^2 + 1)^{\frac{1}{2}}$ $((\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n)$.

The proof of Lemma 2 is based on the following result.

Lemma 3. *Let us the conditions* (14) *are fullfilled. Then the operator* $\left(\bigotimes_{j=1}^{n} (A_j^{l_j} - i1)\right) X^{-l}$ *is bounded in* H.

Proof. Let $E(\cdot)$ be a joint resolution of identity of family $(X_k)_{k=1}^n$ of commuting selfadjoint operators. Put $M = \{E(\alpha) u \mid u \in H$, α is arbitrary bounded Borel set from $\mathbb{R}^n\}$. The set M is as essential domain for X^l therefore it is sufficient to check inclusion $M \subseteq \mathcal{D}\left(\bigotimes_{j=1}^{n} (A_j^{l_j} - i1)\right)$ and the estimate: $\exists c > 0 \ \forall u \in M$

$$\left\| \left(\bigotimes_{j=1}^{n} (A_j^{l_j} - i1) \right) u \right\|_H \leqslant c \| X^l u \|_H.$$

We shall prove more general fact: $\forall (k_i)_{i=1}^n (0 \leqslant k_i \leqslant l_i)$

$$M \subseteq \mathcal{D}(\hat{A}_1^{k_1} \ldots \hat{A}_n^{k_n}) \tag{15}$$

and $\exists c_{k_1, \ldots, k_n} > \ \forall u \in M$

$$\| \hat{A}_{k_1} \ldots \hat{A}_{kn} u \|_H \leqslant c_{k_1, \ldots, k_n} \| X^k u \|_H, \quad k = k_1 + \ldots + k_n. \tag{16}$$

Note that from the interpolation considerations, applying to (14), it is possible to conclude that the operators B_{jk} continuously transform $\mathcal{D}(A_j^{m_j})$ into itself for all $m_j = 1, \dots, l_j - 1$, i. e. $\exists c'_{jk} > 0 \ \forall u \in \mathcal{D}(A_j^{m_j})$

$$\| A_j^{m_j} B_{jk} u \|_{H_j} \leqslant c'_{jk} (\| A_j^{m_j} u \|_{H_j} + \| u \|_{H_j}), \quad m_j = 1, \dots, l_j - 1; \quad j, k = 1, \dots, n. \tag{17}$$

We shall prove (15), (16) by means of induction. These correlations are fulfilled in the case $k_1 = \dots = k_n = 0$. Suppose now that they are fulfilled for the set $(k_i)_{i=1}^n$. For definiteness we shall assume that $k_1 < l_1$ and shall prove (15), (16) for the set $(k_1 + 1, k_2, \dots, k_n)$. From (3) it follows that $\hat{A}_1 u = \sum_{j=1}^n \hat{B}_{1j} u_j$ where $u_j = X_j u \in M \subseteq \mathcal{D}(\hat{A}_1^{k_1} \dots \hat{A}_n^{k_n})$. According to (17) the operators $\hat{B}_{1j}$, $j = 1, \dots, n$, transform the set $\mathcal{D}(\hat{A}_1^{k_1} \dots \hat{A}_n^{k_n})$ into itself. Thus, $\hat{A}_1 u$ also belongs to $\mathcal{D}(\hat{A}_1^{k_1} \dots \hat{A}_n^{k_n})$ and therefore $u \in \mathcal{D}(\hat{A}_1^{k_1+1} \hat{A}_2^{k_2} \dots \hat{A}_n^{k_n})$. Check the corresponding inequality (16). According to (3), (17) and supposition of induction, we get

$$\| \hat{A}_1^{k_1+1} \hat{A}_2^{k_2} \dots \hat{A}_n^{k_n} u \|_H = \left\| \hat{A}_1^{k_1} \dots \hat{A}_n^{k_n} \left(\sum_{j=1}^n \hat{B}_{1j} X_j u \right) \right\|_H \leqslant$$

$$\leqslant \sum_{j=1}^n \| \hat{A}_1^{k_1} \hat{B}_{1j} (\hat{A}_2^{k_2} \dots \hat{A}_n^{k_n} X_j u) \|_H \leqslant \sum_{j=1}^n c'_{1j} \| \hat{A}_1^{k_1} \dots \hat{A}_n^{k_n} X_j u \|_H \leqslant$$

$$\leqslant c_{k_1, \dots, k_n} \sum_{j=1}^n c'_{1j} \| X^k X_j u \|_H \leqslant c_{k_1, \dots, k_n} \sum_{j=1}^n c'_{1j} \| X^{k+1} u \|_H.$$

P r o o f of Lemma 2. It is evidently that the operator $X^{-l} O \in S_2(H_+, H)$ if and only if $O^+ X^{-l} \in S_2(H, H_-)$ where O^+ is the imbedding operator $H \subseteq H_-$. For this operator we have $O^+ = \bigotimes_{j=1}^n O_j^+$ where O_j^+ is the imbedding operator $H_j \subseteq H_{-,j}$, therefore the assertion of lemma is equivalent to inclusion

$$\left(\bigotimes_{j=1}^n O_j^+ \right) X^{-l} \in S_2(H, H_-). \tag{18}$$

But we have

$$\left(\bigotimes_{j=1}^n O_j^+ \right) X^{-l} = \left(\bigotimes_{j=1}^n O_j^+ (A_j^{l_j} - i1)^{-1} \right) \left(\bigotimes_{j=1}^n (A_j^{l_j} - i1) \right) X^{-l}. \tag{19}$$

The last multiplier in (19) is bounded in H according to Lemma 3.

Consider the first multiplier in (19). The condition (13), as above, is equivalent to inclusion $O_j^+ (A_j^{l_j} - i1)^{-1} \in S_2(H_j, H_{-,j})$, therefore for tensor product we have:

$$\bigotimes_{j=1}^n O_j^+ (A_j^{l_j} - i1)^{-1} \in S_2 \left(\bigotimes_{j=1}^n H_j, \bigotimes_{j=1}^n H_{-,j} \right) = S_2(H, H_-).$$

Now from (19) we can conclude that the inclusion (18) is valid.

5. E x p a n s i o n i n e i g e n f u n c t i o n s o f m u l t i p a r a m e t e r p r o b l e m s f o r d i f f e r e n t i a l o p e r a t o r s. In this section we would like to give some applications of Theorems 3, 4 to the differential operators.

Let $H_j = L_2(G_j)$ where G_j is some (perhaps unbounded) domain in space $\mathbb{R}^{d_j}$ with enough smooth boundary ∂G_j ($d_j = 1, 2, \dots$; $j = 1, \dots, n$; the measure on G_j is Lebesgue). Now $H = \bigotimes_{j=1}^n H_j = L_2(G)$, $G = G_1 \times \dots \times G_n \subseteq \subseteq \mathbb{R}^{d_1 + \dots + d_n}$. Denote $x = (x_1, \dots x_n)$ where $x \in G$, $x_j \in G_j$.

Let $\forall j L_j$ be formally selfadjoint arbitrary differential expression of order $r_j = 1, 2, \ldots$ of the form

$$(L_j u)(x_j) = \sum_{|\alpha| \leqslant r_j} a_{j\alpha}(x_j)(D^\alpha u)(x_j), \quad x_j \in G_j, \tag{20}$$

with complex-valued coefficients $a_{j\alpha} \in C^{|\alpha|}(G_j)$ and partial derivatives D^α of order $|\alpha|$. This expression generates the minimal operator in the space H_j which is equal (by definition) to the closure of operator $H_j \supseteq C_0^\infty(G_j) \ni u \mapsto$ $\mapsto L_j u \in H_j$ (C_0^∞ denotes the class of infinitely many differentiable functions, finite with respect to ∂G_j and ∞; shortly: with respect to G_j). Let A_j be some selfadjoint extension of this minimal operator in the space H_j (we suppose that such extension exists).

Now as operator B_{jk} we take the operator of multiplication on real-valued bounded measurable function $b_{jk}(x_j)$ in the space H_j:

$$H_j \ni f(x_j) \mapsto (B_{jk} f)(x_j) = b_{jk}(x_j) f(x_j) \in H_j, \ j, k = 1, \ldots, n. \tag{21}$$

The positive definiteness of operator Δ constructed by (21) in the space H guarantees the following condition:

$$\det((b_{jk}(x_j))_{j,k=1}^n) \geqslant m > 0, \quad x = (x_1, \ldots, x_n) \in G. \tag{22}$$

We shall suppose below that the condition (22) fulfills.

The role of space H_+ in (4) now plays the space

$$H_+ = \bigotimes_{j=1}^n W_2^{l_j}(G_j, p_j(x_j)\, dx_j) \tag{23}$$

where $W_2^{l_j}(G_j, p_j(x_j)\, dx_j)$ denotes the Sobolev space of l_j-times differentiable functions on G_j with some weight $p_j(x_j) \geqslant 1$. The index $l_j = [d_j/2] + 1$. The weight is chosen so that the imbedding $W_2^{l_j}(G_j, p_j(x_j)\, dx_j \subseteq L_2(G_j)$ is quasinuclear (it is possible, see [13, 14]). Thus the imbedding $H_+ \subseteq H$ is also quasinuclear. The negative space with respect to positive $W_2^{l_j}(G_j, p_j(x_j)\, dx_j)$ and zero space $L_2(G_j)$ we denote, as usually, by $W_2^{-l_j}(G_j, p_j(x_j)\, dx_j)$.

As the space D from (4) we take the space $C_0^{s_1, \ldots, s_n}(G)$ which consists of functions on G, s_j times continuously differentiable with respect to variable x_j and finite with respect to ∂G_j and ∞; here $s_j = l_j + r_j$, $j = 1, \ldots n$. This space is provided by corresponding natural topology, which gives the uniform convergence of all derivatives $D_{x_1}^{k_1} \ldots D_{x_n}^{k_n}$, $0 \leqslant k_j \leqslant s_j$, and uniform finiteness.

Theorem 6. *Suppose in addition that $a_{j\alpha} \in C^{l_j}(G_j)$, $b_{jk} \in C^{s_j}(G_j)$, $|\alpha| \leqslant$ $\leqslant r_j$; $j, k = 1, \ldots, n$. Then all assertions of Theorems 3, 4 are valid. Now the generalized eigenfunctions $\varphi = \varphi(\lambda)$ of problem (1) are generalized solutions inside G (from the space $H_- = \bigotimes_{j=1}^n W_2^{-l_j}(G_j, p_j(x)\, dx_j)$) of the system of differential equations*

$$(L_j \varphi)(x) = \sum_{k=1}^n \lambda_k b_{jk}(x_j) \varphi(x), \quad x = (x_1, \ldots, x_n) \in G; \quad \lambda = (\lambda_1, \ldots, \lambda_n);$$

$$j = 1, \ldots, n. \tag{24}$$

Proof. This result immediately follows from Theorems 3, 4 It is necessary only to note that the restrictions putting on coefficients $a_{j\alpha}$, b_{jk} guarantee the conditions (5); the assertion about eigenfunctions follows from (6).

In the case of elliptic differential expressions (20) we can apply the theorem about regularity inside the domain of the kernel, which with respect to every variable satisfies the elliptic equation ([12, Theorem 4.2 from Ch. 3)). From this theorem and (24) the smoothness of eigenfunctions follows. Moreover, it is possible to prove the following theorem.

We denote below by $C^{t_1,\ldots,t_n}(G)$, $t_j = 1, 2, \ldots$, the class of functions $G \ni \ni x \mapsto u(x) \in \mathbb{C}^1$, continuously differentiable t_j times with respect to x_j, $j = = 1, \ldots, n$.

Theorem 7. *Let L_j be an elliptic differential expression in the domain $G_j \subseteq \mathbb{R}^{d_j}$, $d_j = 2, 3, \ldots$, with coefficients $a_{j\alpha} \in C^{|\alpha|+t_j}(G_j)$, $b_{jk} \in C^{t_j}(G_j)$ where $t_j \geqslant r_j + l_j + 1$, $|\alpha| \leqslant r_j$; $j, k = 1, \ldots, n$. Then the assertion of Theorem 6 is valid and the generalized eigenfunctions $\varphi(\lambda) = \varphi(x; \lambda)$ of the problem (1) belong to the class $C^{t_1,\ldots,t_n}(G)$.*

The generalized projector $P(\lambda)$ from theorem 4 acts on finite with respect to G functions u from H_+ by the formula

$$(P(\lambda)u)(x) = \int_G P(x, y; \lambda)\, u(y)\, dy, \quad x \in G; \qquad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n,$$

where spectral kernel $P(x, y; \lambda)$ is a generalized solution inside G from $H_- \otimes H_-$ of the system of equations

$$(L_{j,x_j}(\Delta^{-1}(y)\, P(x, y; \lambda)))(x, y) = \sum_{k=1}^{n} \lambda_k b_{jk}(x_j)\, \Delta^{-1}(y)\, P(x, y; \lambda),$$

$$(\bar{L}_{j,y_j}(\Delta^{-1}(y)\, P(x, y; \lambda)))(x, y) = \sum_{k=1}^{n} \lambda_k b_{jk}(y_j)\, \Delta^{-1}(y)\, P(x, y; \lambda) \tag{25}$$

$$x = (x_1, \ldots, x_n) \in G, \; y = (y_1, \ldots, y_n) \in G; \qquad j = 1, \ldots, n.$$

Here $\bar{L}_j$-differential expression with coefficients conjugate to coefficients of L_j; L_{j,x_j} (L_{j,y_j}) acts with respect to variable x_j (y_j).

Inside $G \times G$ the kernel $P(x, y, \lambda)$ is smooth enough: it belongs to the class $C^{t_1,\ldots,t_n,t_1,\ldots,t_n}(G \times G)$. The kernel $\Delta^{-1}(y)\, P(x, y; \lambda)$ for every λ is a positive definite one, its expansion in generalized eigenfunctions $\varphi_\alpha(\lambda) = \varphi_\alpha(x; \lambda)$ from Theorem 3 of multiparameter problem (1) has the form

$$\Delta^{-1}(y)\, P(x, y; \lambda) = \sum_{\alpha=1}^{N(\lambda)} \varphi_\alpha(x; \lambda)\, \overline{\varphi_\alpha(y; \lambda)}, \quad x, y \in G. \tag{26}$$

The series (26) converges absolutely and uniformly inside $G \times G$ (on every compact from $G \times G$). Moreover, the series (26) can be differentiated: take the derivatives of the form $D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n} D_{y_1}^{\beta_1} \ldots D_{y_n}^{\beta_n}$ where $|\alpha_j|, |\beta_j| \leqslant t_j$. After differentiation the series will again converge absolutely and uniformly inside $G \times G$.

Proof. The assertion about smoothness of eigenfunctions $\varphi(x; \lambda)$ follows from the regularity theorems for elliptic equations and was explained.

Since $P(\lambda): H_+ \to H_-$ is a Hilbert-Schmidt operator, the kernel $P_\lambda \in \in H_- \otimes H_-$ exists for which

$$(P(\lambda)u, v)_H = (P_\lambda, v \otimes \bar{u})_{H \otimes H}, \; u, v \in H_+. \tag{27}$$

This result follows from the general kernel theorem from [12—14]; we shall also write a little symbolically: $P_\lambda = P(x, y; \lambda)$. Since $\forall u \in H_+ \langle P(\lambda)u, u\rangle \geqslant 0$, the kernel $\Delta(x)\, P(x, y; \lambda)$ (and also the kernel $\Delta^{-1}(y)\, P(x, y; \lambda)$) is positive definite. In particular in the sense of generalized functions

$$P(x, y; \lambda) = \frac{\Delta(y)}{\Delta(x)}\, \overline{P(y, x; \lambda)}. \tag{28}$$

Prove (25). Since the range of operator $P(\lambda)$ consists of generalized eigenvectors of the problem (1), according to (6) and (27) we get: $\forall u, v \in C_0^\infty(G)$

$$0 = \Big(P(\lambda)u, \Big(L_j - \sum_{k=1}^{n} \lambda_k b_{jk}(x_j)\Big) v\Big)_H =$$

$$= \left(P_\lambda, \left(\left(L_j - \sum_{k=1}^{n} \lambda_k b_{jk}(x_j)\right) v\right) \otimes \bar{u}\right)_{H \otimes H}.$$

This equality is equivalent to the first equation in (25) understanding as an equation for generalized functions. The second equation in (25) (understanding analogously) follows from the first one and from (28).

Thus the generalized function $\Delta^{-1}(y)\, P(x, y; \lambda)$ satisfies the system (25) (in generalized sense, of course). But with respect to every variable x_j, y_j the differential operators in (25) are elliptic. Therefore we can apply once more the Theorem 4.2, Ch. 3 [12], and conclude that the kernel $\Delta^{-1}(y)\, P(x, y; \lambda)$ is enough smooth and the equations (25) are satisfied in classical sence.

The expansion (26) and possibility to differentiate it are proved as analogous facts for ordinary spectral problems in [12].

If selfadjoint extension A_j of minimal operator (created by L_j) is generated by means of some «good» boundary conditions then it is possible to apply the theorems from [12, 14, 15] about the rise of smoothness up to boundary. These theorems give the smoothness up to boundary of eigenfunctions, spectral kernel and so on for multiparameter problem (1). We shall formulate now some results in this direction.

Fix $j = 1, \ldots, n$ and consider the operator A_j created by the elliptic expression L_j of order $r_j = 2m_j$, $m_j = 1, 2, \ldots$, in the space $L_2(G_j)$ ($G_j \subseteq \mathbb{R}^{d_j}$, possibly is unbounded). Let $G_{0j} \subseteq G_j$ be a bounded domain with boundary ∂G_{0j} and intersection $\partial G_{0j} \cap \partial G_j = \Gamma_j$ is enough smooth piece of surface in $\mathbb{R}^{d_j}$ which is a domain in topology of ∂G_j. In this situation we shall say that $G_{0j} \subseteq G_j$ lies in G_j up to piece Γ_j of boundary ∂G_j. Suppose that the functions u from domain of operator A_j satisfy on Γ_j some normal system of elliptic boundary condition [12, 15]:

$$(C_{ji}u)(x) = 0, \quad x \in \Gamma_j, \ i = 1, \ldots, m_j. \tag{29}$$

$$(C_{ji}u)(x) = \sum_{|\alpha| \leqslant m_{ji}} c_{ji\alpha}(x)\, (D^\alpha u)(x), \quad m_{ji} < 2m_j. \tag{30}$$

Assume that

$$a_{j\alpha}, \ b_{jk} \in C^{\tau_j}(G_j \cup \Gamma_j), \quad |\alpha| \leqslant 2m_j, \ k = 1, \ldots, n; \qquad \tau_j \geqslant l_j + 2m_j, \tag{31}$$

$$c_{ji\alpha} \in C^{\tau_{ji}}(\Gamma_j), \quad |\alpha| \leqslant m_{ji}, \ i = 1, \ldots, n;$$
$$\tau_{ji} \geqslant \max\{2m_j - m_{ji}, \ l_j + m_{ji} + 1\}. \tag{32}$$

We shall say that $u \in W^s_{2,\mathrm{loc}}(G_j, \Gamma_j)$, $s = 1, 2, \ldots$, if for arbitrary bounded domain $G' \subseteq G_j$, which joint with ∂G_j boundary lies strictly inside of piece Γ_j, we have: $u \in W^s_2(G')$. Introduce also the following notations. Let us

$$G^j = G_1 \times \ldots \times G_{j-1} \times G_{j+1} \times \ldots \times G_n$$

for $x = (x_1, \ldots, x_n) \in G$ we shall write $x = (x_j, x^j)$, where $x^j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$, and also: $G = G_j \times G^j$, $j = 1, \ldots, n$.

Theorem 8. *Let us the condition of Theorem 7 be fulfilled and for some $j = 1, \ldots, n$, there exist the bounded domain $G_{0j} \subseteq G_j$ which lies in G_j up to the piece $\Gamma_j \subseteq \partial G_j$. Suppose that $r_j = 2m_j$, the functions u from $\mathcal{D}(A_j)$ satisfy on Γ_j normal system of boundary conditions* (29), (30) *and the conditions of smoothness* (31), (32) *are also satisfied. The piece Γ_j is smooth enough: it belongs to the class $C^{2m_j+\tau_j+1}$.*

Then the generalized eigenfunctions of the problem (1) $\varphi(x; \lambda) = \varphi((x_j, x^j); \lambda) \in H_-$ *for every fixed $x^j \in G^j$ belong to $W^{2m_j+\tau_j}_{2,\mathrm{loc}}(G_{0j}, \Gamma_j)$ and on Γ_j satisfy the boundary condition* (29). *Analogous assertion is valid for the spectral kernel $P((x_j, x^j), y; \lambda)$ with fixed $x^j \in G^j$, $y \in G$ and for the function $\Delta^{-1}((y_j, y^j))\, P(x, (y_j, y^j); \lambda)$ with fixed $x \in G$, $y^j \in G^j$.*

Proof. Assertion of this theorem concerning the eigenfunctions follows from known results about rise of smoothness of solutions of elliptic equations up to boundary [12, 15]. Now the space H_+^n (and respectively H_-) is constructed as in Theorem 6. It is necessary to improve a little the construction of space D. Let $C^{s_1,\dots,s_n}(\tilde{G})$ be the set of all functions from $C^{s_1,\dots,s_n}(\mathbb{R}^{d_1+\dots+d_n})$ restricted on closure $\tilde{G}$ of G; $s_k = l_k + r_k$, $k = 1, \dots, n$. As D we take the set $C_0^{s_1,\dots,s_n}(\tilde{G}, \Gamma_j)$ which consists of all functions from $C^{s_1,\dots,s_n}(\tilde{G})$ which satisfy on $\Gamma_j \times \tilde{G}^j \subset \tilde{G}_j \times \tilde{G}^j$ the boundary conditions (29) and equal to zero in neighborhoods of $\partial G \setminus (\Gamma_j \times \tilde{G}^j)$ and infinity. The set $C_0^{s_1,\dots,s_n}(\tilde{G}, \Gamma_j)$ it is necessary to provide with corresponding topology (see [12, 14]) and then put $D = C_0^{s_1,\dots,s_n}(\tilde{G}, \Gamma_j)$.

Such choice of H_+ and D gives that the generalized eigenfunctions of problem (1) satisfy the j-th equation of system (24) up to the piece Γ_j, on the boundary ∂G_j. Therefore after application of mentioned theorems about rise of smoothness we get the result concerning eigenfunctions.

Assertion concerning the spectral kernel is provided analogously, now we use the system (25) instead of (24).

Remark 1. The first part of this paragraph, including Theorem 6, is true, of course, for the ordinary differential expression (20), when $d_1 = \dots = d_n = 1$. But for the general ordinary linear differential equations there exist the theorems about rise of smoothness of solutions (up to boundary also, i. e. up to the ends of intervals) [14]. Therefore we can apply these theorems in considerations of type Theorems 7, 8. As result, we shall get the Theorems 7, 8 for the case, when $d_1, \dots, d_n = 1, 2, \dots$, i. e. when all differential expressions L_j are ordinary or only some part of them is such. Corresponding domains G_j can be axes or some intervals $(a_j, b_j) \subset \mathbb{R}^1$ $(a_j \geqslant -\infty,\ b_j \leqslant +\infty)$. We shall not formulate the results. They have the form of Theorems 7, 8 but with weaker restrictions on the smoothness of coefficients. It is easy to get these restrictions from the Theorems contained in [14], Ch. 16, § 6.

6. The approach, based on Carleman property of operators. In this section we apply the abstract approach, developed in 4, to the case of elliptic operators A_j, $j = 1, \dots, n$.

Theorem 9. *Let A_j be a selfadjoint extension in the space $H_j = L_2(G_j)$, $G_j \subseteq \mathbb{R}^{d_j}$, $d_j = 2, 3, \dots$, of minimal operator constructed by elliptic differential expression L_j of order r_j with coefficients $a_{j\alpha} \in C^{r_j + t_j}(G_j)$ $(t_j \geqslant r_j + 2d_j + 1)$, B_{jk} is an operator of multiplication on bounded real-valued function $b_{jk} \in C^{t_j'}(G_j)$ $(t_j' \geqslant r_j)$, $j, k = 1, \dots, n$. Suppose that the operator B_{jk} transforms $D(A_j^{n_j})$ $(n_j = [d_j/2r_j])$ into itself and is continuous in the norm of corresponding graph, i. e. $\exists c_{jk} > 0\ \forall u \in \mathcal{D}(A_j^{n_j})$*

$$\| A_j^{n_j}(b_{jk}u) \|_{H_j} \leqslant c_{jk} (\| A_j^{n_j} u \|_{H_j} + \| u \|_{H_j}), \quad j, k = 1, \dots, n. \tag{33}$$

Then $\forall j = 1, \dots, n$ such a function $p_j \in C^\infty(G_j)$, $p_j(x_j) \geqslant 1$ exists (it is possible, tending to ∞ when x_j tends to ∂G_j and ∞) that for expansion in generalized eigenfunctions of multiparameter problem (1) *is suitable the chain $H_- \supseteq H \supseteq H_+$ of the form*

$$L_2(G, p^{-1}(x)\,dx) \supseteq L_2(G) \supseteq L_2(G, p(x)\,dx),$$
$$p(x) = p_1(x_1) \dots p_n(x_n), \quad x = (x_1, \dots, x_n) \in G. \tag{34}$$

Generalized projector $P(\lambda)$ is an integral operator:

$$(P(\lambda) u)(x) = \int_G P(x, y; \lambda)\, u(y)\, dy, \quad u \in L_2(G, p(x)\, dx \tag{35}$$

where the spectral kernel $P(\cdot, \cdot; \lambda) \in L_2(G \times G, p^{-1}(x)\, p^{-1}(y)\, dxdy)$ is smooth enough $(P(\cdot, \cdot; \lambda) \in C^{\tau_1,\dots,\tau_n,\tau_1,\dots,\tau_n}(G \times G)$, $\tau_j = \min\{t_j, t_j'\}$; $j = 1, \dots, n))$ and satisfies inside $G \times G$ the system of equations (25).

Every generalized eigenfunction of problem (1) $\varphi(\cdot, \lambda) \in L_2(G, p^{-1}(x)\,dx)$ *belongs to the class* $C^{\tau_1, \ldots, \tau_n}(G)$ *and satisfies the system* (24).

The expansion of positive definite kernel $\Delta^{-1}(y) P(x, y; \lambda)$ *in eigenfunctions of the problem* (1)

$$\Delta^{-1}(y) P(x, y; \lambda) = \sum_{\alpha=1}^{N(\lambda)} \varphi_\alpha(x; \lambda) \overline{\varphi_\alpha(y; \lambda)}, \quad x, y \in G, \tag{36}$$

converges in the space $L_2(G \times G, p^{-1}(x) p^{-1}(y)\,dxdy)$. *In every strictly internal bounded subdomain of* $G \times G$ *the series* (36) *converges absolutely and uniformly. Moreover, it is possible to differentiate the series* (36): *to take the derivatives* $D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n} D_{y_1}^{\beta_1} \ldots D_{y_n}^{\beta_n}$ *where* $|a_j|, |\beta_j| \leqslant \tau_j$. *After the differentation the series* (36) *also will absolutely and uniformly converge in the mentionned domains.*

P r o o f. Let $l_j = [d_j/2] + 1$. Then [12] $\forall j$ the operator $(A^{l_j} - i1)^{-1}$ is integral operator with enough smooth kernel $K_j(x_j, y_j)$ for which the integral

$$\int_{G_j} |K_j(x_j, y_j)|^2 \, dx_j$$

is bounded when y_j runs in an arbitrary compact including in G_j. Therefore we can select such functions $p_j \in C^\infty(G_j)$, $p_j(y_j) \geqslant 1$, in oder to

$$\int_{G_j} \int_{G_j} |K(x_j, y_j)|^2 p^{-1}(y_j) \, dx_j dy_j < \infty.$$

Thus, the condition (13) is now fulfilled when we take $H_j = L_2(G_j)$, $H_{+,j} = L_2(G_j, p_j(x_j)\,dx_j)$. The condition (14) coincides with (33). Taking $D = C_0^{\tau_1, \ldots, \tau_n}(G)$, we easily check the conditions (5) in our case.

Thus all, assumptions of Theorem 5 are fulfilled and we can apply it with the chain (34) and indicated D.

The representation (35) for $P(\lambda)$ and inclusion $P(\cdot, \cdot; \lambda) \in L_2(G \times G, p^{-1}(x) p^{-1}(y)\,dxdy)$ follow from quasinuclearity of $P(\lambda)$. The smoothness of spectral kernel, generalized eigenfunctions and properties of expansion (36) are established as in Theorem 7 (see also [12]).

R e m a r k 1. Remind [12—14] that selfadjoint operator A, acting in the space $H = L_2(Q, d\mu(x))$, is called Carleman operator if some bounded continuous nonzero function γ, defined on the spectrum of A, exists for which the operator $\gamma(A)$ is integral and for its kernel $K(x, y)$, $x, y \in Q$, for μ-almost all $y \in Q$

$$\int_Q |K(x, y)|^2 \, d\mu(x) < \infty \left((\gamma(A) f)(x) = \int_Q K(x, y) f(y) \, dy, \; f \in H \right).$$

The fundamental step in proof of Theorem 9 is the checking of Carleman property of operators A_j (now $\gamma_j(A_j) = (A^{l_j} - i1)^{-1}$). This property admits to use (34) instead of the chain connected with positive space (23) and therefore to get more detailed information (comparing with Theorem 7) about spectral kernel and generalized eigenfunctions.

R e m a r k 2. The inequalities (33) lead to some restrictions on behaviour of functions b_{jk}. Note that in case of bounded domain G_j the norm of graph $\|u\|_\Gamma = \|A_j^{n_j} u\|_{H_j} + \|u\|_{H_j}$ is equivalent to norm in some Sobolev space («the inequality of coercitivness»). For unbounded domain in some cases it is possible to get analogous equivalence but for Sobolev spaces with weights. Such equivalence permits to get some more concrete restrictions on behaviour of b_{jk}.

R e m a r k 3. As in sec. 5, Remark 1, we can investigate in this section the case when all or some part of expressions L_j are ordinary differential expressions. It is easily possible to reformulate Theorem 9 for such situation, the conditions of this theorem are transformed by means of results from [14], Ch. 16, § 6.

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