UDC 517.986.4
M. M. Bogoliubov ((jr.), member-corr. Acad. Sci. Russia (Steclov Math. Inst., Moscow),
I. V. Mykytiuk, cand. phys.-math. sci. (Inst. appl. probl. mech. and math., Lviv),
B. M. Fil', cand. phys.-math. sci. (Politechn. Inst., Lviv),
A. K. Prykarpatsky, d-r phys.-math. sci. (Inst. appl. probl. mech. and math., Lviv).

Canonical quantization for classical dynamical systems of Neumann type via Moser spectral approach

Канонічне квантування для класичних динамічних систем типу Неймана в рамках спектрального підходу Мозера

The classical Neumann type dynamical systems describe the motion of a particles constrained to live on an N-sphere S^N in $(N+1)$-dimensional space $\mathbb{R}^{N+1}$ and submitted to quasi-harmonic forces. Following the Moser spectral approach to a connection of the infinite dimensional finite-

zoned by Lax dynamical systems with the finite dimensional Neumann type systems on sphere in $\mathbb{R}^{N+1}$, the regular procedure to quantize of them suitably is supposed. The quantum expression of the commuting conserved currents for the quantum Neumann type dynamical systems are determined in a general case via the Dirac canonical quantization procedure.

Класичні динамічні системи типу Неймана описують рух частинки, обмеженої N-вимірною сферою $\mathbf{S}^N$ в $(N+1)$-вимірному просторі $\mathbb{R}^{N+1}$ під дією квазігармонічних сил. Згідно з спектральним підходом Мозера до зв'язку нескінченновимірних по Лаксу динамічних систем із скінченновимірними динамічними системами типу Неймана на сфері в $\mathbb{R}^{N+1}$ запропонована регулярна процедура відповідного їх квантування. В загальному випадку канонічного квантування Дірака визначені квантові вирази комутативних законів збереження для квантових динамічних систем типу Неймана.

1. Hamiltonian analysis of Moser's isomorphism for Korteweg-de Vries and Neumann nonlinear dynamical systems. 1.1. Let's given the following dynamical systems of Korteweg-de Vries [1]

$$du/dt = 6uu_x + u_{xxx} = K[u] \qquad (1)$$

on the infinite dimensional functional manifold $M \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ of 2π-periodical functions on the real axis $\mathbb{R}$, $t \in \mathbb{R}$ is a evolution parameter. In order to describe the related Moser's isomorphism of (1) to finite dimensional Neumann dynamical system on the sphere $\mathbf{S}^N$ (where $N \in \mathbb{N}$ is the number of stable zones in the spectrum $\sigma(L)$ of a associated Lax type spectral problem for the Sturm — Liuville operator $L: C^{(\infty)}(\mathbb{R}; \mathbb{C}) \to C^{(\infty)}(\mathbb{R}; \mathbb{C}^2)$, we shall formulate the special variational properties of functionals on M generated by Lax type representation for (1). Let $F(x, x_0; \lambda) \in GL(2; \mathbb{C})$ be the fundamental solution of the linear matrix differential Lax equation $L[u; \lambda] f(x; \lambda) = 0$, where $L[u; \lambda] = d/dx - \mathcal{A}[u; \lambda]$,

$$\mathcal{A}[u; \lambda] = \left\| \begin{array}{c|c} 0 & 1 \\ \hline -(\lambda + u) & 0 \end{array} \right\|, \qquad (2)$$

$\lambda \in \mathbb{C}$ is spectral parameter, $x, x_0 \in \mathbb{R}$ and $F(x_0, x_0; \lambda) = \mathbf{1}$ being satisfied for all $x_0 \in \mathbb{R}$. The monodromy matrix $S(x_0; \lambda) \in GL(2; \mathbb{C})$, $x \in \mathbb{R}$ is defined by equality $S(x_0; \lambda) = F(x_0 + 2\pi, x_0; \lambda)$ for all $\lambda \in \mathbb{C}$. It si obviously that monodromy matrix $S(x_0; \lambda)$ is a regular functional on the manifold M which is parametric dependent on $x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. Following the results of a book [1] we have main variational formula

$$\delta S(x_0; \mu) = \int_{x_0}^{x_0+2\pi} dx F(x_0 + 2\pi, x; \mu)\, \delta\mathcal{A}[u; \mu]\, F(x, x_0; \mu) \qquad (3)$$

for all $\mu \in \mathbb{C}$, $x_0 \in \mathbb{R}$.

We know [1] that a functional $\Delta(\lambda) = \operatorname{tr} S(x_0; \lambda)$ is invariant via the dynamical system (1), that is $d\Delta(\lambda)/dt = 0$ for all $t \in \mathbb{R}$. It is also clear that $d\Delta(\lambda)/dx_0 = 0$ also for all $x_0 \in \mathbb{R}$. Therefore, the functional $\Delta(\lambda) \in \mathcal{D}(M)$ is regular via Freche't invariant functional on M for all $\lambda \in \mathbb{C}$, defining the generating functional to determine the conservation laws for the dynamical systems (1).

1.2. Let us consider the action on a equality (3) of the Lie's derivative [2] L_K along the vector field (1). Since $L_K F = 0$, $L_K \delta\mathcal{A} = 0$ for all $\mu \in \mathbb{C}$ we obtain that $L_K\, \delta S(x_0; \mu) = 0$. If we shall define the matrix $\Phi(x; \mu)$ as follows:

$$\delta S_{ij}(x_0; \mu) = (\Phi_{ij}(x; \mu), \delta u(x)), \quad i, j = \overline{1, 2}, \qquad (4)$$

we receive that $\operatorname{grad} S(x_0; \mu)(x; \mu) = \Phi(x; \mu)$ for all $x, x_0 \in \mathbb{R}$, $\mu \in \mathbb{C}$, where $(\cdot, \cdot) = \int_{x_0}^{x_0+2\pi} dx \langle \cdot, \cdot \rangle$ is the standard bilinear form on $T^*(M) \times T(M)$.

Now we receive that for all $i, j = \overline{1, 2}$, $t \in \mathbb{R}$, the Lax equality is satisfied

$$d\Phi_{ij}(x; \mu)/dt + K'^* \cdot \Phi_{ij}(x; \mu) = 0. \qquad (5)$$

It is known [1] that the element $\varphi(x;\ \lambda) = \operatorname{grad} \Delta(\lambda)(x;\ \lambda) \in T^*(M)$ satisfies the following determining equations:

$$d\varphi(x;\ \lambda)/dt + K'^* \cdot \varphi(x;\ \lambda) = 0, \quad -4\lambda\theta\varphi(x;\ \lambda) = \eta\varphi(x;\ \lambda) \tag{6}$$

for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, where $\theta, \eta : T^*(M) \to T(M)$ are implectic and noetherian [2] differential operators for dunamical system (1), which are compatible [2] on M and

$$\theta = d/dx = \partial, \quad \eta = \partial^3 + 2u\partial + 2\partial u. \tag{7}$$

Due to a analytical dependence of the matrix $\Phi(x;\ \mu)$ on spectral paramet… $\mu \in \mathbb{C}$ from equations (5) and (6) we find that

$$-4\mu\theta\Phi_{ij}(x;\ \mu) = \eta\Phi_{ij}(x;\ \mu) \tag{8}$$

for all $i, j = \overline{1, 2}$, $x \in \mathbb{R}$. Now we can state following.

P r o p o s i t i o n 1. *The monodromy matrix $S(x_0;\ \mu)$ satisfies commutator relations via Poisson structure $\{\cdot, \cdot\}_\theta$ on M:*

$$\{s_{12}(x_0;\ \lambda),\ s_{12}(x_0;\ \mu)\}_\theta = 0, \quad \left\{s_{12}(x_0;\ \lambda),\ \frac{1}{2}(s_{22} - s_{11})(x_0;\ \mu)\right\}_\theta =$$
$$= \frac{2i}{\lambda - \mu}\left[s_{12}(x_0;\ \lambda)\sqrt{\frac{1 - \Delta^2(\mu)/4}{c(\mu)}} - s_{12}(x_0;\ \mu)\sqrt{\frac{1 - \Delta^2(\lambda)/4}{c(\lambda)}}\right],$$
$$\left\{\frac{1}{2}(s_{22} - s_{11})(x_0;\ \lambda),\ s_{21}(x_0;\ \mu)\right\}_\theta = \frac{2i}{\lambda - \mu}\left[s_{21}(x_0;\ \lambda)\sqrt{\frac{1 - \Delta^2(\mu)/4}{c(\mu)}} -\right.$$
$$\left. - s_{21}(x_0;\ \mu)\sqrt{\frac{1 - \Delta^2(\lambda)/4}{c(\lambda)}}\right], \quad \{s_{21}(x_0;\ \lambda),\ s_{12}(x_0;\ \mu)\}_\theta = \frac{2i}{\lambda - \mu} \times \tag{9}$$
$$\times \left[(s_{22} - s_{11})(x_0;\ \lambda)\sqrt{\frac{1 - \Delta^2(\mu)/4}{c(\mu)}} - (s_{22} - s_{11})(x_0;\ \mu)\sqrt{\frac{1 - \Delta^2(\mu)/4}{c(\lambda)}}\right],$$
$$\{(s_{22} - s_{11})(x_0;\ \lambda),\ (s_{22} - s_{11})(x_0;\ \mu)\}_\theta = 0$$

for all $x_0 \in \mathbb{R}$, $\lambda \neq \mu \in \mathbb{C}$, where $c(\lambda)$ is a special invariant functional on M determined by spectral properties of the Lax L-operator.

The proof of the statement is the direct consequence of the set of formulas (8) added with the obvious evolution equation via a parameter $x_0 \in \mathbb{R}$:

$$dS(x_0;\ \lambda)/dx_0 = [\mathcal{A}[u;\ \lambda],\ S(x_0;\ \lambda)], \tag{10}$$

where $[\cdot, \cdot]$ denotes the ordinary matrix commutator.

1.3. Let us define the following «generating» dynamical system on M:

$$du/dt = \{\Delta(\lambda),\ u\}_\theta, \tag{11}$$

where $\lambda \in \mathbb{C}$ and $t \in \mathbb{R}$ is a new evolution parameter. Following (9) we can state that evolution equations:

$$ds_{12}(x_0;\ \mu)/dt = \frac{2i}{\lambda - \mu}[(s_{22} - s_{11})(x_0;\ \lambda)\, s_{12}(x_0;\ \mu) - s_{12}(x_0;\ \lambda)(s_{22} - s_{11})(x_0;\ \mu)],$$
$$d(s_{22} - s_{11})(x_0;\ \mu)/dt = -\frac{2i}{\lambda - \mu}[s_{12}(x_0;\ \mu)\, s_{21}(x_0;\ \lambda) - s_{12}(x_0;\ \lambda)\, s_{21}(x_0;\ \mu)], \tag{12}$$
$$ds_{21}(x_0;\ \mu)/dt = \frac{2i}{\lambda - \mu}[(s_{22} - s_{11})(x_0;\ \mu)\, s_{21}(x_0;\ \lambda) - (s_{22} - s_{11})(x_0;\ \lambda)\, s_{21}(x_0;\ \mu)],$$
$$i = \sqrt{-1},$$

hold for all $x_0 \in \mathbb{R}$ and $\lambda \neq \mu \in \mathbb{C}$. The system of equations (12) is linear with relation to the functionals $s_{ij}(x_0;\ \mu)$, $i, j = \overline{1, 2}$, as we have that

$$d(s_{11} + s_{22})(x_0;\ \mu)/dt = \{\Delta(\lambda),\ \Delta(\mu)\}_\theta = 0 \tag{13}$$

for all $\lambda, \mu \in \mathbb{C}$. Therefore from (12) and (13) we find that

$$dS(x_0; \mu)/dt = [P(\lambda, \mu), S(x_0; \mu)]; \tag{14}$$

where $P(\lambda, \mu) = \frac{2}{\lambda - \mu} S(x_0; \lambda)$, $x_0 \in \mathbb{R}$, $\lambda \neq \mu \in \mathbb{C}$. As the equation (13) holds, the equation (14) is equivalent to that in the following form:

$$d\tilde{S}(x_0; \mu)/dt = [\tilde{P}(\lambda, \mu), \tilde{S}(x_0; \mu)], \tag{15}$$

where $\tilde{S}(x_0; \mu) = S(x_0; \mu) - \mathbf{1}\Delta(\mu)/2$, $\tilde{P}(\lambda, \mu) = \frac{2}{\lambda - \mu} \tilde{S}(x_0; \lambda)$ for all $\lambda \neq \mu \in \mathbb{C}$.

N o t e. The (λ, μ) — parametric matrix Lax type equation in the form (15) was firstly stated implicitly by authors in the book [1, p. 65, 94], and also in the paper [3] in a context of the K. Neumann problem.

To pass now in the context of the Neumann problem [1, 3] we must suppose that

$$
\begin{gathered}
s_{12}(x_0; \mu) = -i\tilde{s}_{12}(x_0; \mu)\tilde{c}(\mu), \\
s_{21}(x_0; \mu) = i\tilde{s}_{21}(x_0; \mu)\tilde{c}(\mu), \\
\frac{1}{2}(s_{22} - s_{11})(x_0; \mu) = -i\tilde{s}(x_0; \mu)\tilde{c}(\mu), \quad i = \sqrt{-1}.
\end{gathered} \tag{16}
$$

Here $\tilde{c}(\mu) = \sqrt{(1 - \Delta^2(\mu)/4)/c(\mu)}$, $\mu \in \mathbb{C}$, is a some function and $c(\mu) := = \prod_{j=1}^{N} (\mu - \nu_j) \times \prod_{j=1}^{N+1} (\lambda - \omega_j)^{-1}$, where $\{\nu_j \in \mathbb{R}_+ : j = \overline{1, N}\}$ is a spectrum of antiperiodic spectral problem, $N \in \mathbb{Z}_+$ is the number of related Bloch zones of stability [1], $\omega_i \neq \omega_j$ for all $i \neq j$.

1.4. Let the periodic spectrum $\{\omega_j \in \mathbb{R}_+ : j = \overline{1, N+1}\}$ is fixed. Therefore due to *J*. Moser [4] we can pass in (16) to the K. Neumann canonical representation:

$$
\begin{gathered}
\tilde{s}_{12}(x_0; \mu) = \prod_{j=1}^{N} (\lambda - \mu_j(x_0)) / \prod_{j=1}^{N+1} (\lambda - \omega_j) := \sum_{j=1}^{N+1} \frac{q_j^2(x_0)}{\lambda - \omega_j}, \\
\tilde{s}_{21}(x_0; \mu) = \sum_{j=1}^{N+1} \frac{p_j^2(x_0)}{\lambda - \omega_j} + 1, \\
\tilde{s}(x_0; \mu) = d\tilde{s}_{12}(x_0; \mu)/dx_0 = \sum_{j=1}^{N+1} \frac{q_j(x_0)\, p_j(x_0)}{\lambda - \omega_j},
\end{gathered} \tag{17}
$$

provided that the following equalities for $\mu \in \mathbb{C}$,

$$
\begin{gathered}
-\tilde{s}^2(x_0; \mu) + \tilde{s}_{12}(x_0; \mu)\tilde{s}_{21}(x_0; (\mu) = \tilde{\gamma}(\mu), \\
u(x_0) = 2\sum_{j=1}^{N} \mu_j(x_0) - \sum_{j=1}^{N+1} \omega_j - \sum_{j=1}^{N} \nu_j,
\end{gathered} \tag{18}
$$

being held, where $\{\mu_j(x_0) \in \mathbb{R}_+ : j = 1, N\}$ is divisor of the initial Lax L-operator as in [1, p. 57]. Computing the evolution (10) via variable $x_0 := \tau \in \mathbb{R}$ from (17) and (18) we obtain the classical K. Neumann equations on the sphere S^N:

$$d^2 q_j/d\tau^2 + \omega_j q_j = 0, \quad dq_j/d\tau = p_j, \quad \sum_{j=1}^{N+1} q_j^2 = 1, \tag{19}$$

$j = \overline{1, N+1}$, or equivalently the following Hamiltonian system on $M^{2N+2} =$

$= T\mathbb{R}^{N+1} \cong \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ with the canonical symplectic structure $\omega^{(2)} = \sum_{j=1}^{N+1} dp_j \wedge \wedge dq_j \in \Lambda^2(M^{2N+2})$:

$$dq_j/d\tau = \{H, q_j\}_{(N+1)}, \quad dp_j/d\tau = \{H, p_j\}_{(N+1)}, \tag{20}$$

where $H = \frac{1}{4}\sum_{j\neq k}^{N+1}(q_i p_k - q_k p_j)^2 + \frac{1}{2}\sum_{j=1}^{N+1}\omega_j q_j^2$ is the Hamiltonian and $\{\cdot, \cdot\}_{(N+1)}$ is corresponding Poisson structure on $T\mathbb{R}^{N+1}$.

The formula (18) also admits the simple construction of the alternative to the set $\{\nu_j \in \mathcal{D}(M): j = \overline{1, N}\}$ hierarchy of the conservation laws as follows [4]:

$$\tilde{\gamma}_j = q_j^2 + \sum_{j\neq k}^{N+1}(q_j p_k - q_k p_j)^2/(\omega_j - \omega_k), \tag{21}$$

where $\tilde{\gamma}(\mu) = \sum_{j=1}^{N+1}\tilde{\gamma}_j/(\mu - \omega_j)$ has place due to (18) for all $\mu \in \mathbb{C}$.

2. The quantization problem. 2.1. Let us consider the quantum Neumann model (19) where the canonical Poisson brackets on the symplectic phase manifold M^{2N+2} are replaced [3] due to Dirac [5] by the corresponding canonical commutation relations

$$\begin{gathered}
\{\tilde{s}_{12}(\tau;\lambda), \ \tilde{s}_{12}(\tau;\mu)\}_{(N+1)} = 0 = \{\tilde{s}(\tau;\lambda), \ \tilde{s}(\tau;\mu)\}_{(N+1)}, \\
\{\tilde{s}_{12}(\tau;\lambda), \ \tilde{s}(\tau;\mu)\}_{(N+1)} = -\frac{2}{\lambda-\mu}[\tilde{s}_{12}(\tau;\lambda) - \tilde{s}_{12}(\tau;\mu)], \\
\{\tilde{s}(\tau;\lambda), \ \tilde{s}_{21}(\tau;\mu)\}_{(N+1)} = -\frac{2}{\lambda-\mu}[\tilde{s}_{21}(\tau;\lambda) - \tilde{s}_{21}(\tau;\mu)], \\
\{\tilde{s}_{21}(\tau;\lambda), \ \tilde{s}_{12}(\tau;\mu)\}_{(N+1)} = \frac{4}{\lambda-\mu}[\tilde{s}(\tau;\lambda) - \tilde{s}(\tau;\mu)],
\end{gathered} \tag{22}$$

where the canonical identification is used: $\{\cdot, \cdot\}_\theta \to \{\cdot, \cdot\}_{(N+1)}$ as the tangent sphere $T\mathbf{S}^N$ is invariant space of the dymanical system (19) and diffeomorfic to that of the initial dynamical system Korteweg-de Vries (1) due to Novicov-Lax theory [1]. After the quantization procedure we obtain the following operator quantities for all $\tau \in \mathbb{R}$, $\lambda \in \mathbb{C}$:

$$\begin{gathered}
:\tilde{s}_{12}(\tau;\lambda): = \hat{s}_{12}(\tau;\lambda), \quad :\tilde{s}_{21}(\tau;\lambda): = \hat{s}_{21}(\tau;\lambda), \\
:\tilde{s}(\tau;\lambda): = \hat{s}(\tau;\lambda), \quad :\tilde{\gamma}_j: = \hat{\gamma}_j, \quad j = \overline{1, N+1},
\end{gathered} \tag{23}$$

acting in the related Hilbert space of functions, where by means of a operator : : we denoted the appropriate law of operator ordering. Due to (22) we have identity

$$[\hat{s}_{21}(\tau;\lambda), \ \hat{s}_{12}(\tau;\mu)]\frac{i}{\hbar} = \frac{4}{\lambda-\mu}[\hat{s}(\tau;\lambda) - \hat{s}(\tau;\mu)], \tag{24}$$

which gives us

$$\hat{s}(\tau;\lambda) = \frac{1}{2}\sum_{j=1}^{N+1}(q_j p_j + p_j q_j)/(\lambda - \omega_j), \tag{25}$$

that is the operation of ordering $\tilde{s}(\tau;\lambda) \to :\tilde{s}(\tau;\lambda):$ which acts as the operator symmetrization as firstly was stated in [3]. The corresponding ordering to quantities $:\tilde{s}_{12}(\tau;\lambda):$ and $:\tilde{s}_{21}(\tau;\lambda):$ is obviously trivial.

2.2. To reconstruct the operator ordering in the quantity: $\tilde{\gamma}(\lambda) := \hat{\gamma}(\lambda)$, $\lambda \in \mathbb{C}$, it is useful to consider the equation (10) in the quantum Heisenberg form [5]:

$$d\hat{S}(\tau;\lambda)/d\tau = \frac{i}{\hbar}[\hat{\mathcal{A}}(\tau;\lambda), \hat{S}(\tau;\lambda)], \tag{26}$$

where $\hat{S}(\tau;\lambda) := \| : \tilde{s}_{ij}(\tau;\lambda) : \|_{i,j=\overline{1,2}}$, $\hat{\mathcal{A}}(\tau;\lambda) := :\mathcal{A}(\tau;\lambda):$ for all $\tau \in \mathbb{R}$, $\lambda \in \mathbb{C}$. As the byproduct of the Heisenberg equation (26) we get that operators

$$\hat{\Delta}(\lambda) = \mathrm{Tr}\,\hat{S}(\tau;\lambda), \quad \hat{\gamma}(\lambda) = \frac{1}{2}\mathrm{Tr}\,\hat{S}^2(\tau;\lambda) - \frac{1}{4}\hat{\Delta}^2(\lambda) \tag{27}$$

are conserved quantities for the quantized Neumann systems (19) on the sphere $\mathbf{S}^N$. Computing due to (27) the operator $\hat{\gamma}(\lambda)$, $\lambda \in \mathbb{C}$, we obtain that

$$\hat{\gamma}(\lambda) = -\hat{s}^2(\tau;\lambda) + \frac{1}{2}[\hat{s}_{12}(\tau;\lambda)\hat{s}_{21}(\tau;\lambda) + \hat{s}_{21}(\tau;\lambda)\hat{s}_{12}(\tau;\lambda)] \tag{28}$$

that is also the symmetrized law of the canonical operator ordering. In particular from (28) we recast the convenient form (21) after identification $\tilde{\gamma}_j \to \hat{\gamma}_j$, $j = \overline{1, N+1}$, for the conserved operator quantities of the quantum Neumann model:

$$\hat{\gamma}_j = q_j^2 + \sum_{j \neq k}^{N+1} (q_j p_k - q_k p_j)^2/(\omega_j - \omega_k). \tag{29}$$

The conserved quantities (29) are commuting [3] due to quantum operator relation (24) and as following of (22): for all $\lambda \neq \mu \in \mathbb{C}$, $\tau \in \mathbb{R}$

$$\begin{gathered} [\hat{s}_{12}(\tau;\lambda), \hat{s}(\tau;\mu)]\frac{i}{\hbar} = -\frac{2}{\lambda-\mu}[\hat{s}_{12}(\tau;\lambda) - \hat{s}_{12}(\tau;\mu)], \\ [\hat{s}(\tau;\lambda), \hat{s}_{21}(\tau;\mu)]\frac{i}{\hbar} = -\frac{2}{\lambda-\mu}[\hat{s}_{21}(\tau;\lambda) - \hat{s}_{21}(\tau;\mu)], \\ [\hat{s}_{12}(\tau;\lambda), \hat{s}_{12}(\tau;\mu)]\frac{i}{\hbar} = 0 = [\hat{s}(\tau;\lambda), \hat{s}(\tau;\mu)]\frac{i}{\hbar}. \end{gathered} \tag{30}$$

2.3. The corresponding to (19) quantum Neumann model is obtained by the following procedure. Let $\hat{H}$ be the quantum Hamiltonian of the form

$$\hat{H} = \frac{1}{4}\sum_{j \neq k}^{N+1}(q_j p_k - q_k p_j)^2 + \frac{1}{2}\sum_{j=1}^{N+1}\omega_j q_j^2. \tag{31}$$

Then the quantum Neumann dynamical system will obtain the following form

$$\begin{gathered} dq_j/d\tau = \frac{i}{\hbar}[\hat{H}, q_j] = p_j - q_j\sum_{k=1}^{N+1}(q_k p_k), \\ dp_j/d\tau = \frac{i}{\hbar}[\hat{H}, p_j] = -\omega_j q_j - q_j\sum_{k=1}^{N+1}p_k^2 + \sum_{k=1}^{N+1}(q_k p_k)p_j, \end{gathered} \tag{32}$$

where $j = \overline{1, N+1}$, $\tau \in \mathbb{R}$. In the standard Dirac type representation we may to recast the following operators

$$q_j \to q_j, \quad p_j \to \frac{\hbar}{i}\frac{\partial}{\partial q_j}, \tag{33}$$

acting in a dens domain in the Hilbert space $L_2(\mathbb{R}^{N+1};\mathbb{C})$, where for all $j, k = \overline{1, N+1}$ the conditions $[p_j, q_k]\frac{i}{\hbar} = \delta_{j,k}$ are satisfied obviously. The corresponding Hamiltonian (31) is a symmetric operator in $L_2(\mathbb{R}^{N+1}; \mathbb{C})$, that is the related spectrum of it is positive. Finally one easily verifies that the sphere $S^N = \left\{(q_1, q_2, \ldots, q_{N+1})^\tau \in \mathbb{R}^{N+1} : \sum_{j=1}^{N+1} q_j^2 = 1\right\}$ is invariant in relation to the quantum Neumann dynamical system (32), but the TS^N is not that, since the operator quantity $\sum_{j=1}^{N+1} (q_j p_j + p_j q_j) \neq 0$ for all $\tau \in \mathbb{R}$ in spite of a identity $\sum_{j=1}^{N+1} q_j^2 = 1$. But if the quantum parameter $\hbar$ tends to zero the classical Neumann dynamical system recasts of the (32) immediately. As the unsolved problem up to now one remains to find spectrum description of the Hamiltonian (31).

3. Quantization of the general Neumann type dynamical systems on the sphere. 3.1. The paper [6] contains the list of the new integrable by Liouville nonlinear dynamical systems of the Neumann type on the sphere S^N for any $N \in \mathbb{N}$, the Lax type formulation of its being found in a each case. The some of them are included also in the book [1].

Let us consider the following dynamical system of the Neumann-Bogoliubov type:

$$d^2 q_j/d\tau^2 + \omega_j^2 q_j = \omega_j q_j v(q, p) + u(q, p)\, q_j, \tag{34}$$

on the sphere $\mathbf{S}^N$, where $\omega_j \neq \omega_k \in \mathbb{R}$ if $j \neq k = \overline{1, N+1}$,

$$p_j = dq_j/d\tau, \quad v(q, p) = 2\sum_{j=1}^{N+1} \omega_j q_j^2 + \alpha_0,$$
$$u(q, p) = \sum_{j=1}^{N+1} \omega_j^2 q_j^2 - \frac{1}{2} v^2(q, p) + \frac{1}{2} v(q, p)\, \alpha_0 - \sum_{j=1}^{N+1} p_j^2, \tag{35}$$

$\alpha_0 \in \mathbb{R}$ being any constant quantity. It is a very light exercise to proof tha $\sum_{j=1}^{N+1} q_j^2 = 1$ is invariant via dynamical system (34). In order to pass in the quantization structure of the section 2 above, we recast the dynamical system (34) in the canonical Hamiltonian form on the phase manifold $M^{2N+2} = T^*\mathbb{R}^{N+1}$:

$$dq_j/d\tau = \{H, q_j\}_{(N+1)}, \quad dp_j/d\tau = \{H, p_j\}_{(N+1)}, \tag{36}$$

where $j = \overline{1, N+1}$, $(q, p)^\tau \in M^{2N+2}$ and on $\mathbf{S}^N$

$$H = \frac{1}{2}\sum_{j=1}^{N+1} p_j^2 + \frac{1}{2}\sum_{j=1}^{N+1} \omega_j^2 q_j^2 - \frac{1}{8} v^2(q, p). \tag{37}$$

Let us define the following functionals on the manifold M^{2N+2}:

$$\tilde{\gamma}_j = \sum_{j\neq k}^{N+1} (q_j p_k - q_k p_j)^2/(\omega_j - \omega_k) + q_j^2\left(\omega_j^2 - \alpha_0 - \sum_{k=1}^{N+1} \omega_k q_k^2\right). \tag{38}$$

Then we can compute that $H = \frac{1}{4}\sum_{j=1}^{N+1} \tilde{\gamma}_j$. This result is useful in further analysis. The full description [1, 6] of the dynamical system (34) is given by the following statement: the hierarchy of functionals (38) is invariant by means of the evolution (36) and commutative on the phase manifold $T^*\mathbf{S}^N \subset_{\rightarrow} M^{2N+2}$.

3.2. As is stated in [1], the associated to the dynamical system (36) Lax type representation has the form (10), where the matrix $\mathcal{A}(\tau;\lambda)$ is given by expression:

$$\mathcal{A}(\tau;\lambda)=\left\|\begin{array}{c|c} 0 & 1 \\ \hline u(q,p)+\lambda v(q,p)-\lambda^2 & 0\end{array}\right\| \tag{39}$$

for all $\lambda\in\mathbb{C}$ and $\tau\in\mathbb{R}$. Therefore the associated with (39) reduced monodromy matrix $\tilde{S}(\tau;\lambda)=\|\tilde{s}_{ij}(\tau;\lambda)\|_{i,j=\overline{1,2}}$ satisfies the commutation relations as ones in (9), the quantity

$$-\tilde{s}^2(\tau;\lambda)+\tilde{s}_{12}(\tau;\lambda)\tilde{s}_{21}(\tau;\lambda)=\tilde{\gamma}(\lambda) \tag{40}$$

being conservative for all $\lambda\in\mathbb{C}$ according to the evolution (36), and

$$\tilde{s}_{12}(\tau;\lambda)=\sum_{j=1}^{N+1}q_j^2/(\lambda-\omega_j),$$

$$\tilde{s}_{21}(\tau;\lambda)=\frac{1}{2}\,d\tilde{s}_{12}(\tau;\lambda)/d\tau^2-\tilde{s}_{12}(\tau;\lambda)[\lambda v(q,p)+u(q,p)-\lambda^2]=$$

$$=\sum_{j=1}^{N+1}p_j^2/(\lambda-\omega_j)+\sum_{j=1}^{N+1}\omega_j q_j^2+\lambda, \tag{41}$$

$$\tilde{s}(\tau;\lambda)=\frac{1}{2}\,d\tilde{s}_{12}(\tau;\lambda)/d\tau.$$

Now the canonical quantization procedure due to Dirac [5] on the manifold M^{2N+2} is used: $[p_j,q_k]\,\frac{i}{\hbar}=\delta_{jk}$ for $j,k=\overline{1,N+1}$, and $\tilde{s}_{mn}(\tau;\lambda)=$ $=:\tilde{s}_{mn}(\tau;\lambda)$: for $m\neq n=\overline{1,2}$, $\hat{S}(\tau;\lambda)=:\tilde{S}(\tau;\lambda):$, where $::$ is a special operator ordering on the space of polynomial symbols defined on M^{2N+2}. In particular we can state that for all $\lambda\in\mathbb{C}$

$$\hat{S}(\tau;\lambda)=\frac{1}{2}\sum_{j=1}^{N+1}(q_jp_j+p_jq_j)/(\lambda-\omega_j) \tag{42}$$

is the symmetrized operator ordering. Since the quantum monodromy matrix $\hat{S}(\tau;\lambda)=\|:\tilde{s}_{ij}(\tau;\lambda):\|_{i,j=\overline{1,2}}$ for all $\tau\in\mathbb{R}$, $\lambda\in\mathbb{C}$, satisfies the evolution Heisenberg type equation (26), we can to construct the operator $\hat{\gamma}(\lambda)=$ $=:\tilde{\gamma}(\lambda):=\sum_{j=1}^{N+1}\hat{\gamma}_j/(\lambda-\omega_j)$, where operators $\hat{\gamma}_j$, $j=\overline{1,N+1}$, are given exactly by the formula (38) after a standard identification.

3.3. There is the alternative procedure of quantization for the dynamical systems on the manifolds with nontrivial topological structure — the geometrical quantization [7]. Namely, if we are given the dynamical system on the manifold M^{2N} which is hamiltonian in relation to the symplectic structure $\omega^{(2)}\in\Lambda^2(M^{2N})$, $2N=\dim M^{2N}\in\mathbb{N}$, for any differentiable function $\gamma\in\mathcal{D}(M^{2N})$ we can build the predquantized operator $\hat{\gamma}\in\operatorname{Hom}(L_2)$, following the van Hove-Sigal expression:

$$\hat{\gamma}=\gamma+\frac{\hbar}{i}K_\gamma-\varphi^{(1)}(K_\gamma), \tag{43}$$

where by definition $i_{K_\gamma}\omega^{(2)}:=-d\gamma\in\Lambda^1(M^{2N})$ and $\varphi^{(1)}=d^{-1}\omega^{(2)}\in\Lambda^1(M^{2N})$ according to local Poincare lemma [2]; L_2 is

Hilbert space of a representation the algebra of quantized operators, the function $\gamma \in \mathcal{D}(M^{2N})$ in (43) being considered as the operator of a multiplication on the function in L_2. The map $\gamma: \rightarrow \hat{\gamma}$ (43) is named by a predquantization.

Let us consider the Hilbert space $L_2 := L_2(M^{2N}; \mathbb{C})$, where the scalar product is defined as folloms:

$$(f_1, f_2) = \int_{M^{2N}} (\omega^{(2)})^N f_1^* f_2 \tag{44}$$

for any $f_1, f_2 \in L_2$. Due to the condition (43) we have obtained the Lie algebra homomorfism of the Poisson commuting functions on M^{2N} and that of the commutator structurized algebra of the quantized operators of Hom (L_2).

But it is easy to state that the expression (43) not always determines the well posed operator of Hom (L_2), that is the second cohomology class of the symplectic form $\omega^{(2)} \in \Lambda^2(M^{2N})$ should be integer [7]. Correspondingly the Hilbert space L_2 transforms into the linear vector bundle L_2 with the manifold M^{2N} as a base. Moreover, the 1-differential form $\varphi^{(1)} = d^{-1}\omega^{(2)}$ in local charts is interpreted as the local expression for the connection ∇ in L_2. Namely, et us given the connection ∇ in L_2 in accordance with the scalar product (44), that is

$$K\langle f_1, f_2\rangle = \langle \nabla_K f_1, f_2\rangle + \langle f_1, \nabla_K f_2\rangle, \tag{45}$$

where $\langle \cdot, \cdot \rangle$ is hermitian structure on L_2, the formula $(f_1, f_2) := \int_{M^{2N}} (\omega^{(2)})^N \langle f_1, f_2\rangle$ being specified one in (44), $K: M^{2N} \rightarrow T(M^{2N})$ is any vector field on M^{2N}.

If the line bundle L_2 on a neighbourhood $U_\alpha \subset M$ admits a nonzero section $s_\alpha: M^{2N} \rightarrow L_2$, then the section space $\mathcal{I}(U_\alpha; L_2)$ is identified with the space $C^{(\infty)}(U_\alpha; \mathbb{C})$ due to the formula

$$C^{(\infty)}(U_\alpha; \mathbb{C}) \ni f \rightleftarrows f s_\alpha \in \mathcal{I}(U_\alpha; L_2). \tag{46}$$

The corresponding operator ∇_K acts as following

$$\nabla_K f = Kf - \frac{i}{\hbar}\varphi^{(1)}(K) f, \tag{47}$$

where the differential 1-form $\varphi^{(1)} \in \Lambda^1(M^{2N})$ is determined by the equality

$$\nabla_K s_\alpha = -\frac{i}{\hbar}\varphi_\alpha^{(1)}(K) s_\alpha. \tag{48}$$

Comparing the equalities (47) and (43) gives the following predquantization formula of Souriau-Kostant

$$\hat{\gamma} = \gamma + \frac{\hbar}{i}\nabla_{K_\gamma}. \tag{49}$$

It is observed easy of (49) that the curvature differential 2-form of the connection ∇ on L_2 is identified with the symplectic 2-form $\frac{1}{2\pi\hbar}\omega^{(2)} \in \Lambda^2(M^{2N})$, the cohomology class of the 2-form $\omega^{(2)}$ being integer. The quantization approach based on the formula (49) also gives the certain opoprtunity to describe the spectrum structure of the quantum Hamiltonian for the Neumann type dynamical systems as well as the structure of the eigenfunctions space of it.

1. *Integrable* dynamical systems / Yu. A. Mytropolsky, N. N. Bogoliubov (jr.), A. K. Prykarpatsky, V. G. Samoilenko.— K.: Naukova dumka, 1987.— 286 p. (in Russian).
2. *Prykarpatsky A., Mykytiuk I.* Algebraic aspects of integrabilily of nonlinear dynamical systems on the manifolds.— K. : Naukova dumka, 1991.— 286 p. (in Russian).

3. *Avan J., Talon M.* Alternative Lax structures for the classical and quantum Neumann model // Phys. Lett. B.— 1991.— **268**, N 2.— P. 209—216.
4. *Moser J.* Geometry of quadric and spectral theory // Proc. Chern Symp., 1979.— New York: Springer, 1980.— P. 147—188.
5. *Dirac P. A. M.* Principles of quantum mechanics.— Oxford, 1935. 300 p.
6. *Samoilenko V. Gr., Prykarpatsky A. K., Mykytiuk I. V.* Abelian integrals, integrable dynamical systems of the Neuman-Rosohatius type and Lax representation // Ukr. Math. J.— 1989.— **41**, N 8.— P. 1094—1100 (in Russian).
7. *Hurt N.* Geometric quantization in action.— Reidel, 1983.— 336 p.

Recieved 23.03.92