UDC 517.9

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A remark about a norm estimate for white noise distributions

Зауваження до оцінки норми розподілів білого шуму

The description of the space $(S)^*$ of white noise distributions is received in the terms of the S-transform well-known in the white noise analysis.

Одержано опис простору (S^*) узагальнених функцій від білого шуму в термінах відомого в аналізі білого шуму S-перетворення.

1. Introduction and main result. Let us consider the real Schwartz space $S(\mathbb{R})$ and its dual $S'(\mathbb{R})$ equipped with the σ-algebra $\mathcal{B}$ generated by its cylinder sets and with the white noise measure μ given by

$$C(f) = \int_{S'(\mathbb{R})} \exp(i\langle x, f\rangle)\, d\mu(x) = \exp\left(-\frac{1}{2}|f|_2^2\right)$$

for $f \in S(\mathbb{R})$. Here $|\cdot|_2$ denotes the norm $L^2(\mathbb{R})$ and $\langle\cdot,\cdot\rangle$ dual pairing. Below, we shall shortly recall the construction of the space $(S)^*$ of generalized functionals of white noise, i. e. generalized functionals on $S'(\mathbb{R})$, and some necessary facts from White Noise Analysis. For notation, definitions, more background and references, we refer the reader to [1—3].

Let $\mathcal{P}$ denote the algebra generated by the smooth linear functionals $L_f = \langle\cdot, f\rangle$, $f \in S(\mathbb{R})$. Consider the self-adjoint operator A on $L^2(\mathbb{R})$ which is the closure of

$$(Af)(t) = -f''(t) + (1+t^2) f(t), \quad t \in \mathbb{R}, \quad f \in S(\mathbb{R}).$$

For $p \geq 0$ let $S_p(\mathbb{R})$ denote the completion $S(\mathbb{R})$ with respect to the norm $|f|_{2,p} = = |A^p f|_2$. Then $S(\mathbb{R}) = \operatorname{pr}\lim_{p\in\mathbb{N}} S_p(\mathbb{R})$. The dual space $S_{-p}(\mathbb{R})$ corresponding to $S_p(\mathbb{R})$ is the completion of $S(\mathbb{R})$ with respect to the norm $|f|_{2,-p} = |A^{-p} f|_2$ and $S'(\mathbb{R}) = = \bigcup_{p\in\mathbb{N}} S_{-p}(\mathbb{R})$.

Every element $\varphi \in (L^2) \equiv L^2(S'(\mathbb{R}), \mathcal{B}, \mu)$ admits a chaos decomposition [2]

$$\varphi = \sum_{n=0}^{\infty} I_n(f^{(n)}) \tag{1}$$

where $f^{(n)} \in \hat{L}^2(\mathbb{R}^n)$, $\cdot$ denoting symmetrization, and $I_n(f^{(n)})$ is the multiple

Wiener integral of $f^{(n)}$ of order n. We have the equality

$$\|\varphi\|_2^2 = \sum_{n=0}^{\infty} n! \, |f^{(n)}|_2^2$$

($\|\cdot\|_2$ denoting the norm of (L^2)) from which we obtain the isomorphism between (L^2) and the «Fock space» $\bigoplus_{n=0}^{\infty} \hat{L}^2(\mathbb{R}^n, n!\, d^n t)$.

For $p \in \mathbb{N}_0$ let $(S)_p$ denote the Hilbert space which is the completion of $\mathcal{P}$ with respect to the norm

$$\|\varphi\|_{2,p}^2 = \sum_{n=0}^{\infty} n! \, |f^{(n)}|_{2,p}^2 = \sum_{n=0}^{\infty} n! \, (A^{\otimes n})^p f^{(n)}|_2^2. \tag{2}$$

Then we define the space (S) of test functionals of white noise as the projective limit of the family $\{(S)_p, p \in \mathbb{N}_0\}$.

The dual space $(S)_{-p}$ corresponding to $(S)_p$ by (2) is the completion of $\mathcal{P}$ with respect to the norm

$$\|\varphi\|_{2,-p}^2 = \sum_{n=0}^{\infty} n! \, |f^{(n)}|_{2,-p}^2 = \sum_{n=0}^{\infty} n! \, |(A^{\otimes n})^{-p} f^{(n)}|_2^2. \tag{3}$$

The space $(S)^*$ of white noise distributions is the dual to (S) and has the representation $(S)^* = \bigcup_{p \in \mathbb{N}_0} (S)_{-p}$.

On (L^2) we define the S-transform of an element φ by

$$(S\varphi)(f) = \int \varphi(x + f) \, d\mu(x), \quad f \in S(\mathbb{R}).$$

Note that we have the formula

$$(S\varphi)(f) = \int \varphi(x) : \exp\langle x, f\rangle : d\mu(x)$$

where we have set

$$: \exp\langle x, f\rangle : = \exp(\langle x, f\rangle - \frac{1}{2}|f|_2^2).$$

For $\varphi \in (L^2)$ with the chaos decomposition (1) one has

$$(S\varphi)(f) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f^{(n)}(t_1, \dots, t_n) f(t_1) \dots f(t_n) \, dt_1 \dots dt_n \tag{4}$$

as an easy direct computation shows.

Since, for all $\lambda \in C$ and $f \in S(\mathbb{R})$, $: \exp\langle \lambda f, \cdot\rangle : \in (S)$, we may extend the S-transform to $(S)^*$ by the following dual pairing of $\Phi \in (S)^*$

$$(S\Phi)(\lambda f) = \langle\langle \Phi, : \exp \lambda \langle f, \cdot\rangle : \rangle\rangle, \quad \lambda \in \mathbb{C}, \quad f \in S(\mathbb{R}).$$

It is not hard to see that for any $\Phi \in (S)^*$ the functional $\Psi(f) = (S\Phi)(f)$, $f \in S(\mathbb{R})$ is ray entire on $S(\mathbb{R})$, i. e. for every $f, g \in S(\mathbb{R})$ the function $\Psi(\lambda f + g)$, $\lambda \in \mathbb{C}$ is entire analytic. Moreover, there exist $p \in \mathbb{N}_0$ and $C > 0$, $K > 0$ such that for all $\lambda \in \mathbb{C}$, $f \in S(\mathbb{R})$

$$|\Psi(\lambda f)| \leqslant C \, e^{K|\lambda|^2 |f|_{2,p}^2}. \tag{5}$$

In [2] a ray entire Ψ with the estimate (5) on its growth for some $p \in \mathbb{N}_0$ and $C, K > 0$ was named $\mathcal{U}$-functional. The main result of [2] was the fact that any element in $(S)^*$ has an S-transform which is a $\mathcal{U}$-functional, and conversely, for any $\mathcal{U}$-functional Ψ there is a unique element $\Phi \in (S)^*$ with $S\Phi = \Psi$, i. e. $\Phi = S^{-1}$T. A similar result (in terms of the inequality (11) below) had been obtained in [4], see also [5, Ch. 2] and [6].

The main result of the paper is an estimate for the **norm** $\|\Phi\|_{2,-q}$ of the distribution $\Phi = S^{-1}\Psi \in (S)^{*} = \bigcup_{q\in\mathbb{N}_0} (S)_{-q}$ by the coefficients C and K from (5).

Theorem. *Let Ψ be a ray entire functional on $S(\mathbb{R})$ such that for some $C > 0$, $K > 0$ and $p \in \mathbb{N}_0$ we have the estimate* (5) *and let $m \in \mathbb{N}_0$ be such that $e^2K < 2^{2m-1}$. Then for any $q \in \mathbb{N}_0$, $q \geqslant p + m + 1$ we have the inclusion*

$$\Phi = S^{-1}\Psi \in (S)_{-q}$$

and

$$\|\Phi\|_{2,-q} \leqslant \frac{C}{\sqrt{1 - \frac{e^2K}{2^{2m-1}}}}. \tag{6}$$

Remark. It is not hard to see that by Theorem we obtain the inclusion $\Phi \in (S)_{-q}$ for any $q > \frac{2 + \ln K}{2\ln 2} + p + \frac{3}{2}$.

Alternatively to the S-transform one frequently considers the «$\mathscr{T}$-transform»

$$(\mathscr{T}\Phi)(f) = \langle\langle \Phi, e^{i\langle\cdot, f\rangle}\rangle\rangle, \quad f \in S(\mathbb{R}) \tag{7}$$

(see e. g. [3]). These are again $\mathcal{U}$-functionals, since

$$(\mathscr{T}\Phi)(f) = (S\Phi)(if)\,C(f), \quad f \in S(\mathbb{R}). \tag{8}$$

As a consequence we have the obvious but useful the next Corollary.

Corollary. *Let Ψ be a ray entire functional on $S(\mathbb{R})$ such that for some $C > 0$, $K > 0$ and $p \in \mathbb{N}_0$ we have the estimate* (5) *and let $m \in \mathbb{N}_0$ be such that $e^2(K + \frac{1}{2}) < 2^{2m-1}$. Then for any $q \in \mathbb{N}_0$, $q \geqslant p + m + 1$ we have the inclusion*

$$\Phi = \mathscr{T}^{-1}\Psi \in (S)_{-q}$$

and

$$\|\Phi\|_{2,-q} \leqslant \frac{C}{\sqrt{1 - \frac{e^2\left(K + \frac{1}{2}\right)}{2^{2m-1}}}}. \tag{9}$$

2. Proof of Theorem. Because $A \geqslant 2$ from (5) we obtain the estimate

$$|\Psi(\lambda f)| \leqslant Ce^{K|\lambda|^2|f|^2_{2,p}} \leqslant Ce^{2^{-2m}K|\lambda|^2|f|^2_{2,p+m}}$$
$$f \in S(\mathbb{R}), \quad \lambda \in \mathbb{C}. \tag{10}$$

In [4, 5, Ch. 2] it had been proved that the functional Ψ can be extended onto the complexification $S_{p+m,c}(\mathbb{R})$ of the Hilbert space $S_{p+m}(\mathbb{R})$ with the estimate of growth

$$|\Psi(z)| \leqslant Ce^{K/2^{2m}|z|^2_{2,p+m}} \quad (z \in S_{p+m,c}(\mathbb{R})), \tag{11}$$

where $|z|^2_{2,p+m}$ is the norm of $S_{p+m,c}(\mathbb{R})$.

The functional Ψ has Taylor decomposition

$$\Psi(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\, d^n\Psi(0)(z), \; z \in S_{p+m,c}(\mathbb{R}), \tag{12}$$

where $d^n\Psi(0)\;(z)$ is the n-th differential of Ψ in the point $0 \in S_{p+m,c}(\mathbb{R})$ [7].

For any $p \in \mathbb{N}$ the embedding operator $i_{p+1,p} : S_{h+1}(\mathbb{R}) \to S_p(\mathbb{R})$ belongs to the Hilbert — Schmidt class. By the kernel theorem we have the representation for any $n \in \mathbb{N}_0$

$$d^n \Psi(0)(z) = \langle \Psi^{(n)}, z^{\otimes n} \rangle \, (z \in S_{p+m+1,c}(\mathbb{R}))$$

with kernels $\Psi^{(n)} \in S_c'(\mathbb{R}^n)$ such that

$$|(A^{\otimes n})^{-(p+m+1)} \Psi^{(n)}|_2 < \infty$$

(e. g. [5, Ch. 1]).

Let us denote $F^{(n)} = \frac{1}{n!} \Psi^{(n)}, \quad n \in \mathbb{N}_0$. Then

$$\Psi(z) = \sum_{n=0}^{\infty} \langle F^{(n)}, z^{\otimes n} \rangle, \quad z \in S_c(\mathbb{R})$$

and due to the definition of the S-transform we must prove that the distribution

$$\Phi = S^{-1}\Psi = \sum_{n=0}^{\infty} I_n(F^{(n)})$$

belongs to $(S)_{-q}$ for all $q \geqslant m + p + 1$.

L e m m a. *For every $q \geqslant p + m + 1$, $n \in \mathbb{N}$ one has the estimate.*

$$\left| \frac{1}{n!} \Psi^{(n)} \right|_{2,-q} \leqslant \frac{C}{\sqrt{n!}} (2a)^{n/2}, \tag{13}$$

where we denote $a = 2^{-m} e^2 K > 0$.

P r o o f. By the fact that $|i_{p+m+1,p+m}| < 1$ we have the inequality

$$|(A^{\otimes n})^{-(p+m+1)} \Psi^{(n)}|_2 \leqslant \| d\Psi^{(n)}_{(0)} \|_{\mathscr{L}^n(S_{p+m}(\mathbb{R}))},$$

where $\| d\Psi^{(n)}_{(0)} \|_{\mathscr{L}^n(S_{p+m}(\mathbb{R}))}$ is the norm of the n-linear form $d\Psi^{(n)}(0)(z)$, $z \in S_{p+m,c}(\mathbb{R})$ (see e. g. the proof of Theorem 5.3 in [5, Ch. 2]). By the Cauchy inequality for the entire functional Ψ on $S_{p+m,c}(\mathbb{R})$ we obtain [7]

$$\frac{1}{n!} \| d\Psi^{(n)}_{(0)} \|_{\mathscr{L}^n(S_{p+m}(\mathbb{R}))} \leqslant \frac{n^n}{n!} \frac{1}{R^n} \sup_{|z|_{2,p+m} = R} |\Psi(z)| \leqslant \frac{n^n}{n!} \frac{1}{R^n} C e^{2^{-m} K R^2}.$$

By minimizing the last expression with respect to $R > 0$ we get

$$\frac{1}{n!} |\Psi^{(n)}|_{2,-q} \leqslant \frac{1}{n!} |\Psi^{(n)}|_{2,-(p+m+1)} \leqslant \frac{C}{\sqrt{n!}} (2a)^{n/2}, \quad n \in \mathbb{N}.$$

By using the Lemma we have for the norm $\|\cdot\|_{2,-q}$ of Φ the estimate (see (3))

$$\|\Phi\|^2_{2,-q} = \sum_{n=0}^{\infty} n! \, |F^{(n)}|_{2,-q} \leqslant C^2 \sum_{n=0}^{\infty} (2a)^n = \frac{C^2}{1 - e^2 K / 2^{m-1}}.$$

R e m a r k. The main result is the norm estimate (6). The inclusion in $(S)_{-q}$, $q > \frac{2 + \ln K}{2 \ln 2} + p + \frac{3}{2}$ is not optimal. A simple application of Schwartz' inequality [8] gives $q > \frac{\ln K}{2 \ln 2} + p + 1$. Other estimates for q have been given in [6, 9].

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Received 17.09.91