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Contitional symmetry and reduction of partial differential equations

Умовна симетрія і редукція диференціальних рівнянь з частинними похідними

We establish sufficient conditions of reduction of partial differential equations admitting nontrivial conditional symmetry. The results obtained generalize the classical conditions of reduction of differential equations by using group-invariant solutions. Some examples of reduction of systems of partial differential equations both by number of independent and dependent variables are considered.

Встановлені достатні умови редукції диференціальних рівнянь з частинними похідними, які мають нетривіальну умовну симетрію. Одержані результати узагальнюють класичні умови редукції диференціальних рівнянь за допомогою інваріантно-групових розв'язків. Розглянуто ряд прикладів редукції систем диференціальних рівнянь з частинними похідними за числом незалежних і залежних змінних.

Analysing the already-known methods of construction of exact solutions of nonlinear partial differential equations (PDE) (such as the methods of group-theoretical reduction [1, 2], differential constraints [3], ansatzes [4—6]) we came to conclusion that the majority of them was based on the idea of narrowing the set of solutions, i. e. choosing from the whole set of solutions of the equation under study specific subsets that admitted analytical description. To realize this idea one has to impose on the set of solutions some additional constraints (equations) picking out such subsets. Clearly, additional equations are supposed to be simpler than the initial one. Supplementing the initial equation with additional conditions we come, as a rule, to over-determined system of PDE. Consequently, there arises a problem of investigating the matter of its compatibility. Another restriction on the choice of additional conditions is that the resulting system of PDE has to have wider (or another) symmetry than the initial equation has.

In the present paper we establish sufficient conditions of reduction of differential equations which generalize the classical conditions of reduction of PDE admitting non-trivial Lie transformation group. The subject of the study is an over-determined system of PDE

$$U_A(x, u, \underset{1}{u}, \ldots, \underset{r}{u}) = 0, \quad A = \overline{1, M}, \tag{1}$$

$$\xi_{a\mu}(x, u)\, u^{\alpha}_{x_\mu} - \eta^{\alpha}_{a}(x, u) = 0, \quad a = \overline{1, N}, \tag{2}$$

where $x = (x_0, x_1, \ldots, x_{n-1})$, $u(x) = (u^0(x), \ldots, u^{m-1}(x))$, $\underset{s}{u} = \{\partial^s u^\alpha/\partial x_{\mu_1} \ldots \partial x_{\mu_s},\ 0 \leqslant \mu_i \leqslant n-1\}$, $s = \overline{1, r}$, U_A, $\xi_{a\mu}$, η^α_a are smooth enough functions, $N \leqslant n-1$.

Hereafter the summation over repeated indices is understood. Let us introduce designations

$$R_1 = \operatorname{rank} \| \xi_{a\mu}(x, u) \|_{a=1\ \mu=0}^{N\ n-1},$$

$$R_2 = \operatorname{rank} \| \xi_{a\mu}(x, u), \quad \eta^\alpha_a(x, u) \|_{a=1\ \mu=0\ \alpha=0}^{N\ n-1\ m-1}.$$

It is evident that inequality $R_1 \leqslant R_2$ holds. We shall prove that the case $R_1 = R_2$ leads to reduction of PDE (1) by the number of independent variables and the case $R_1 < R_2$ — to reduction of PDE (1) by the number of independent and dependent variables.

1. Reduction of PDE by the number of independent variables. In this point we suppose that the equality $R_1 = R_2$ holds.

Definition 1. *The set of the first-order differential operators*

$$Q_a = \xi_{a\mu}(x, u)\, \partial_{x_\mu} + \eta^\alpha_a(x, u)\, \partial_{u^\alpha}, \tag{3}$$

where $\partial_{x_\mu} = \partial/\partial x_\mu$, $\partial_{u^\alpha} = \partial/\partial u^\alpha$; $\xi_{\alpha\mu}$, η^α_a are smooth functions, is called involutive one, provided there exist such functions $f^c_{ab}(x, u)$, that

$$[Q_a, Q_b] = f^c_{ab} Q_c, \quad a, b = \overline{1, N}. \tag{4}$$

Here $[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1$.

The simplest example of the involutive set of operators is a Lie algebra.

It is common knowledge that conditions (4) provide compatibility of over-determined system of PDE (2) (the Frobenius theorem [7]). The general solution of system (2) is given by formulas

$$F^\alpha(\omega_1, \omega_2, \ldots, \omega_{n+m-R_1}) = 0, \quad \alpha = \overline{0, m-1}, \tag{5}$$

where $\omega_j = \omega_j(x, u)$ are functionally-independent first integrals of system of PDE (2), F_α are arbitrary smooth functions.

By force of the condition $R_1 = R_2$, one can choose first integrals (say, $\omega_1, \ldots, \omega_m$) satisfying the following condition:

$$\det \| \partial\omega_j/\partial u^\alpha \|_{j=1\ \alpha=0}^{m\ m-1} \neq 0. \tag{6}$$

On resolving relations (5) with respect to ω_j, $j = \overline{1, m}$, we have

$$\omega_j = \varphi_j(\omega_{m+1}, \omega_{m+2}, \ldots, \omega_{m+n-R_1}), \quad j = \overline{1, m}, \tag{7}$$

where φ_j are arbitrary smooth functions.

D e f i n i t i o n 2. *Expression* (7) *is called the ansatz for field* $u^\alpha = u^\alpha(x)$ *invariant under the involutive set to operators* (3) *provided relation* (6) *holds.*

Formulas (7) take especially simple and clear form if

$$\partial \xi_{a\mu}/\partial u^\alpha = 0, \quad \eta_a^\alpha = f_a^{\alpha\beta}(x)\, u^\beta,$$

$$a = \overline{1, N}, \quad \mu = \overline{0, n-1}, \quad \alpha, \beta, \gamma = \overline{0, m-1}. \tag{8}$$

Under conditions (8) operators (3) can be rewritten in non-Lie form [8]

$$Q_a = \xi_{a\mu}(x)\, \partial_{x_\mu} + \eta_a(x), \quad a = \overline{1, N}, \tag{9}$$

where $\eta_a = \| -\partial \eta_a^\alpha/\partial u^\beta \|_{\alpha,\beta=0}^{m-1}$ are $(m \times m)$-matrices, system (2) taking the form

$$\xi_{a\mu}(x)\, u_{x_\mu} + \eta_a(x)\, u = 0, \quad a = \overline{1, N}. \tag{10}$$

Here $u = (u^0, u^1, \dots, u^{m-1})^T$ is the function-column.

In such a case, the set of functionally-independent first integrals of system (2) with $R_1 = R_2$ can be chosen as follows [7]

$$\omega_j = b_{j\alpha}(x)\, u^\alpha, \quad j = \overline{1, m}, \tag{11}$$

$$\omega_i = \omega_i(x), \quad i = \overline{m+1, m+n+R_1}$$

and what is more $\det \|b_{j\alpha}(x)\|_{j=1}^{m}{}_{\alpha=0}^{m-1} \neq 0$.

Substituting (11) into (7) and resolving with respect to the variables u^α, $\alpha = \overline{0, m-1}$, we have

$$u^\alpha = A^{\alpha\beta}(x)\, \varphi^\beta(\omega_{m+1}, \omega_{m+2}, \dots, \omega_{m+n-R_1})$$

or (in the matrix notation)

$$u = A(x)\, \varphi(\omega_{m+1}, \omega_{m+2}, \dots, \omega_{m+n-R_1}). \tag{12}$$

It is not difficult to verify that the matrix

$$A(x) = (\| b_{j\alpha}(x) \|_{j=1}^{m}{}_{\alpha=0}^{m-1})^{-1}$$

satisfies system of PDE

$$Q_a A \equiv \xi_{a\mu}(x)\, A_{x_\mu} + \eta_a(x)\, A = 0, \quad a = \overline{1, N}, \tag{13}$$

while the functions $\omega_{m+1}(x), \omega_{m+2}(x), \dots, \omega_{m+n-R_1}(x)$ form the complete set of functionally-independent first integrals of system of PDE

$$\xi_{a\mu}(x)\, \omega_{x_\mu} = 0, \quad a = \overline{1, N}. \tag{14}$$

We say that ansatz (7) reduces system of PDE (1) if substitution of formulas (7) into (1) yields system of PDE for functions $\varphi^0, \varphi^1, \dots, \varphi^{m-1}$ that contains only new independent variables $\omega_{m+1}, \omega_{m+2}, \dots, \omega_{m+n-R_1}$.

D e f i n i t i o n 3. *System of PDE* (1) *is conditionally-invariant under the involutive set of differential operators* (3) *provided over-determined system of PDE* (1), (2) *is invariant, in Lie's sense, under the one-parameter transformation groups having generators* Q_a, $a = \overline{1, N}$.

Before formulating the reduction theorem, we shall prove some auxiliary assertions.

L e m m a 1. *Suppose that operators* (3) *form an involutive set. Then the set of differential operators*

$$Q'_a = \lambda_{ab}(x)\, Q_b, \quad a = \overline{1, N}, \tag{15}$$

with $\det \| \lambda_{ab}(x, u) \|_{a,b=1}^{N} \neq 0$ *is also involutive one.*

Proof is carried out by direct computation. Really,

$$[Q'_a, Q'_b] = [\lambda_{ac}Q_c, \lambda_{bd}Q_d] = \lambda_{ac}(Q_c\lambda_{bd})Q_d - \lambda_{bd}(Q_d\lambda_{ac})Q_c + + \lambda_{ac}\lambda_{bd}f^{d_1}_{cd}Q_{d_1} = \tilde{f}^c_{ab}Q_c = \tilde{f}^c_{ab}\lambda^{-1}_{cd}Q'_d.$$

Here λ^{-1}_{cd} are the elements of the inverse of the matrix $\|\lambda_{ab}(x, u)\|^N_{a,b=1}$.

L e m m a 2. *Let differential operators* (3) *satisfy the condition $R_1 = R_2$ and besides the conditions*

$$[Q_a, Q_b] = 0, \quad a, b = \overline{1, N} \tag{16}$$

hold. Then there exists the change of variables

$$x'_\mu = f_\mu(x, u), \quad \mu = \overline{0, n-1}, \qquad u'^\alpha = g^\alpha(x, u), \quad \alpha = \overline{0, m-1} \tag{17}$$

reducing operators Q_a to the form $Q'_a = \partial_{x'_{a-1}}$.

P r o o f. It is known that for any first-order differential operator

$$Q = \xi_\mu(x, u)\,\partial_{x_\mu} + \eta^\alpha(x, u)\,\partial_{u^\alpha},$$

where ξ_μ, η^α are smooth enough functions, there exists change of variables (17) reducing it to the form $Q' = \partial_{x'_0}$ (see, for example, [1]). Consequently, the operator Q_1 from the set (3) with the change of variables (17) is reduced to the form $Q'_1 = \partial_{x'_0}$. From conditions $[Q_1, Q_a] = 0$, $a = \overline{2, N}$, it follows that the coefficients of operators $Q'_2, Q'_3, \ldots, Q'_N$ do not depend on the variable x'_0. That is why the operator Q'_2 with the change of variables

$$x''_0 = x'_0,$$

$$x''_\mu = f'_\mu(x'_1, \ldots, x'_{n-1}, u'), \ \mu = \overline{1, n-1},$$

$$u''^\alpha = g'^\alpha(x'_1, \ldots, x'_{n-1}, u'), \quad \alpha = \overline{0, m-1}$$

not changing the form of the operator Q'_1 is reduced to the operator $Q''_2 = \partial_{x''_1}$.

On repeating the above procedure $(N-2)$ times we complete the proof.

L e m m a 3. *System of PDE of the form* (1), *conditionally-invariant under the set of differential operators ∂_{x_μ}, $\mu = \overline{0, N-1}$, has the following structure*:

$$U_A = F_{AB}W_B(x_N, x_{N+1}, \ldots, x_{n-1}, u, \underset{1}{u}, \ldots, \underset{r}{u}) + + F^\alpha_{A\mu}u^\alpha_{x_\mu}, \ A = \overline{1, M}, \ \alpha = \overline{0, m-1}, \mu = \overline{0, N-1}, \tag{18}$$

where F_{AB}, $F^\alpha_{A\mu}$ are arbitrary smooth functions on $x, u, \underset{1}{u}, \ldots, \underset{r}{u}$, W_B are arbitrary smooth functions and besides $\det\|F_{AB}\|^M_{A,B=1} \neq 0$.

We shall prove the lemma under $N = 1$. By force of the definition 3, system (1) is conditionally-invariant under the operator $Q = \partial_{x_0}$ if the system

$$U_A(x, u, \underset{1}{u}, \ldots, \underset{r}{u}) = 0, \quad A = \overline{1, M}, \tag{19}$$
$$u^\alpha_{x_0} = 0, \quad \alpha = \overline{1, m-1}$$

is invariant in Lie's sense under the one-parameter translation group with respect to the variable x_0. After denoting by the symbol $\tilde{Q}$ the r-th prolongation of the operator Q, the Lie criteria of the invariance of system of PDE (19) under this group reads (see, for example, [1, 2])

$$\tilde{Q}U_A\Big|_{\substack{U_B=0\\ u^\alpha_{x_0}=0}} = 0, \quad A, B = \overline{1, N}, \quad \alpha = \overline{0, m-1}, \tag{19a}$$

$$\tilde{Q}u^{\alpha}_{x_0}\Big|_{\substack{U_B=0\\ u^{\beta}_{x_0}=0}} = 0, \quad B=\overline{1, N}, \quad \alpha, \beta = \overline{0, m-1}\,. \tag{19b}$$

As a direct computation shows, the relations

$$\tilde{Q} \equiv \partial_{x_0}, \quad \tilde{Q}u^{\alpha}_{x_0} \equiv \partial_{x_0}(u^{\alpha}_{x_0}) = 0$$

hold (let us recall that in the prolonged space $x, u, \underset{1}{u}, \ldots, \underset{r}{u}$ the variables x_0 and $u^{\alpha}_{x_0}$ are independent) that is why, using the undefined coefficients method we can rewrite (19a), (19b) in the form

$$\partial U_A/\partial x_0 = R_{AB}U_B + P^{\alpha}_{A}u^{\alpha}_{x_0}, \quad A = \overline{1, M}\,, \tag{19c}$$

where R_{AB}, P^{α}_{A} are arbitrary smooth functions on $x, u, \underset{1}{u}, \ldots, \underset{r}{u}$.

System (19 c) can be considered as a system of inhomogeneous ordinary differential equations for the functions U_A, $A=\overline{1, M}$. Integrating (19 c) with $P^{\alpha}_{A} = 0$, we have

$$U^{(0)}_{A} = F_{AB}W_B, \quad A = \overline{1, M},$$

where W_B, $B=\overline{1, M}$, are arbitrary smooth functions on $x_1, x_2, \ldots, x_{n-1}, u, \underset{1}{u}, \ldots, \underset{r}{u}$; $F = \| F_{AB} \|^{M}_{A,B=1}$ is the fundamental matrix of system (19 c) (which, as is well-known, satisfies the condition $\det F \neq 0$).

Further, applying the method of variation of arbitrary constant we get the formula (18) with $N = 1$, where

$$F^{\alpha}_{A0} = F_{AB} \int (F)^{-1}_{BC} P^{\alpha}_{C} dx_0, \quad A = \overline{1, M}, \quad \alpha = \overline{0, m-1}.$$

The lemma is proved.

Theorem 1. *Let system of PDE* (1) *be conditionally-invariant under the involutive set of operators* (3). *Then the ansatz invariant under the set of operators* (3) *reduces this system.*

Proof. By definition of the quantity R_1 the inequality $R_1 \leqslant N$ holds. We denote by the symbol δ the difference $N - R_1$. Then R_1 equations of system (2) are linearly-independent (without loosing generality, we can suppose that the first R_1 equations are linearly-independent) and the rest δ equations are their linear combinations.

By force of the condition $R_1 = R_2$, there exists such non-singular $(R_1 \times \times R_1)$-matrix $\| \lambda_{ab}(x, u) \|^{R_1}_{a,b=1}$ that

$$\lambda_{ab}(\xi_{b\mu}u^{\alpha}_{x_\mu} - \eta^{\alpha}_{b}) = u^{\alpha}_{x_{a-1}} + \sum_{\mu=R_1}^{n-1} \tilde{\xi}_{a\mu}u^{\alpha}_{x_\mu} - \tilde{\eta}^{\alpha}_{a}, \quad a = \overline{1, R_1}, \quad \alpha = \overline{0, m-1}.$$

By definition of the conditional invariance, system of PDE (1), (2) is invariant under the one-parameter transformation groups having the generators (3). That is why, the equivalent system of PDE

$$U_A(x, u, \underset{1}{u}, \ldots, \underset{r}{u}) = 0, \quad A = \overline{1, M},$$

$$u^{\alpha}_{x_{a-1}} + \sum_{\mu=R_1}^{n-1} \tilde{\xi}_{a\mu}u^{\alpha}_{x_\mu} - \tilde{\eta}^{\alpha}_{a} = 0, \quad a = \overline{1, R_1}, \quad \alpha = \overline{0, m-1} \tag{20}$$

is invariant under the one-parameter group having the generators

$$Q'_a = \lambda_{ab}Q_b = \partial_{x_{a-1}} + \sum_{\mu=R_1}^{n-1} \tilde{\xi}_{a\mu}\partial_{x_\mu} + \tilde{\eta}^{\alpha}_{a}\partial_{u^{\alpha}}\,. \tag{21}$$

Really, the action of the one-parameter transformation group having the infinitesimal operator Q_a on solution manifold of system (20) is equivalent to the identity transformation.

As the set of operators (21) is involutive (the lemma 1), there exist such functions $f^c_{ab}(x, u)$ that

$$[Q'_a, Q'_b] = f^c_{ab} Q'_c, \quad a, b, c = \overline{1, R_1}. \tag{22}$$

Computating commutators in the left parts of equalities (22) and equating coefficients of linearly-independent operators $\partial_{x_0}, \partial_{x_1}, \ldots, \partial_{x_{R_1-1}}$, we have $f^c_{ab} = 0$, $a, b, c = \overline{1, R_1}$. Consequently, the operators Q'_a commute. Hence, by force of the lemma 2, it follows that there exists a change of variables (17) reducing these operators to the form $Q''_a = \partial/\partial x'_{a-1}$.

In the new variables x', $u'(x')$ system (20) reads

$$U'_A(x', u', \underset{1}{u'}, \ldots, \underset{r}{u'}) = 0, \quad A = \overline{1, M},$$

$$u'^{\alpha}_{x'_{a-1}} = 0, \quad \alpha = \overline{0, m-1}, \quad a = \overline{1, R_1}. \tag{23}$$

And besides, system of PDE (23) is conditionally-invariant under the set of operators $Q''_a = \partial_{x'_{a-1}}$, $a = \overline{1, R_1}$. That is why, by force of the lemma 3 system (23) is rewritten in the form

$$U'_A = F_{AB} W_B(x'_{R_1}, \ldots, x'_{n-1}, \underset{r}{u'}, \underset{1}{u'}, \ldots, \underset{r}{u'}) +$$

$$+ F^{\alpha}_{A\mu} u'^{\alpha}_{x'_{\mu}}, \quad A = \overline{1, M}, \quad \alpha = \overline{0, m-1}, \quad \mu = \overline{0, R_1 - 1},$$

$$u'^{\alpha}_{x'_{a-1}} = 0, \quad \alpha = \overline{0, m-1}, \quad a = \overline{1, R_1},$$

where $\det \| F_{AB} \|^{R_1}_{A,B=1} \neq 0$, whence

$$W_A(x'_{R_1}, \ldots, x'_{n-1}, u', \underset{1}{u'}, \ldots, \underset{r}{u'}) = 0, \tag{24}$$

$$u'^{\alpha}_{x'_{a-1}} = 0, \quad A = \overline{1, R_1}, \quad \alpha = \overline{0, m-1}, \quad a = \overline{1, R_1}.$$

The ansatz for field $u'^{\alpha} = u'^{\alpha}(x')$ invariant under the involutive set of operators $Q''_a = \partial_{x'_{a-1}}$, $a = \overline{1, R_1}$, is given by the following formulas:

$$u'^{\alpha} = \varphi^{\alpha}(x'_{R_1}, x'_{R_1+1}, \ldots, x'_{n-1}), \quad \alpha = \overline{0, m-1}. \tag{25}$$

Here φ^{α} are arbitrary smooth enough functions.

Substituting expressions (25) into (24), we get

$$W_A(x'_{R_1}, \ldots, x'_{n-1}, u', \underset{1}{u'}, \ldots, \underset{r}{u'}) \equiv W'_A(x'_{R_1}, \ldots, x'_{n-1}, \varphi, \underset{1}{\varphi}, \ldots, \underset{r}{\varphi}) = 0, \tag{26}$$

where $\underset{s}{\varphi}$ is the set of partial derivatives of the functions $\varphi^{\alpha} = \varphi^{\alpha}(x'_{R_1}, \ldots$ $\ldots, x_{n-1})$ of the order s.

Rewritting the ansatz (25) in the initial variables x, $u(x)$

$$g^{\alpha}(x, u) = \varphi^{\alpha}(f_{R_1}(x, u), \ldots, f_{n-1}(x, u)), \quad \alpha = \overline{0, m-1}, \tag{27}$$

we get the ansatz for field $u^{\alpha} = u^{\alpha}(x)$, $\alpha = \overline{0, m-1}$, invariant under the involutive set of operators (3), that reduces system (1) to system of PDE with $n - R_1$ independent variables. Theorem is proved.

C o n s e q u e n c e. *Let the operators*

$$Q_a = \xi_{a\mu}(x, u)\, \partial_{x_{\mu}} + \eta^{\alpha}(x, u)\, \partial_{u^{\alpha}}, \quad a = \overline{1, N}, \quad N \leqslant n - 1$$

be basis elements of a subalgebra of the invariance algebra of system of equations (1) *and besides the condition* $R_1 = R_2$ *holds. Then the ansatz invariant under the*

Lie algebra $\langle Q_1, Q_2, \ldots, Q_N\rangle$ reduces system (1) *to system of PDE having $n - N$ independent variables.*

P r o o f. From the definition of the Lie algebra it follows that operators Q_a satisfy (4) with $f^c_{ab} = \text{const}$. Consequently, they form an involutive set of the first-order differential operators. That is why, the above assertion is the direct consequence of the theorem 1.

By force of the above proved assertion, the classical theorem about reduction of differential equations by using group-invariant solutions [1, 2, 9] is the particular case of the theorem 1. Provided one of the operators Q_a do not belong to the invariance algebra of the equation under study and conditions of the theorem 1 hold, we have the reduction via Q_a conditionally-invariant ansatzes (numerous examples of conditionally-invariant solutions were constructed in [4—6, 10—14]).

In the following, we shall consider some examples.

E x a m p l e 1. The maximal, in Lie's sense, invariance algebra of the Schrödinger equation

$$\triangle_3 u + U(\vec{x}^2)\, u = 0 \tag{28}$$

with arbitrary function U is the Lie algebra of the rotation group having the following basis elements:

$$J_{ab} = x_a \partial_{x_b} - x_b \partial_{x_a}, \quad a, b = \overline{1, 3}. \tag{29}$$

To obtain the ansatz invariant under the set of operators (29) one has to construct the complete set of the first integrals of system of PDE

$$x_a u_{x_b} - x_b u_{x_a} = 0, \quad a, b = \overline{1, 3}. \tag{30}$$

The above set contains $3 - R_1$ functionally-invariant first integrals, where

$$R_1 = \text{rank}\, \| \xi_{ab}(x) \|^3_{a,b=1} = \text{rank} \begin{Vmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{Vmatrix} = 2.$$

Consequently, the ansatz for field $u = u(\vec{x})$ invariant under the Lie algebra having basis elements (29) has the form

$$u(\vec{x}) = \varphi(\omega), \tag{31}$$

where $\varphi \in C^2(\mathbb{R}^1, \mathbb{C}^1)$ is an arbitrary smooth functions, $\omega = \omega(\vec{x})$ is the first integral of system of PDE (30). It is not difficult to become convinced of that $\omega = \vec{x}^2$ satisfies (30) and, consequently, is the first integral. Substitution of (31) into (28) yields the ordinary differential equation for the function $\varphi(\omega)$

$$4\omega\ddot{\varphi} + 6\dot{\varphi} + U(\omega)\,\varphi = 0.$$

Thus, the ansatz for field $u = u(\vec{x})$ invariant under the three-dimensional Lie algebra having the basis elements (29), reduces equation (28) to $(3 - R_1)$-dimensional PDE (in the case involved, to ordinary differential equation).

E x a m p l e 2. Consider the nonlinear eikonal equation

$$u^2_{x_0} - u^2_{x_1} - u^2_{x_2} - u^2_{x_3} + 1 = 0. \tag{32}$$

As is established in [15], the maximal invariance algebra of equation (32) is the 21-parameter conformal algebra $AC(2, 3)$. This algebra contains, in particular, one-dimensional subalgebra generated by the operator $Q = x_0\partial_u - u\partial_{x_0}$.

To obtain the ansatz invariant under the operator Q, one has to construct the complete set of the first integrals of PDE

$$u u_{x_0} + x_0 = 0. \tag{33}$$

Solution of (33) is looked for in the implicit form $f(x, u) = 0$, whence $u f_{x_0} - x_0 f_u = 0$.

The complete set of the first integrals of the above PDE is as follows $\omega_0 = u^2 + x_0^2$, $\omega_1 = x_1$, $\omega_2 = x_2$, $\omega_3 = x_3$. Resolving the relation $f(\omega_0, \omega_1, \omega_2, \omega_3) = 0$ with respect to ω_0, we have

$$u^2 + x_0^2 = \varphi(\omega_1, \omega_2, \omega_3) \tag{34}$$

Consequently, formula (34) gives the ansatz for field $u^\alpha = u^\alpha(x)$ invariant under the operator Q. On resolving (34) with respect to u we get

$$u = \{-x_0^2 + \varphi(\omega_1, \omega_2, \omega_3)\}^{1/2} . \tag{35}$$

Let us emphasize that ansatz (34) can not be represented in the form (12) since coefficients of the operator Q do not satisfy condition (8).

Substituting (35) into (32) we get three-dimensional PDE for a function $\varphi = \varphi(\vec{\omega})$

$$\varphi_{\omega_1}^2 + \varphi_{\omega_2}^2 + \varphi_{\omega_3}^2 - \varphi^2 = 0.$$

E x a m p l e 3. In [16] the detailed group-theoretical analysis of the nonlinear wave equation

$$u_{tt} = (a^2(u)\, u_x)_x, \tag{36}$$

where $a(u)$ is some smooth function, was carried out. It was established that the maximal invariance algebra of equation (36) had the following basis operators:

$$Q_1 = \partial_t, \quad Q_2 = \partial_x, \quad Q_3 = t\partial_t + x\partial_x. \tag{37}$$

That is why, the most general group-invariant ansatz for PDE (36) is given by the formula $u = \varphi(\omega)$, where $\omega = \omega(t, x)$ is the first integral of PDE

$$\{\alpha\partial_t + \beta\partial_x + \delta(t\partial_t + x\partial_x)\}\, \omega(t, x) = 0. \tag{38}$$

Here α, β, δ are arbitrary real constants. Using transformations from the group G with generators of the form (37) one can reduce equation (38) to one of the equations

1) $\alpha\omega_t + \beta\omega_x = 0$ (under $\delta = 0$);
2) $t\omega_t + x\omega_x = 0$ (under $\delta \neq 0$).

The first integrals of the above equations are given by the formulas $\omega = \alpha x - \beta t$ and $\omega = xt^{-1}$, accordingly.

Thus, there are two inequivalent group-invariant ansatzes for PDE (36) with arbitrary function $a(u)$

1) $u(t, x) = \varphi(\alpha x - \beta t)$,
2) $u(t, x) = \varphi(xt^{-1})$. (39)

Substitution of the above ansatzes into equation (36) yields ordinary differential equations

1) $(\beta^2 - \alpha^2 a^2(\varphi))\, \ddot{\varphi} - 2\alpha^2 a(\varphi)\, \dot{a}(\varphi)\, \dot{\varphi}^2 = 0$,

2) $(\omega^2 - a^2(\varphi))\, \ddot{\varphi} - 2\omega\dot{\varphi} - 2a(\varphi)\, \dot{a}(\varphi)\, \dot{\varphi}^2 = 0$.

It was established not long ago [17] that ansatzes (39) do not exhaust all possible ansatzes reducing PDE (36) to ordinary differential equations. This fact is the consequence of the conditional symmetry that can not be found within the framework of the infinitesimal Lie method.

Let us show following [17] that equation (36) is conditionally-invariant under the operator

$$Q = \partial_t - \varepsilon a(u)\, \partial_x, \tag{40}$$

where $\varepsilon = \pm 1$.

Acting by the second prolongation of the operator Q on (36), we have

$$\tilde{Q}\{u_{tt} - (a^2(u)\, u_x)_x\} = \varepsilon \dot{a} u_x \{u_{tt} - (a^2 u_x)_x\} + \varepsilon(\ddot{a} u_x + \dot{a}\partial_x)(u_t^2 - a^2 u_x^2), \tag{41}$$

whence it follows that PDE (36) is non-invariant, in Lie's sense, under the group having infinitesimal operator (40). But, if we impose on the function

$u\ (t, x)$ the additional constraint

$$Qu \equiv u_t - \varepsilon a(u) u_x = 0, \tag{42}$$

the right part of (41) vanishes. Consequently, system (36), (42) is invariant, in Lie's sense under the group having generator (40), whence we conclude that the initial PDE (36) is conditionally-invariant under the operator Q.

The complete set of functionally-independent first integrals of equation (42) can be chosen in the form $\omega_1 = u$, $\omega_2 = x + \varepsilon a(u)\, t$.

Consequently, the ansatz invariant under the operator Q is given by the formula $\omega_2 = \varphi(\omega_1)$ or

$$x + \varepsilon a(u) t = \varphi(u), \tag{43}$$

where $\varphi(u)$ is an arbitrary smooth enough function.

Substituting (43) into (36) we come to conclusion that PDE (36) is identically satisfied. Saying it another way, formula (43) gives solution of nonlinear equation (36) under arbitrary function $\varphi(u)$. Let us recall that solutions obtained by using group-invariant ansatzes (39) contain two arbitrary integration constants and can not, in principle, contain arbitrary function.

Thus, conditional symmetry of PDE essentially extends our possibilities to reduce it.

E x a m p l e 4. Consider the system of nonlinear Dirac equations

$$\{i\gamma_\mu \partial_\mu - \lambda (\bar{\psi}\psi)^{1/2k}\}\, \psi = 0, \tag{44}$$

where γ_μ, $\mu = \overline{0, 3}$, are (4×4)-Dirac matrices, $\psi = \psi(x_0, x_1, x_2, x_3)$ is the four-dimensional complex function-column, $\bar{\psi} = (\psi^*)^T \gamma_0$, λ, k are real constants, $\partial_\mu = \partial/\partial x_\mu$, $\mu = \overline{0, 3}$.

As is known (see, for example, [5]), the maximal, in Lie's sense, invariance group of system of PDE (44) is the eleven-parameter extended Poincaré group supplemented by the three-parameter group of linear transformations in the space ψ^α, $\psi^{*\alpha}$. In [5, 10] it is established that conditional symmetry of the nonlinear Dirac equation is essentially wider. From. [10] it follows that system (44) is conditionally-invariant under the involutive set of operators

$$Q_1 = \frac{1}{2}\,(\partial_0 - \partial_3), \quad Q_2 = w_1 \partial_2 - \{B_1 \psi\}^\alpha \partial_{\psi^\alpha},$$

$$Q_3 = \frac{1}{2}\,(\partial_0 + \partial_3) - \dot{w}_1 (x_1 \partial_1 + x_2 \partial_2) - \dot{w}_2 \partial_1 - \{B_2 \psi\}^\alpha \partial_{\psi^\alpha}, \tag{45}$$

where B_1, B_2 are variable (4×4)-matrices of the form

$$B_1 = \frac{1}{2}\,(1 - 2k)\, \dot{w}_1 \gamma_2 (\gamma_0 + \gamma_3), \quad B_2 = -k\dot{w}_1 + (2w_1)^{-1} (2\dot{w}_1^2 -$$

$$- w_1 \ddot{w}_1)(\gamma_1 x_1 + 2(k-1)\, \gamma_2 x_2)(\gamma_0 + \gamma_3) + (2w_1)^{-1} \times$$

$$\times ((2\dot{w}_1 \dot{w}_2 - w_1 \ddot{w}_2)\, \gamma_1 + 2(w_3 \dot{w}_1 - w_1 \dot{w}_3)\, \gamma_2)(\gamma_0 + \gamma_3),$$

w_1, w_2, w_3 are arbitrary smooth functions on $x_0 + x_3$, by the symbol $\{\Psi\}^\alpha$ the α-th component of the function Ψ is designated.

As coefficients of the operators (45) satisfy conditions (8) they can be rewritten in non-Lie form

$$Q_1 = \frac{1}{2}\,(\partial_0 - \partial_3), \quad Q_2 = w_1 \partial_2 + B_1,$$

$$Q_3 = \frac{1}{2}\,(\partial_0 + \partial_3) - \dot{w}_1 (x_1 \partial_1 + x_2 \partial_2) - \dot{w}_2 \partial_1 + B_2.$$

Consequently, the ansatz for field $\psi(x)$ invariant under the set of operators Q_1, Q_2, Q_3 is looked for in the form (12), where $A(x)$ is a (4×4)-matrix and $\omega = \omega(x)$ is a real function satisfying the system of PDE

$$\frac{1}{2}(A_{x_0} - A_{x_3}) = 0, \quad w_1 A_{x_2} + B_1 A = 0,$$

$$\frac{1}{2}(A_{x_0} + A_{x_3}) - (\dot{w}_1 x_1 + \dot{w}_2) A_{x_1} - \dot{w}_1 x_2 A_{x_2} + B_2 A = 0,$$

$$\omega_{x_0} - \omega_{x_3} = 0, \quad \omega_{x_2} = 0,$$

$$\omega_{x_0} + \omega_{x_3} - 2(\dot{w}_1 x_1 + \dot{w}_2)\,\omega_{x_1} - 2\dot{w}_1 x_2 \omega_{x_2} = 0.$$

Not going into details of integration of the above system we write down the final result-the ansatz for field $\psi = \psi(x)$ invariant under the involutive set of operators (45)

$$\psi(x) = w_1^k \exp\{(2w_1)^{-1}(\dot{w}_1 x_1 + \dot{w}_2)\,\gamma_1(\gamma_0 + \gamma_3) + \tag{46}$$

$$+ (2w_1)^{-1}((2k-1)\,\ddot{w}_1 x_2 + w_3)\,\gamma_2(\gamma_0 + \gamma_3)\}\,\varphi(w_1 x_1 + w_2).$$

The above ansatz reduce system of PDE (44) to system of ordinary differential equations for four-component function $\varphi = \varphi(\omega)$

$$i\gamma_1\dot{\varphi} - \lambda(\bar{\varphi}\varphi)^{1/2k}\varphi = 0. \tag{47}$$

The general solution of system (47) has the form [5]

$$\varphi = \exp\{i\lambda\gamma_1(\bar{\chi}\chi)^{1/2k}\omega\}\,\chi,$$

where χ is arbitrary constant four-component column. On substituting the obtained expression for $\varphi = \varphi(\omega)$ into (46) we get the class of exact solutions of the nonlinear Dirac equation that contains three arbitrary functions.

Analysis of nonlinear mathematical and theoretical physics equations admitting non-trivial conditional symmetry had been carried out in [14].

3. Reduction of PDE by the number of independent and dependent variables. Let (3) be involutive set of operators satisfying the condition $R_2 - R_1 = \delta > 0$. In such a case the above technique of reduction of PDE by using ansatzes invariant under the involutive set (3) needs some modification. It is worth noting that the case when (3) are basis operators of some subalgebra of the Lie invariance algebra of the equation under study satisfying the condition $R_1 < R_2$ leads to «partially-invariant solutions» [18].

Solution of the initial system of PDE is looked for in implicit form

$$w^\alpha(x, u) = 0, \quad \alpha = \overline{0, m-1}, \tag{48}$$

where w^α are smooth functions satisfying the condition

$$\det \| \partial w^\alpha / \partial u^\beta \|_{\alpha,\beta=0}^{m-1} \neq 0. \tag{49}$$

As a result, equations (1), (2) take the form

$$H_A(x, u, w, \underset{1}{w}, \ldots, \underset{r}{w}) = 0, \quad A = \overline{1, M}, \tag{50}$$

$$\xi_{a\mu}(x, u)\, w^\alpha_{x_\mu} + \eta^\beta_a(x, u)\, w^\alpha_{u^\beta} = 0, \quad a = \overline{1, N}, \tag{51}$$

where $\underset{s}{w} = \{\partial^s w / \partial x_{\mu_1} \ldots \partial x_{\mu_p} \partial u^{\alpha_1} \ldots \partial u^{\alpha_q},\ p + q = s\}$.

It is clear, that operators (3) being defined in the space of variables x, u, $w(x, u)$ satisfy the condition $R_1' = R_2'$ (since coefficients of ∂_{w^α} are equal to zero). Using the same arguments as those applied to prove the theorem 1, we establish the following fact: there is a change of variables (17) that reduces system (51) to the form

$$w^{\alpha}_{x'_{\mu}} = 0, \quad \mu = \overline{0, R_1 - 1}, \quad w^{\alpha}_{u'^{\beta}} = 0, \quad \beta = \overline{0, \delta - 1}. \tag{52}$$

Provided system (48), (50) is conditionally-invariant under the set of operators (3) and (52) holds, it is rewritten as follows

$$w^{\alpha}(x', u') = 0, \quad \alpha = \overline{0, m-1},$$
$$H'_A(x'_{R_1}, \ldots, x'_{n-1}, u'^{\delta}, \ldots, u'^{m-1}, w, \underset{1}{w}, \ldots, \underset{r}{w}) = 0, \tag{53}$$

where the symbol $\underset{s}{w}$ designates collection of partial derivatives of the function w of the order s with respect to the variables $x'_{R_1}, \ldots, x'_{n-1}, u'^{\delta}, \ldots, u'^{m-1}$.

Integrating equations (52) we get the ansatz for field w^{α}

$$w^{\alpha} = F^{\alpha}(x'_{R_1}, \ldots x'_{n-1}, u'^{\delta}, \ldots, u'^{m-1}), \quad \alpha = \overline{0, m-1}, \tag{54}$$

where F^{α} are arbitrary smooth functions. But, one can not obtain the ansatz for field $u'^{\alpha}(x')$ by substituting (54) into relations $w^{\alpha}(x', u'(x')) = 0$, $\alpha = \overline{0, m-1}$, because the inequality $R_2 - R_1 = \delta > 0$ break the condition (49) (under $\delta > 0$ the matrix $\| \partial w^{\alpha}/\partial u'^{\beta} \|_{\alpha,\beta=0}^{m-1}$ has null columns).

To avoid this difficulty we shall consider, by definition, the expressions

$$F^{\alpha}(x'_{R_1}, \ldots, x'_{n-1}, u'^{\delta}, \ldots, u'^{m-1}) = 0, \quad \alpha = \overline{\delta, m-1},$$
$$u'^{j} = C_j, \quad j = \overline{0, \delta - 1}$$

to be the ansatz for field $u'^{\alpha} = u'^{\alpha}(x')$ invariant under the set of operators

$$Q_j = \partial_{x'_{j-1}}, \quad j = \overline{1, R_1}, \quad X_i = \partial_{u'^{i-1}}, \quad i = \overline{1, \delta}. \tag{55}$$

The above ansatz is rewritten in the form

$$u'^{\alpha} = C_{\alpha}, \quad \alpha = \overline{0, \delta - 1}, \quad u'^{\alpha+\beta} = \varphi^{\beta}(x'_{R_1}, \ldots, x'_{n-1}), \quad \beta = \overline{0, m - \delta - 1}, \tag{56}$$

where φ^{β} are arbitrary smooth functions, C_{α} are arbitrary constants.

On rewritting (56) in the initial variables, we have

$$g^{\alpha}(x, u) = C_{\alpha}, \quad \alpha = \overline{0, \delta - 1}, \quad g^{\beta+\delta}(x, u) = \varphi^{\beta}(f_{R_1}(x, u), \ldots, f_{n-1}(x, u)),$$
$$\beta = \overline{0, m - \delta - 1}. \tag{57}$$

And what is more, substitution of expressions (57) into the initial system of PDE (1) or, equivalently, expressions $w^{\alpha} = g^{\alpha} - C_{\alpha}$, $\alpha = \overline{0, \delta - 1}$, $w^{\beta} = g^{\beta+\delta} - \varphi^{\beta}$, $\beta = \overline{0, m - \delta - 1}$ into PDE (50) yields system of M differential equations for $m - \delta$ functions. Consequently, the dimension of system (1) is decreased by R_1 independent and δ dependent variables.

Let us rewrite formulas (57) in the form that is more convinient in applications. For this purpose, we note that, without loss of generality, operators (3) satisfying the condition $R_2 - R_1 = \rho > 0$ can be renumerated in such a way that the first R_1 operators satisfy the condition

$$\operatorname{rank} \| \xi_{a\mu} \|_{a=1}^{R_1} {}_{\mu=0}^{n-1} = \operatorname{rank} \| \xi_{a\mu}, \eta_a^{\alpha} \|_{a=1}^{R_1} {}_{\alpha=0}^{m-1} {}_{\mu=0}^{n-1}$$

and the last $N - R_2$ operators are linear combinations of the previous R_2 operators.

Let $\omega_j(x, u)$, $j = \overline{1, m + n - R_2}$ be the complete set of functionally-independent first integrals of system (51) and besides

$$\operatorname{rank} \| \partial\omega_j/\partial u^{\alpha} \|_{j=1}^{m-\delta} {}_{\alpha=0}^{m-1} = m - \delta$$

and $\rho_i(x, u)$ be solutions of equations $Q_{i+R_1}\rho(x, u) = 1$ with $i = 1, 2, \ldots, \delta$. Then formulas (57) can be expressed in the equivalent form

$$\rho_i(x, u) = C_i, \quad i = \overline{1, \delta},$$

$$\omega_j(x, u) = \varphi^j(\omega_{R_1}(x, u), \ldots, \omega_{n-1}(x, u)), \quad j = \overline{1, m-\delta}. \tag{58}$$

D e f i n i t i o n 4. *Expressions* (58) *are called the ansatz for field* $u^\alpha = u^\alpha(x)$ *invariant under the involutive set of operators* (3) *provided* $R_2 - R_1 \equiv \delta > 0$.

The above arguments can be summarized in the form of the following assertion.

T h e o r e m 2. *Let system of PDE* (1) *be conditionally-invariant under the involutive set of operators* (3) *and besides* $R_1 < R_2$. *Then the ansatz invariant under the set of operators* (3) *reduce this system.*

E x a m p l e 1. System of two wave equations

$$\Box u = 0, \quad \Box v = 0 \tag{59}$$

is invariant under the one-parameter group having the infinitesimal operator $Q = \partial_v$. Since $R_1 = 0$, $R_2 = 1$, the parameter δ is equal to 1. The complete set of the first integrals of equation $\partial w(x, u, v)/\partial v = 0$ is given by the functions

$$\omega_\mu = x_\mu, \quad \mu = \overline{0, 3}, \quad \omega_4 = u.$$

That is why, the ansatz for field $u(x)$, $v(x)$ invariant under the operator Q has the form (58)

$$u = \varphi(\omega_0, \omega_1, \omega_2, \omega_3), \quad v = C, \quad C = \text{const}.$$

Substituting the above expressions into (59), we get

$$\varphi_{\omega_0\omega_0} - \varphi_{\omega_1\omega_1} - \varphi_{\omega_2\omega_2} - \varphi_{\omega_3\omega_3} = 0$$

i. e. the reduction of the initial system (59) by the number of dependent variables takes place.

E x a m p l e 2. Consider the system of nonlinear Thirring equations

$$\mathrm{i}v_x = mu + \lambda_1 |u|^2 v, \quad \mathrm{i}u_y = mv + \lambda_2 |v|^2 u, \tag{60}$$

where u, v are complex functions on x, y and m, λ_1, λ_2 are real constants.

The above system admits the one-parameter transformation group having the generator

$$Q = \mathrm{i}u\partial_u + \mathrm{i}v\partial_v - \mathrm{i}u^*\partial_{u^*} - \mathrm{i}v^*\partial_{v^*}.$$

After the change of variables

$$u(x, y) = H_1(x, y) \exp\{\mathrm{i}Z_1(x, y) + \mathrm{i}Z_2(x, y)\},$$

$$v(x, y) = H_2(x, y) \exp\{\mathrm{i}Z_1(x, y) - \mathrm{i}Z_2(x, y)\},$$

where H_j, Z_j are new dependent variables, the operator Q takes the form $Q' = \partial_{Z_1}$. Consequently, the ansatz invariant under the operator Q reads

$$u(x, y) = H_1(x, y) \exp\{\mathrm{i}C + \mathrm{i}Z_2(x, y)\}, \tag{61}$$

$$v(x, y) = H_2(x, y) \exp\{\mathrm{i}C - \mathrm{i}Z_2(x, y)\}.$$

Substitution of (61) into (60) yields system of four PDE for three functions H_1, H_2, Z_2

$$H_{2x} = mH_{1x} \sin 2Z_2, \quad H_{1y} = -mH_2 \sin 2Z_2,$$

$$H_2 Z_{2x} = mH_1 \cos 2Z_2 + \lambda_1 H_1 H_2^2,$$

$$-H_1 Z_{2y} = mH_2 \cos 2Z_2 + \lambda_2 H_2 H_1^2.$$

E x a m p l e 3. Group analysis of the one-dimensional gas dynamics equations

$$u_t + uu_x + \rho^{-1} p_x = 0, \quad \rho_t + (u\rho)_x = 0, \quad p_t + (up)_x + (\gamma - 1) pu_x = 0 \tag{62}$$

had been carried out by Ovsjannikov [1]. He established, in particular, that invariance algebra of system of PDE (62) contains the basis element

$$Q = p\partial_p + \rho\partial_\rho. \tag{63}$$

The complete set of functionally-independent first integrals of the equation $Qw(t, x, u, p, \rho) = 0$ is as follows $\omega_1 = u$, $\omega_2 = p\rho^{-1}$, $\omega_3 = t$, $\omega_4 = x$. Consequently, the ansatz invariant under the operator Q (63) can be chosen in the form

$$u = \varphi^1(t, x), \quad p\rho^{-1} = \varphi^2(t, x), \quad \ln\rho + F(p\rho^{-1}) = C, \tag{64}$$

where $C = \text{const}$, F is some smooth function.

Substituting the ansatz (64) into system of PDE (62) we get system of three differential equations for two unknown functions $\varphi^1(t, x)$, $\varphi^2(t, x)$:

$$\begin{gathered}\varphi^1_t + \varphi^1\varphi^1_x - \varphi^2\dot{F}(\varphi^2)\,\varphi^2_x = 0,\\ \varphi^2_t + \varphi^1\varphi^2_x + (\gamma - 1)\,\varphi^2\varphi^1_x = 0,\\ \varphi^1_x((1-\gamma)\,\varphi^2\dot{F}(\varphi^2) - 1) = 0.\end{gathered} \tag{65}$$

Thus, the reduction of gas dynamics equations by the number of dependent variables takes place.

It is interesting to note that under $\varphi^1_x \neq 0$ it follows from the third equation of system (65) that $F = \lambda + (1-\gamma)^{-1}\ln(\rho^{-1}p)$. Substituting this expression into (62) we obtain $p = k\rho^\gamma$, $k \in \mathbb{R}^1$ — the relation characterizing polytropic gas.

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