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# On the Liapunov convexity theorem with applications to sign-embeddings

# Теорема Ляпунова про опуклість та її застосування до знако-вкладень

It is proved (Theorem 1) that for a Banach space $X$ the following statements are equialent: i) the range of every $X$-valued $\sigma$-additive non-atomic measure of finite variation has convex closure; ii) $L_1$ does not sign-embed in $X$.

Доведено (теорема 1), що для банахового простору $X$ еквівалентні такі твердження: i) множина значень будь-якої $X$-значної σ-адитивної безатомної міри з скінченною варіацією має опукле замикання; ii) простір $L_1$ не можна знако-вкласти в $X$,

**1. Introduction.** Throughout this paper by «$X$-valued measure» we mean a σ-additive $X$-valued measure $\mu$ defined on a σ-field $\Sigma$ of subsets of a set $\Omega$. The variation of $\mu$ is the scalar measure $|\mu|$ defined as

$$|\mu|(A) = \sup \sum_{i=1}^{n} \|\mu(A_i)\|$$

where the supremum is taken over all collections of disjoint subsets $A_i \in \Sigma$, $\bigcup_{i=1}^{n} A_i = A$. The Lebesgue measure on Borel σ-field $\mathfrak{B}$ on (0, 1) we denote by $\lambda$. By $\mathcal{L}(X, Y)$ we denote the space of all continuons linear operators from $X$ into $Y$; $\chi(A)$ is the characteristic function of a set $A \in \Sigma$; $L_p = L_p(0, 1)$.

The classical theorem of A. A. Liapunov [1] states that the range (=the set of values) of an arbitrary $X$-valued non-atomic measure is convex if $X$ is finite-dimensional. The converse is also valid [2, p. 265]: if the range of every $X$-valued measure is convex then $X$ is finite-dimensional (a simple proof: suppose that $X$ is infinite-dimensional; let $T \in \mathcal{L}(L_1, X)$ be an operator with zero kernel; then $\mu(A) = T\chi(A)$ is an $X$-valued measure of finite variation with non-convex range. Indeed, putting $x = \mu((0, 1))$ we obtain that $x/2$ does not belong to the range of $\mu$ because $x/2 = \mu(A)$ implies $\mu(0, 1) \setminus A) - = \mu((0, 1)) - \mu(A) = x/2$, hence $T(\chi(A) - \chi((0, 1) \setminus A)) = x/2 - x/2 = 0$. It is impossible since $T$ has zero kernel).

The question whether the range of every $X$-valued measure has convex closure is substantial in the infinite-dimensional case. If we consider measures of bounded variation, the answer is yes for spaces $X$ having the Radon-Nikodym property [2, p. 266]. Theorem 1 stated in Abstract characterizes those Banach spaces.

It was an interesting problem of H. P. Rosenthal [3] whether Condition ii) is equivalent to the statement that $L_1$ does not isomorphically embed in $X$. This problem was solved in negative by M. Talagrand [4]. In fact, he has constructed an example of a subspace $X$ of $L_1$ such that $L_1$ does not embed both in $X$ and in $L_1/X$. If we suppose that $L_1$ does not sign-embed in $L_1/X$ then every operator from $L_1$ into $L_1/X$ is narrow (the definition is given below) and hence so is the quotient-map $T: L_1 \to L_1/X$. This easily implies that $X$ contains a subspace isomorphic to $L_1$ (spanned by a sequence equivalent to the Haar system on which $T$ has sufficiently small norm). Thus, since $L_1$ in fact does not embed in $X$ we have that $L_1$ sign-embeds in $L_1/X$ while $L_1$ does not isomorphically embed in $L_1/X$.

Then in Section 2 we prove a non-separable generalization of the Liapunov theorem (Theorem 2) and give a quantitative version of it in the setting of measures admitting atoms (Theorem 3) estimating the degree of non-convexity of the range of an $X$-valued measure by norms of atoms.

Section 3 is devoted to a study of narrow operators which in some sense generalize compact operators and some other kinds of operators.

Definition 1. *An operator $T \in \mathcal{L}(L_p(\Omega, \Sigma, \mu), X)$, $1 \leqslant p < \infty$, is said to be narrow if for each $A \in \Sigma$ and $\varepsilon > 0$ there exists $x \in L_p(\Omega, \Sigma, \mu)$ such that*

$$1)\ x^2 = \chi(A); \qquad 2) \int_\Omega x d\mu = 0; \qquad 3)\ \|Tx\| < \varepsilon.$$

Note that Condition 2) in the above definition is superfluous:

Proposition 1 [5]. *Suppose $T \in \mathcal{L}(L_p(\Omega, \Sigma, \mu), X)$. If for each $A \in \Sigma$ and $\varepsilon > 0$ there is $x \in L_p(\Omega, \Sigma, \mu)$ with (1) and (3) then $T$ is narrow.*

In [6] non-narrow operator from $L_1$ were called norm-sign-preserving.

We describe some properties of narrow operators and also talk about the relations between narrow operators and another kinds of operators; then **we** review the known sufficient conditions on operators in order to be narrow.

In the next section we study the classes $\mathfrak{M}_p$, $1 \leqslant p < \infty$, of those separable Banach spaces $X$ for which every operator $T \in \mathcal{L}(L_p, X)$ is narrow. In particular, if $X \in \mathfrak{M}_p$ then $L_p$ does not sign-embed in $X$. For $p = 1$ the converse also holds.

D e f i n i t i o n 2. *An operator $T \in \mathcal{L}(L_p, X)$ is called a sign-embedding if there is $\delta > 0$ such that for every «sign» $x$, i. e. for every $x \in L_p$ taking values in the set $\{-1, 0, 1\}$ we have that $\|Tx\| \geqslant \delta\|x\|$.*

For example, $c_0 \in \mathfrak{M}_p$ for each $p$, $1 \leqslant p < \infty$. Theorem 1 asserts that if $1 \leqslant p < 2$ and $p < q$ then $L_q \in \mathfrak{M}_p$. But if $p \geqslant 2$ then for each q the statement is false.

We are gratefull to M. I. Ostrovskii for the idea to consider the nontion of infratype as applied to narrow operators that has simplified the proof of Theorem 1 and also has allowed to prove it for all values of $p$ and $q$, $1 \leqslant p < 2$, $p < q$.

2. O n t h e r a h g e s o f v e c t o r m e a s u r e s. First we need some lemmas.

L e m m a 1. *Let $X$ be a Banach space, $\mu : \Sigma \to X$ be an $X$-valued measure of bounded variation and let $A_0 \in \Sigma$ satisfies*

$$\inf\left\{\left\|\mu(B) - \frac{1}{2}\mu(A_0)\right\| : B \in \Sigma|_{A_0}\right\} > 0.$$

*Then there is a set $A \in \Sigma|_{A_0}$, $\mu(A) \neq 0$, and a number $a > 0$ such that for every $B \subset A$ and $C \subset B$, $B, C \in \Sigma$; $|\mu|(B) \neq 0$ we have*

$$\left\|\mu(C) - \frac{1}{2}\mu(B)\right\| > a|\mu|(B). \tag{1}$$

P r o o f. Choose $a > 0$ so that for every $B \subset A_0$, $B \in \Sigma$

$$\left\|\mu(B) - \frac{1}{2}\mu(A_0)\right\| > a|\mu|(A_0). \tag{2}$$

We shall call in the sequel a set $B \in \Sigma$ to be narrow if there is a subset $C \in \Sigma|_B$ for whch the converse to (1) inequality holds. Our lemma states the existence of a set $A$ without non-trivial narrow subsets. For each $A \in \Sigma$ we define the value

$$\varepsilon(A) = \sup\{|\mu|(B) : B \subset A,\ B \text{ is narrow}\}.$$

To prove the lemma we should construct two sequences of measurable sets: $A_0 \supset A_1 \supset A_2 \supset \ldots$ and $B_0, B_1, B_2, \ldots$, where $B_n$ are disjoint narrow sets, $B_n \subset A_n$, $|\mu|(B_n) \geqslant 1/2\varepsilon(A_n)$, $A_{n+1} = A_n \setminus B_n$. This construction can be realized easily by induction: the existence of $B_0 \subset A_0$, $|\mu|(B_0) \geqslant 1/2\varepsilon(A_0)$ follows from the definition of $\varepsilon(A_0)$; then we put $A_1 = A_0 \setminus B_0$; there is $B_1 \subset A_1$ with $|\mu|(B_1) \geqslant 1/2\varepsilon(A_1)$ and we set $A_2 = A_1 \setminus B_1$ and so on. Now we define $B_\infty = \bigcup_{n=0}^{\infty} B_n$. $B_\infty$ is a narrow set (as a union of disjoint narrow sets) and since $A_0$ is a non-narrow set by (2), rhe set $A = A_0 \setminus B_\infty$ has non-zero measure. Since $A \subset A_n$ for each $n$, then $\varepsilon(A) \leqslant \inf_n \varepsilon(A_n) \leqslant 2\inf_n|\mu|(B_n)$. But $B_n$ are disjoint and hence $\inf_n|\mu|(B_n) = 0$. We have got $\varepsilon(A) = 0$, i. e. $A$ has no non-trivial narrow subsets.

R e m a r k s. 1. As a consequence of Lemma 1 we obtain Lemma 3 from [3]: If $T \in \subset \mathcal{L}(L_1, X)$ is a non-narrow operator then $L_1$ sign-embeds in $X$.

2. One can use Lemma 3 from [3] in the proof of Theorem 1 instead of our Lemma 1.

L e m m a 2. *Let $\mu$ be an $X$-valued measure. If for each $A \in \Sigma$ and $\varepsilon > 0$ there is $B \in \Sigma$, $B \subset A$, such that $\|\mu(B) - 1/2\,\mu(A)\| < \varepsilon$, then the range $\mu(\Sigma)$ of $\mu$ has convex closure.*

For the convenience of the proof of Theorem 3 we shall prove the following more general lemma.

For a subset $F$ of a Banach space $X$ we define a number $C(F)$ as a degree of non-convexity:

$$C(F) = \sup \left\{ \text{dist} \left( F, \frac{x+y}{2} \right) : x, y \in F \right\}.$$

Obviously, $C(F) = 0$ if and only if $F$ has convex closure.

Lemma 2a. *Let $\mu$ be an $X$-valued measure and $r$ be a number such that for every $\varepsilon > 0$ and every $A \in \Sigma$ there is a $\Sigma$-measurable subset $A_\varepsilon \subset A$ with*

$$\left\| \mu(A_\varepsilon) - \frac{1}{2} \mu(A) \right\| \leqslant r + \varepsilon.$$

*Then $C(\mu(\Sigma)) \leqslant 2r$.*

Proof of Lemma 2a. Suppose $x, y \in \mu(\Sigma)$, i. e. $x = \mu(A)$ and $y = \mu(B)$ where $A, B \in \Sigma$. Put $\hat{A} = A \setminus (A \cap B)$ and $\hat{B} = B \setminus (A \cap B)$ and choose $A_\varepsilon \subset \hat{A}$ and $B_\varepsilon \subset \hat{B}$ for a fixed $\varepsilon > 0$ so that $\| \mu(A_\varepsilon) - 1/2 \mu(\hat{A}) \| \leqslant r + \varepsilon$ and $\| \mu(B_\varepsilon) - 1/2 \mu(\hat{B}) \| \leqslant r + \varepsilon$. Then for $z = \mu(A_\varepsilon \cup B_\varepsilon \cup (A \cap B))$ we obtain

$$\text{dist} \left( \mu(\Sigma), \frac{x+y}{2} \right) \leqslant \left\| z - \frac{x+y}{2} \right\| =$$

$$= \left\| \mu(A_\varepsilon) + \mu(B_\varepsilon) - \frac{1}{2} (\mu(\hat{A}) + \mu(\hat{B})) \right\| \leqslant 2(r + \varepsilon).$$

This proves the lemma by arbitrariness of $\varepsilon$.

Corollary 1. *Let an $X$-valued measure $\mu$ of bounded variation has non-convex range closure. Then there are $\varepsilon > 0$ and $A \in \Sigma$, $\mu(A) \neq 0$ such that for each disjoint $A_1, A_2 \in \Sigma|_A$*

$$\| \mu(A_1) - \mu(A_2) \| \geqslant \varepsilon |\mu| (A_1 \cup A_2). \tag{3}$$

Proof. By Lemma 2 there is a set $A_0 \in \Sigma$ for which the conditions of Lemma 1 hold. Hence there are: a set $A$ and a number $a > 0$, $\mu(A) \neq 0$, such that for every $C \subset B \subset A$, $\mu(B) \neq 0$ the inequality (1) holds. Then (3) follows from (1) if we set $\varepsilon = 2a$, $B = A_1 \cup A_2$, $C = A_1$:

$$\| \mu(A_1) - \mu(A_2) \| = 2 \left\| \mu(C) - \frac{1}{2} \mu(B) \right\| \geqslant 2a |\mu| (B).$$

Theorem 1. *For a Banach space $X$ the following statements are equivalent:*

i) *the range of every $X$-valued non-atomic measure $\mu$ of bounded variation has convex closure;*

ii) *$L_1$ does not sign-embed in $X$;*

iii) *each bounded operator from $L_1$ into $X$ is narrow.*

Note that the equivalence of ii) and iii) can be obtained from Lemma 3 of [3].

Proof. i) $\rightarrow$ iii). Let $T \in \mathcal{L}(L_1, X)$. Then the Borel $X$-valued measure $\mu$ on (0, 1) defined as $\mu(A) = T\chi(A)$ is non-atomic and of bounded variation. Fix a Borel subset $A \subset 0, 1)$ and consider the restriction of $\mu$ to subsets of $A$. Then by i) the element $1/2\, T\chi(A) = 1/2 (\mu(A) + \mu(\varnothing))$ can be approximated by values of $\mu$: for every $\varepsilon > 0$ there is $A_\varepsilon \subset A$ with

$$\left\| \frac{1}{2} T\chi(A) - T\chi(A_\varepsilon) \right\| < \varepsilon.$$

By arbitrariness of $\varepsilon$, the equality

$$\| T\chi(A \setminus A_\varepsilon) - T\chi(A_\varepsilon) \| = 2 \left\| \frac{1}{2} T\chi(A) - T\chi(A_\varepsilon) \right\|$$

implies that $T$ is narrow.

iii) $\rightarrow$ ii) is evident.

ii) $\to$ i). Let $\mu : \Sigma \to X$ be a non-atomic measure of bounded variation such that $\mu(\Sigma)$ has non-convex closure. Let $\mathring{A}$ be as in Corollary 1. Then $\langle \mathring{A}, \Sigma|_{\mathring{A}}, |\mu| \rangle$ is a non-atomic measure space. Define an operator $T$: $L_1(\mathring{A}, \Sigma|_{\mathring{A}}, |\mu|) \to X$ so that $T\chi(B) = \mu(B)$ for each $B \in \Sigma|_{\mathring{A}}$. If $B_i$ are disjoint then

$$\left\| \sum a_i T\chi(B_i) \right\|_X = \left\| \sum a_i \mu(B_i) \right\|_X \leqslant \sum |a_i| \, |\mu|(B_i) = \left\| \sum a_i \chi(B_i) \right\|_{L_1}$$

for each scalars $(a_i)$ and hence $T$ can be extended by linearity and continuity to a bounded operator defined on $L_1(\mathring{A}, \Sigma|_{\mathring{A}}, |\mu|)$. By (3) it is a sign-embedding. Since $|\mu|$ is non-atomic, we obtain evidently the existence of a sign-embedding of $L_1$ in $X$ which contradicts iii).

Now we shall prove a non-separable generalization of the Liapunov theorem. Let $\langle \Omega, \Sigma, \mu_0 \rangle$ be a measure space with a nonatomic positive scalar measure $\mu_0$. Put

$$\aleph_\alpha = \min\{\dim L_1(A) : A \in \Sigma, \ \mu_0(A) > 0\}.$$

Here $\dim X$ denotes the least cardinality of subsets of $X$ with dense linear span. Denote by $\dim \operatorname{alg} X$ the cardinality of Hamel basis of a linear space $X$.

Theorem 2 (The case of real scalars field). *Let $X$ be a Banach space with*

$$\dim \operatorname{alg} X < \aleph_\alpha^{\aleph_0}$$

*and let $\mu$ de an $X$-valued $\sigma$-additive measure on $\Sigma$ of finite variation, absolutely continuous with respect to $\mu_0$. Then the range of $\mu$ is convex.*

Proof. By Hahn Decomposition Theorem decompose $\Omega$ into disjoint $\Sigma$-measurable subsets $\Omega_n$ of $\Omega$, $n \geqslant 1$, so that if $\mathring{A} \subset \Omega_n$ then

$$(n-1)\mu_0(A) \leqslant |\mu|(A) < n\mu_0(A). \tag{4}$$

Define an operator $T_n \in \mathcal{L}(L_1(\mu_0), X)$ for each $n \geqslant 1$. For a simple function $x = \sum_{k=1}^{m} a_k \chi(A_k)$ where $A_k \in \Sigma$, $A_i \cap A_j = \varnothing$ for $i \neq j$; $\bigcup_{k=1}^{m} A_k = \Omega$, we put

$$T_n x = \sum_{k=1}^{m} a_k \mu(A_k \cap \Omega_n) = \int_{\Omega_n} x d\mu.$$

By (4) $T_n$ can be extended on $L_1(\mu_0)$ by linearity and continuity. Note that $T_n \chi(A) = \mu(A \cap \Omega_n)$ for each $A \in \Sigma$. Put $Y = \bigcap_{n=1}^{\infty} \ker T_n$. Show that if $\mu_0(A) > 0$ then $Y \cap L_\infty(A) \neq \{0\}$. Let $m$ be such that $\mu_0(A \cap \Omega_m) > 0$. Note the following simple fact: if S is a linear operator acting between linear spaces $E_1$ and $E_2$ then

$$\dim \operatorname{alg}(E_1/\ker S) \leqslant \dim \operatorname{alg} E_2$$

where by $E_1/\ker S$ we denote the quotient space. Hence

$$\dim \operatorname{alg}(L_1(A \cap \Omega_m)/\ker T_m|_{L_1(A \cap \Omega_m)}) \leqslant \dim \operatorname{alg} X.$$

On the other hand

$$\dim \operatorname{alg} L_\infty(A \cap \Omega_m) = \dim \operatorname{alg} L_1(A \cap \Omega_m) \geqslant \aleph_\alpha^{\aleph_0}.$$

Now use the following elementary fact: if $E_1$ and $E_2$ are subspaces of a linear space $E$ with

$$\dim \operatorname{alg}(E/E_1) < \operatorname{diam} \operatorname{alg} E_2$$

then $E_1 \cap E_2 \neq \{0\}$. Thus, by the theorem assumptions there exsists

$$x \in \ker T_m|_{L_1(A \cap \Omega_m)} \cap L_\infty(A \cap \Omega_m)$$

and $x \neq 0$. By the definition of operators $T_n$ it is clear that since supp $x \subset \Omega_m$, we have $T_n x = 0$ for all $n \geqslant 1$. Thus, $x \in Y \subset L_\infty(\mathring{A})$.

List the formulation of a part of Theorem 1, § 10 from [5] we need. Tet $E$ be a symmetric Banach space on $\langle \Omega, \Sigma, \mu \rangle$ over the reals and let $X$ be a (closed) subspace of $E$. The following statements are equivalent:

i) $X \cap L_\infty (A) \neq \{0\}$ whenever $\mu (A) > 0$,

iii) for each $A \in \Sigma$ and each number $0 < h < \infty$ there exists $x \in X$ such that $x = \nu\chi (A') - \chi (A'')$ where $A' \cup A'' = A$, $A' \cap A'' = \varnothing$ and $\mu (A') = \mu (A)/(\nu + 1)$. This theorem gives us for subspace $Y$ and measure $\mu_0$ that iii) holds.

Let $A, B \in \Sigma$; $0 < t < 1$. In order to prove the convexity of the range of $\mu$ we shall construct a set $C \in \Sigma$ so that

$$\mu (C) = t\mu (A) + (1 - t) \mu (B).$$

Put

$$\nu_1 = \frac{1-t}{t}, \quad \nu_2 = \frac{t}{1-t}.$$

Choose by iii) $x_1$ and $x_2$ from $Y$ so that

$$x_1 = \nu_1\chi (C_1') - \chi (C_1''),$$

where $C_1' \cup C_1'' = A\setminus B$, $C_1' \cap C_1'' = \varnothing$ and

$$x_2 = \nu_2\chi (C_2') - \chi (C_2''),$$

where $C_2' \cup C_2'' = B\setminus A$, $C_2' \cap C_2'' = \varnothing$. Since $x_1, x_2 \in Y$, we have $T_n x_1 = T_n x_2 = 0$ for all $n$, i. e.

$$0 = \int\limits_{\Omega_n} x_i d\mu = \nu_i \mu (C_i') - \mu (C_i''), \qquad i = 1, 2.$$

Hence $\mu (C_1'') = \nu_i \mu (C_i')$. Taking into account that $\mu (A\setminus B) = \mu (C_1') + \mu (C_1'')$ and $\mu (B\setminus A) = \mu (C_2') + \mu (C_2'')$, we conclude

$$\mu (C_1') = \frac{\mu (A\setminus B)}{\nu_1 + 1} = t\mu (A\setminus B),$$

$$\mu (C_2') = \frac{\mu (B\setminus A)}{\nu_2 + 1} = (1 - t) \mu (B\setminus A).$$

Finally put $C = C_1' \cup C_2' \cup (A \cap B)$. Then

$$\mu (C) = t\mu (A\setminus B) + (1 - t) \mu (B\setminus A) + \mu (A \cap B).$$

R e m a r k. When $\aleph_\alpha = \aleph_0$, we obtain the Liapunov convexity theorem.

Now consider a measure space $\langle \Omega, \Sigma, \mu \rangle$ with an $X$-valued $\sigma$-additive measure $\mu$ containing atoms. Recall that a set $A \in \Sigma$ is called an atom for $\mu$ f $\mu (A) \neq 0$ and $B \subset A$, $B \in \Sigma$ imply that either $\mu (B) = 0$ or $\mu (\,') = \mu (A)$.

D e f i n i t i o n 3. *The degree of atomlessness of* $\mu$ *is the number*

$$at (\mu) = \sup \{\| \mu (A)\| : A \text{ is an atom for } \mu\}.$$

It is not hard to see that if $a > at (\mu)$ then every $A \in \Sigma$ can be decomposed into a finite union of disjoint $\Sigma$-measurable subsets: $A = \bigcup\limits_{i=1}^{n} A_i$ such that $\max_i \| \mu (A_i)\| \leqslant a$.

T h e o r e m 3. *Let* $\dim X < \infty$. *Then there exists a contant* $K > 0$ *such that for each* $X$*-valued measure*

$$C (\mu (\Sigma)) \leqslant K at (\mu).$$

P r o o f. Let $A \in \Sigma$ and $\varepsilon > 0$. Choose disjoint subsets $A_i \in \Sigma$, $\bigcup\limits_{i=l}^{n} A_i = A$, such that $\max \| \mu (A_i) \| < at (\mu) + \varepsilon$. For each $Q \subset \{1, 2, \ldots, n\}$ denote by $A_Q$ the set $\bigcup_{i\in Q} A_i$. Then

$$\min_Q \left\| \mu(A_Q) - \frac{1}{2}\mu(A) \right\| = \min_Q \left\| \sum_{i\in Q} \mu(A_i) - \frac{1}{2}\sum_{j=1}^{n} \mu(A_j) \right\| =$$

$$= \frac{1}{2}\min_Q \left\| \sum_{i\in Q} \mu(A_i) - \sum_{i\notin Q} \mu(A_i) \right\| = \frac{1}{2}\min_{\alpha_i=\pm 1} \left\| \sum_{i=1}^{n} \alpha_i \mu(A_i) \right\|.$$

Using the Steinitz estimation [7, p. 27], Lemma 2.1.2 with $\lambda_i = \frac{1}{2}$

$$\min_{\alpha_i=\pm 1} \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leqslant \dim X \max_i \| x_i \|$$

we obtain the conditions of Lemma 2a with $r = 1/2 \dim X$ at $(\mu)$. This imply Theorem 3 with $K = \dim X$.

3. Narrow operators. It is evident that every compact operator is narrow (considering a Rademacher type system $\{r_n(A)\}_{n=1}^{\infty}$ in $L_p(A)$, $A \in \mathfrak{B}$, we obtain that $\| Tr_n(A) \|$ tends to 0 and $r_n^2(A) = \varkappa(A)$). On the other hand, if $T \in \mathscr{L}(L_p, X)$ is narrow then there exists a subspace $E \subset L_p$ isometric to $L_p$ and such that the restriction $T|_E$ is compact (of a suitable small norm). Such a subspace could by generated a system which is isometrically equivalent to the Haar system on which $T$ has sufficiently small norm (cf. [5]). But an analogy with compact operators does not past too far. If $p > 1$ then the sum of two narrow operators in $\mathscr{L}(L_p) = \mathscr{L}(L_p, L_p)$ need not be narrow. Moreover, every operator in $\mathscr{L}(L_p)$ is a sum of two narrow operators [5]. But in $\mathscr{L}(L_1)$ the sum of two narrow operators is always narrow [5]. Recall that an operator $T \in \mathscr{L}(X, Y)$ is said to be a Dunford-Pettis operator provided the image under $T$ of a weakly compact set has compact closure. An operator $T \in \mathscr{L}(X, Y)$ is called absolutely summing if for every sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ such that $\sum_{n=1}^{\infty} x_n$ converges unconditionally, the series $\sum_{n=1}^{\infty} Tx_n$ converges absolutely $\left(\text{i. e. } \sum_{n=1}^{\infty} \| Tx_n \| < \infty\right)$.

Proposition 2. a). *Every Dunford-Pettis operator $T \in \mathscr{L}(L_p, X)$ is narrow.* b). *Every absolutely summing operator $T \in \mathscr{L}(L_p, X)$ is narrow.*

Proof. Let $A \in \mathfrak{B}$ and let $\{r_n\}_{n=1}^{\infty}$ be a Rademacher type system on $A$. a). Since $\{r_n, n \in N\}$ is a weakly compact set in $L_p$, we have that $\{Tr_n\}_{n=1}^{\infty}$ is relatively compact in $X$. Suppose that $\{Tr_{n_k}\}_{k=1}^{\infty}$ converges. But since $w - \lim_n r_n = 0$, we have $w - \lim_k Tr_{n_k} = 0$, hence $\lim_k \| Tr_{n_k} \| = 0$. b). Since $\{r_n\}_{n=1}^{\infty}$ is equivalent to an orthonormal basis of a Hilbert space [8, p. 66], the series $\sum_{n=1}^{\infty} (1/n)\, r_n$ converges unconditionally. Thus, $\sum_{n=1}^{\infty} (1/n)\, Tr_n$ converges, hence $\lim \inf_n \| Tr_n \| = 0$.

Problem 1 [5]. Is every strictly singular operator $T \in \mathscr{L}(L_p, X)$ narrow $(p \neq 1, 2)$?

Recall that an operator $T \in \mathscr{L}(X, Y)$ fixes a copy of a Banach space $Z$ if there is a subspace $E \subset X$ isomorphic to $Z$ for which the restriction $T|_E$ is an isomorphism. A deep result annonced in [9, p. 54] and obtained partially in [10] and [11] asserts that if $1 \leqslant p < 2$ and $T \in \mathscr{L}(L_p)$ fixes no copy of $L_p$ then $T$ is narrow (the same is false when $p \geqslant 2$). It is shown in [12] that if $T \in \mathscr{L}(L_1, X)$ fixes no copy of $l_1$ then $T$ is narrow. For a large class of spaces $X$ it is proved in [6] that if $T \in \mathscr{L}(L_1, X)$ fixes no copy of $L_1$ then $T$ is narrow (but not every separable Banach spaces posses the above property [4]).

Note that all above given sufficient conditions on operator in order to be narrow are not necessary ones. An example of a narrow operator $T \in \mathcal{L}(L_p)$ with $1 \leqslant p < \infty$ which fixes a copy of $L_p$ one can obtain considering the operator of conditional expectation with respect to an arbitrary non-atomic sub-σ-field of $\mathfrak{B}$ which has infinite codimension in $\mathfrak{B}$ (cf. [5]. Example 1 from § 8] ∍. Convenient necessary and sufficient conditions of narrowness of operators from $\mathcal{L}(L_1)$ are obtained in [13]: an operator $T \in \mathcal{L}(L_1)$ is narrow if and only if for every $A \in \mathfrak{B}$ the restriction $T|_{L_1(A)}$ is not an isomorphism. According to [3, 6] an operator $T \in \mathcal{L}(L_1, X)$ is narrow if and only if for each $A \in \mathfrak{B}$ and $\varepsilon > 0$ there exists $x \in L_1$ which takes alues in $\{-1, 0, 1\}$ on $A$ and 0 off $A$ and such that $\| Tx \| < \varepsilon \| x \|$.

Problem 2 [5]. Let $T \in \mathcal{L}(L_p, X)$ fixes no copy of $l_2$, $1 \leqslant p < \infty$, $p \neq 2$. Is it obligatory that $T$ is narrow?

Problem 3 [5]. Let $1 \leqslant p < 2$ and suppose that an infinite-dimensional Banach space $Z$ embeds in $L_p$ and contains no subspace isomorphic to $L_p$. Let $T \in \mathcal{L}(L_p, X)$ fixes no copy of $Z$. Is $T$ obligatory a narrow operator?

4. Banach spaces $X$ for which every operator $T \in \mathcal{L}(L_p, X)$ is narrow. The class of all infinite-dimensional separable Banach space $X$ for which every operator $T \in \mathcal{L}(L_p, X)$ is narrow, we denote by $\mathfrak{M}_p$. The most studied class among them is $\mathfrak{M}_1$. So $\mathfrak{M}_1$ contains all spaces which do not contain subspaces isomorphic to $l_1$ [12]. For a large class of separable Banach spaces $X$ (which in particular contains all separable duals) it is proved that if $L_1$ does not embed in $X$ then $X \in \mathfrak{M}_1$ [6]. One can see that $\mathfrak{M}_1$ contains all spaces with the Radon-Nikodym property. Indeed, let $X$ be a Banach space with the RNP. Then every operator $T \in \mathcal{L}(L_1, X)$ is representable [2, p. 63] and hence each operator is Dunford-Pettis [2, p. 74]. This yields by Proposition 2. a) $X \in \mathfrak{M}_1$. It is clear that if $X \in \mathfrak{M}_1$ then $L_1$ does not embed in $X$. But the converse is false as it has been observed above in Section 1 (cf. [4]).

It is not hard to see that for each $1 \leqslant r < 2$ and $1 \leqslant p < \infty$ $l_r \in \mathfrak{M}_p$ [5].

Recall that a Banach space $X$ is said to be of infratype $q > 1$ if there is a constant $C$ such that for each $\{X_k\}_{k=1}^n \subset X$

$$\min_{\theta_k = \pm 1} \Big\| \sum_{k=1}^{n} \theta_k x_k \Big\| \leqslant C \Big( \sum_{k=1}^{n} \| x_k \|^q \Big)^{1/q}.$$

Theorem 4. *Suppose that $X$ is of infratype $q > 1$. Then for each $p$, $1 \leqslant p < q$, we have $X \in \mathfrak{M}_p$.*

Proof. Suppose $T \in \mathcal{L}(L_p, X)$, $A \in \mathfrak{B}$ and let $n \geqslant 1$ be an integer. Partition $A$ into $n$ equimeasurable subsets of equal measures $A_1, \ldots, A_n$. Then

$$\min_{\theta_k = \pm 1} \Big\| \sum_{k=1}^{n} \theta_k T\varkappa(A_k) \Big\| \leqslant C \Big( \sum_{k=1}^{n} \| T\varkappa(A_k) \|^q \Big)^{1/q} \leqslant$$

$$\leqslant C \| T \| \Big( \sum_{k=1}^{n} \| \varkappa(A_k) \|_{L_p}^q \Big)^{1/q} = C \| T \| \Big( \sum_{k=1}^{n} \Big( \frac{\lambda(A)}{n} \Big)^{q/p} \Big)^{1/q} =$$

$$= C \| T \| (\lambda(A))^{1/p} (n^{1-q/p})^{1/q} \to 0 \text{ as } n \to \infty.$$

This means that for every $\varepsilon > 0$ there is an integer n and a collection $\theta_1, \ldots, \theta_n$ of signs such that for

$$x = \sum_{k=1}^{n} \theta_k \varkappa(A_k)$$

we have $\|Tx\| < \varepsilon$. It is enough to conclude by Proposition 1 that $T$ is narrow.

Theorem 5. *$L_q \in \mathfrak{M}_p$ for $1 \leqslant p < 2$ and $p < q$.*

This follows from Teheorem 4 and the fact that $L_g$ is of infratype min $(q, 2)$ [14].

Remark that if $p \geqslant 2$ then $L_g \notin \mathfrak{M}_p$ for every $q$. Indeed, denote by $I_p$ the identity map from $L_p$ into $L_2$ and by $S$ an arbitrary isomorphic embedding from $L_2$ into $L_q$. Then $T = S \cdot I_p$ is a non-narrow operator from $L_p$ into $L_q$.

The next lemma was proved in [12] for $p = 1$ but in a long way.

Lemma 3. *Let* $1 \leqslant p < \infty$ *and* $T \in \mathcal{L}(L_p, c_0)$. *Then for every* $\varepsilon > 0$ *there is* $x \in L_p$ *such that* $|x(t)| \equiv 1$, $\int xd\,\lambda = 0$ *and* $\|Tx\| < \varepsilon$.

Proof. Let $T \in \mathcal{L}(L_p, c_0)$ and $\varepsilon > 0$. Put

$$K = \left\{x \in L_\infty \colon \|x\|_{L_\infty} \leqslant 1, \quad \|Tx\| \leqslant \varepsilon, \quad \int xd\,\lambda = 0\right\}.$$

Note that $K$ is a convex and a weakly compact subset of $L_p$. By the Krein-Milman theorem there is an extremal point $x_0 \in K$. Let us prove that $|x_0(t)| \equiv 1$. Supposing the contrary, choose $\delta > 0$ and $B \in \mathcal{B}$ with $\lambda(B) > 0$ so that $|x_0|(t) \leqslant 1 - \delta$ for all $t \in B$. Now we show that $|Tx_0| = \varepsilon$. Indeed, if we suppose that $\|Tx_0\| < \varepsilon$ then putting $x = h(\chi(B_1) - \chi(B_2))$ where $h$ is some positive number satisfying $h < \delta$ and

$$h\,\|T(\varkappa(B_1) - \varkappa(B_2))\| < \varepsilon - \|Tx_0\|$$

and $B_1 \cup B_2 = B$; $\lambda(B_1) = \lambda(B_2) = \lambda(B)/2$, we would obtain that $(x_0 + x) \in K$ and $(x_0 - x) \in K$, that is impossible since $x_0$ is an extremal point of $K$. Thus, $\|Tx_0\| = \varepsilon$. Set $Tx_0 = \sum_{n=1}^{\infty} z_n e_n$ where $\{e_n\}_{n=1}^{\infty}$ is the unit vector basis of $c_0$. Choose $n_0$ so that $|z_n| \leqslant \varepsilon/2$ for any $n \geqslant n_0$. The subspace

$$E = T^{-1}\left(\overline{\mathrm{Lin}\,\{e_{n_0+1}, e_{n_0+2}, \ldots\}}\right) \cap \left\{x \in L_p : \int xd\,\lambda = 0\right\}$$

has finite codimension in $L_p$, hence $L_\infty(B) \cap E \neq \{0\}$. Choose $x \in E \cap L_\infty(B)$, $x \neq 0$ so that $\|x\|_{L_\infty} \leqslant \delta$ and $\|Tx\| \leqslant \varepsilon/2$. Thus, $(x_0 + x) \in K$ and $(x_0 - x) \in K$. This contradiction proves the lemma.

In the case of reals where $|x|(t) \equiv 1$ means $x^2(t) \equiv 1$ Lemma 3 implies the next theorem.

Theorem 6. $c_0 \in \mathfrak{M}_p$ *whenever* $1 \leqslant p < \infty$.


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