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Hamiltonian analysis of exact solvability of the quantum 3-level supperradiance Dicke model

Гамільтоновий аналіз точної інтегровності трирівневої квантової моделі надвипромінювання Дікке

It is proved, that the quantum 3-level supperradiance Dicke model is exactly solvable. The Lax representation of the evolutionary equations system is derived basing on the theory of current Lie algebras. The quantum inverse scattering problem method is used to obtain quantum analogues of the action-angle variables. The spectra of the energy operator and the other quantum integrals of motion is constructed as well as the exact one- and multiparticles excitation eigenstates of the model. It is shown, that model possesses the states of constrained quasiparticles (quantum solitons), which involve the supperradianse pulses.

Доведена точна інтегровність трирівневої квантової моделі надвипромінювання Дікке. На основі теорії алгебр Лі струмів побудовано зображення Лакса операторної системи еволюційних рівнянь. Методом оберненої квантової задачі розсіювання знайдено квантовий аналог змінних дія-кут. Побудовано спектри інтегралів руху та точні одно- і багаточастинкові збуджені стани моделі. Показано, що моделі притаманні стани зв'язаних квазічастинок (квантові солітони), що спричиняють імпульси надвипромінювання.

1. Introduction. The supperradiance Dicke model play an essential role in quantum optics as well as in the general theories of multiparticles systems with interaction between the Fermi- and Bose- subsystems. This model was investigated by many authors (see review article [1], and [2]), wich applied to the problem different approximate methods. The aim of this work is to show, that 3-level Dicke model is exactly solvable using the inverse scattering problem method.

2. Description of the model and the basic operator evolutionary equations. We shall consider the one-dimension system of the 3-level atoms interacting with the electromagnetic field. The Hamiltonian of the system can be expressed as follows (we shall use the second quantisation representation)

$$H = -i\int_{+\infty}^{-\infty} dx \sum_{j=1}^{3} \varepsilon_j^+(x)\, d_x \varepsilon_j(x) -$$

$$-\int_{+\infty}^{-\infty} dx \sum_{j=1}^{3} \sqrt{\varkappa_j}\,[\varepsilon_j^+(x)\, p(x) + \varepsilon_j(x)\, p_j^+(x)] \tag{1}$$

where we note: $\varkappa_j$ — the interaction constant, the field operators of the electric field obey the Bose-commutation relations

$$[\varepsilon_j(x),\ \varepsilon_k^+(x)] = \delta_{jk}\delta(x-y). \tag{2}$$

The transition operators $p_j(x)$, the occupation operators $n_j(x)$ of the volume unit, the transition operators p_j^k and the occupation operators n_j^k of the k-number atom are related with the following expressions

$$p_j(x) = \sum_{k=1}^{N} p_j^k\, \delta(x-x_k), \qquad n_j(x) = \sum_{k=1}^{N} n_j^k\, \delta(x-x_k). \tag{3}$$

It is easy to show, they satisfy the next commutation relations

$$[p_1(x),\ p_1^+(y)] = (n_0(x) - n_1(x))\,\delta(x-y),$$
$$[p_2(x),\ p_2^+(y)] = (n_1(x) - n_2(x))\,\delta(x-y),$$
$$[p_3(x),\ p_3^+(y)] = (n_0(x) - n_2(x))\,\delta(x-y),$$
$$[p_1(x),\ p_2(y)] = p_3(x)\,\delta(x-y), \quad [p_1^+(x),\ p_3(y)] = p_2(x)\,\delta(x-y), \tag{4}$$
$$[p_3(x),\ p_2(y)] = p_1(x)\,\delta(x-y), \quad [p_1(x),\ n_1(y)] = p_1(x)\,\delta(x-y),$$
$$[n_1(x),\ p_1^+(y)] = p_1^+(x)\,\delta(x-y), \quad [p_2(x),\ n_2(y)] = p_2(x)\,\delta(x-y),$$
$$[n_2^+(x),\ p_2^+(y)] = p_2^+(x)\,\delta(x-y), \quad [n_1(x),\ p_2(y)] = p_2(x)\,\delta(x-y),$$
$$[p_3(x),\ n_2(y)] = p_3(x)\,\delta(x-y),$$
$$[p_1(x),\ p_2^+(y)] = [n_1(x),\ p_3(y)] = [n_2(x),\ p_1(y)] =$$
$$= [n_1(x),\ n_2(y)] = [p_1(x),\ p_3(y)] = [p_2(x),\ p_3(y)] = 0.$$

We want to emphasize here that application of the one-dimension model is in agreement with the experimental data, that the supperradiance phenomena is observable in long narrow needle-shaped mediums. Next approximation named «the rotation wave approximation» refers to the cancellation of the terms $\varepsilon_j^+(x)\,p_j^+(x) + \varepsilon_j(x)\,p_j(x)$, $j = \overline{1, 3}$, describing the effects of the simultaneous creation (annihilation) of the photons and excitations of the atoms.

The particles number operator of the model can be expressed as follows

$$N = \int_{-\infty}^{+\infty} dx \left[\sum_{j=1}^{3} \varepsilon_j^+(x)\,\varepsilon_j(x) + \sum_{j=1}^{2} n_j(x)\right]. \qquad (5)$$

Using (1), (2), (4) it is easy to derive the evolutionary equations for the operators $\varepsilon_j(x)$, $p_j(x)$, $n_j(x)$:

$$\begin{aligned}
i\,(d_t\varepsilon_j + d_x\varepsilon_j) &= -\sqrt{\varkappa_j}\,p_j,\\
i\,(d_t\varepsilon_j^+ + d_x\varepsilon_j^+) &= \sqrt{\varkappa_j}\,p_j,\\
i\,d_t p_1 &= -\sqrt{\varkappa_1}\varepsilon_1(n_0 - n_1) - \sqrt{\varkappa_2}\varepsilon_2^+ p_3 + \sqrt{\varkappa_3}\varepsilon_3 p_2^+,\\
i\,d_t p_2 &= \sqrt{\varkappa_1}\varepsilon_1^+ p_3 - \sqrt{\varkappa_2}\varepsilon_2(n_1 - n_2) - \sqrt{\varkappa_3}\varepsilon_3 p_1^+,\\
i\,d_t p_3 &= \sqrt{\varkappa_1}\varepsilon_1 p_2 - \sqrt{\varkappa_2}\varepsilon_2 p_1 - \sqrt{\varkappa_3}\varepsilon_3(n_0 - n_2),\\
i\,d_t n_1 &= \sqrt{\varkappa_1}(\varepsilon_1^+ p_1 - p_1^+\varepsilon_1) - \sqrt{\varkappa_2}(\varepsilon_2^+ p_2 - p_2^+\varepsilon_2),\\
i\,d_t n_2 &= \sqrt{\varkappa_2}(\varepsilon_2^+ p_2 - p_2^+\varepsilon_2) - \sqrt{\varkappa_3}(\varepsilon_3^+ p_3 - p_3^+\varepsilon_3),\\
i\,d_t p_3^+ &= -\sqrt{\varkappa_1}\varepsilon_1^+ p_2^+ + \sqrt{\varkappa_2}\varepsilon_2^+ p_1^+ + \sqrt{\varkappa_3}\varepsilon_3^+(n_0 - n_2),\\
i\,d_t p_2^+ &= -\sqrt{\varkappa_1}\varepsilon_1 p_3^+ + \sqrt{\varkappa_2}\varepsilon_2^+(n_1 - n_2) + \sqrt{\varkappa_3}\varepsilon_3^+ p_1,\\
i\,d_t p_1^+ &= \sqrt{\varkappa_1}\varepsilon_1^+(n_0 - n_1) + \sqrt{\varkappa_2}\varepsilon_2 p_3^+ - \sqrt{\varkappa_3}\varepsilon_3^+ p_2.
\end{aligned} \qquad (6)$$

Notise, that after cancelations $\varepsilon_1(x) = \varepsilon(x)$, $\varepsilon_2(x) = \varepsilon_3(x) = 0$, $p_1(x) = p(x)$, $p_2(x) = p_3(x) = 0$, $n_1(x) = n(x)$, $n_2(x) = 0$ equations system (6) will take the form of the Maxwell-Bloch equations often used in the theories of interaction of the radiation with 2-level atoms medium. As it is proved [3] in the classical case and [4] for the quantum model, the Maxwell-Bloch system of equations is exactly solvable. The aim of the present work is to show the exactly solvability of the more general system (6), to give its Lax representation and to describe the excitation eigenstates of the quantum 3-level model using the quantum inverse scatering problem method.

3. Lax operator for the classical system of the evolutionary equation. In this chapter we shall construct the Lax operator for the classical analogues of the equations (6) using the ideas formulated in [5].

Let g be a Lie algebra and C_{ab}^c the structure constants of g with respect to the basis X_a $(a, b, c = 1, \ldots, n)$,

$$[X_a, X_b] = C_{ab}^c X_c. \qquad (7)$$

We shall introduce the co-ordinates in the linear space g^* conjugated to spase g as follows: if $\xi = \xi^a X_a \in g$, then $u(\xi) = (u, \xi) = u_a\xi^a$. We shall define bracket $\{.,.\}$ in the algebra $\mathcal{A}$ of smooth functions $f(u)$ on g^*

$$\{f_1, f_2\} = -C_{ab}^c \frac{df_1}{du_a}\frac{df_2}{du_b} u_c, \qquad (8)$$

which gives a poissonian structure on g^*. The Lie-Poisson bracket for the coordinates can be expressed as follows

$$\{u_a, u_b\} = -C_{ab}^c u_c. \qquad (9)$$

We shall consider the current algebra $C(g)$ associated with Lie algebra g

$$\xi(\lambda) = \sum_{k\gg-\infty}^{+\infty} \xi_k \lambda^k, \quad [\xi(\lambda), \eta(\lambda)] = \sum_{k\gg-\infty} \sum_{i+j=k} [\xi_i, \eta_j], \tag{10}$$

where $\xi_i, \eta_j \in g$, $\xi(\lambda), \eta(\lambda) \in C(g)$, and symbol $k \gg -\infty$ means that the row is finite on powers of the λ^{-1}. Generators of the current algebra $C(g)$ can be defined:

$$X_{a,k} = X_a \lambda^k, \quad [X_{a,k}, X_{b,l}] = C^c_{ab} X_{c,k+l}, \tag{11}$$

$$a, b, c = 1, \dots, n, \quad l, k = -\infty, \dots, +\infty.$$

The Lie-Poisson bracket for the coordinates in the conjugated spase is given by expressions

$$\{u_{a,k}, u_{b,l}\} = - C^c_{ab} u_{c,k+l}. \tag{12}$$

We shall introduce the generating function $u_a(\lambda)$ for the co-ordinates of the element $u \in C^*(g)$ in the formal Laurant series form

$$u_a(\lambda) = \sum_{k=-\infty}^{k<+\infty} u_{a,k} \lambda^{-k-1}, \tag{13}$$

$$u(\xi) = (u, \xi) = \sum_k u_{a,k} \xi^a_k = \operatorname{Res} u_a(\lambda) \xi^a(\lambda).$$

We can decompose $C(g)$ in the linear sum of two subalgebras

$$C(g) = C_+(g) + C_-(g), \text{ where } C_+(g) = \sum_{k=0} g\lambda^k, \quad C_-(g) = \sum_{-\infty<k}^{k=-1} g\lambda^k. \tag{14}$$

Analogical decomposition will take place for the conjugated spase $C^*(g)$. We shall introduce in $C^*(g)$ the new poissonian structure. According to the decomposition (14) we shall introduse the new Lie algebra structure in the vector spase $C(g)$ with commutator $[.,.]_0$ as follows

$$[\xi_\pm, \eta_\pm]_0 = \pm [\xi_\pm, \eta_\pm], \quad [\xi_\pm, \eta_\mp]_0 = 0, \tag{15}$$

where $\xi = \xi_+ + \xi_-$, $\eta = \eta_+ + \eta_+ \in C(g)$.

Accordant Lie-Poisson bracket in the space $C^*(g)$ in terms of the generating functions can be expressed by the formulae:

$$\{u_a(\eta_+), u_b(\mu)\}_0 = C^c_{ab} \frac{u_c(\lambda) - u_c(\mu)}{\lambda - \mu}, \quad \{u_a(\lambda), u_b(\mu)\}_0 = 0. \tag{16}$$

Expression (16) can be united into one. If we admit the existence of the nongenerative Kiling form of the Lie algebra g, and introduce quantities

$$K_{ab} = (K^{ab})^{-1} = \langle X_a, X_b \rangle = \operatorname{Tr}(X_a X_b), \quad A^a = K^{ab} X_b, \tag{17}$$

$$r(\lambda) = \lambda^{-1} K^{ab} X_a \otimes X_b,$$

$$U(\lambda) = u_a(\lambda) A^a, \tag{18}$$

where we denote tensor product with the symbol $\otimes$, we can ascertain that the Lie-Poisson bracket (16) in terms of this quantities will take the form of the Yang-Baxter equation:

$$\{U(\lambda) \otimes U(\mu)\}_0 = [r(\lambda - \mu), U(\lambda) \otimes 1 + 1 \otimes U(\mu)]. \tag{19}$$

We can easily introduce in (19) the quantity dependence from the spase coordinate x, which took place in the Yang-Baxter equation using the current algebra $\mathcal{G}(g)$ of the Laurant series $\xi(x, \lambda)$ with coefficient, which depend from x and satisfy certainly boundary conditions. It is evident, that

$$[X_{a,k}(x), X_{b,l}(y)] = C^c_{ab} X_{c,k+l} \delta(x-y). \tag{20}$$

The Yang-Baxter equation contains the concrete matrix U (x, λ) in the auxiliary space; this matrix is a rational function of the spectral parametre λ, but in the (19) the formal Laurant series with coeffisient in the given Lie algebra g is present. The agreement of this problem is following: fundamental Poisson bracket for the concrete model are the realisation of the (19) for the concrete matrix representation of the given Lie algebra g restricted on the orbit of the respective algebra $\mathfrak{G}_0(g)$ in the phase spase $\mathfrak{G}_0^*(g)$.

In order to describe the finite-dimension orbits of the Lie algebra $C_0(g)$ we can introduce the finite-dimension Poisson submanifolds in $C_9^*(g)$, puting invariant relatively the Poisson action of the Lie algebra $C_0(g)$ restriction on the coordinates $u_{a,k}$ (or on its generation functions $u_a(\lambda)$, $U(x, \lambda)$. After the respective restrictions the matrix $U(x, \lambda)$ will be the rational function of variable λ and will define the Lax operator L ($L(x, \lambda) = d_x - U(x, \lambda)$).

Let g be Lie algebra SU (3); we can express its generators X_a, $a = 1, \ldots, \ldots, 8$ as follows:

$$X_1 = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix},$$
$$X_4 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{21}$$
$$X_7 = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad X_8 = \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Let us concretize the expression (18), taking the Lax operator in the following form

$$L(x, \lambda) = d_x - \left(\lambda J_a(x) A^a + Q_a(x) A^a + \frac{1}{\lambda} M_a(x) A^a\right). \tag{22}$$

We shall define the orbits on the phase space by the conditions:

$$\begin{gathered}
J_1(x) = \ldots = J_6(x) = J_8(x) = 0, \quad J_7(x) = 2, \\
Q_1(x) = \varepsilon_1(x) + \varepsilon_1^*(x), \quad Q_4(x) = i(\varepsilon_1^*(x) - \varepsilon_1(x)), \\
Q_2(x) = \varepsilon_2(x) + \varepsilon_2^*(x), \quad Q_5(x) = i(\varepsilon_2(x) - \varepsilon_2^*(x)), \\
Q_3(x) = \varepsilon_3(x) + \varepsilon_3^*(x), \quad Q_6(x) = i(\varepsilon_3(x) - \varepsilon_3^*(x)), \\
Q_7(x) = Q_8(x) = 0, \\
M_1(x) = p_1(x) + p_1^*(x), \quad M_4(x) = i(p_1^*(x) - p_1(x)), \\
M_2(x) = p_2(x) + p_2^*(x), \quad M_5(x) = i(p_2(x) - p_2^*(x)), \\
M_3(x) = p_3(x) + p_3^*(x), \quad M_6(x) = i(p_3(x) - p_3^*(x)), \\
M_7(x) = n_1(x) - n_2(x), \quad M_8(x) = -3(n_1(x) + n_2(x)),
\end{gathered} \tag{23}$$

where $\varepsilon_j(x)$, $p_j(x)$, $n_j(x)$ — field functions of the classical model with respect to the operator equations system of the quantum field model. It is possible to draw this conclusion using the following argumentation. The orbits defined by equations (25) are the Poisson submanifold relatively to the Lie-Poisson brackets:

$$\begin{gathered}
\{J_a(x), J_b(y)\}_0 = \{J_a(x), Q_b(y)\}_0 = \{J_a(x), M_b(y)\}_0 = \{Q_a(x), M_b(y)\}_0 = 0, \\
\{Q_a(x), Q_b(y)\}_0 = C_{ab}^c J_c(x) \delta(x - y), \\
\{M_a(x), M_b(y)\}_0 = -C_{ab}^c M_c(x) \delta(x - y).
\end{gathered} \tag{24}$$

If we express using (23) the Poisson bracket (24) in terms of the field functions $\varepsilon_j(x)$, $p_j(x)$, $n_j(x)$ and realize the transition to the commutators of the quantumfield operators in agreement with the formula $\{.,.\} \to \frac{i}{\hbar}[.,.]$, we shall receive the complete set of the commutator relations (2), (4).

So, expressions (21)—(23) give a possibility to construct the Lax operator $L(x, \lambda)$ for the classical analogy of the operator equations system

$$L(x,\lambda) = d_x - i \begin{pmatrix} \lambda & \varepsilon_1^*(x) & \varepsilon_2(x) \\ \varepsilon_1(x) & 0 & \varepsilon_3(x) \\ \varepsilon_2^*(x) & \varepsilon_3^*(x) & -\lambda \end{pmatrix} - \frac{i}{\lambda} \begin{pmatrix} n_1(x) & p_1^*(x) & p_2(x) \\ p_1(x) & -(n_1(x)+n_2(x)) & p_3(x) \\ p_2^*(x) & p_3^*(x) & n_2(x) \end{pmatrix}. \qquad (25)$$

4. The auxiliary quantum eigenvalue problem. Let us take to consideration the three-component operator field $\psi_j(x)$, $j = \overline{0, 2}$, obeying the Bose-commutation relations and the completety condition

$$[\psi_j(x), \psi_k^+(x)] = \delta_{jk}\delta(x-y), \qquad \sum_{j=1}^{2} \psi_j^+(x)\psi_j(x) = 1. \qquad (26)$$

We shall express the atom operators of the model in terms of the components of this field as follows

$$p_1(x) = \psi_0^+(x)\psi_1(x), \quad p_2(x) = \psi_1^+(x)\psi_2(x)_2 \quad p_3(x) = \psi_0^+(x)\psi_2(x),$$
$$p_1^+(x) = \psi_1^+(x)\psi_0^+(x), \quad p_2^+(x) = \psi_2^+(x)\psi_1(x), \quad p_3^+(x) = \psi_2^+(x)\psi_0(x), \qquad (27)$$
$$n_j(x) = \psi_j^+(x)\psi_j(x), \qquad j = \overline{0, 2},$$

so, each of the levels of the atoms medium we shall represent by the own oscillators. As it is easy to show, operator products (27) satisfy the commutation relations (4), so the dynamical properties of the model will not be changed by using the ossillator representation (26), (27). We shall construct quantum form of the Lax operator using quantum operators in the oscillator representation instead of the classical field functions and puting the interaction constants

$$L(x,\lambda) = d_x - i \begin{pmatrix} \lambda & \sqrt{\varkappa_1}\,\varepsilon_1^+(x) & \sqrt{\varkappa_2}\,\varepsilon_2(x) \\ \sqrt{\varkappa_1}\,\varepsilon_1(x) & 0 & \sqrt{\varkappa_3}\,\varepsilon_3(x) \\ \sqrt{\varkappa_2}\,\varepsilon_2^+(x) & \sqrt{\varkappa_3}\,\varepsilon_3^+(x) & -\lambda \end{pmatrix} -$$
$$- \frac{i}{\lambda} \begin{pmatrix} \psi_1^+(x)\psi_1(x) & \varkappa_1\psi_1^+(x)\psi_0(x) & \varkappa_2\psi_1^+(x)\psi_2(x) \\ \varkappa_1\psi_0^+(x)\psi_1(x) & \psi_0^+(x)\psi_0(x) - 1 & \varkappa_3\psi_0^+(x)\psi_2(x) \\ \varkappa_1\psi_2^+(x)\psi_1(x) & \varkappa_3\psi_2^+(x)\psi_0(x) & \psi_2^+(x)\psi_2(x) \end{pmatrix}. \qquad (28)$$

We shall consider the auxiliary quantum eigenvalue problem on the infinite interval $-\infty < x < +\infty$:

$$d_x\Phi(x,\lambda) = :U(x,\lambda)\Phi(x,\lambda):, \qquad (29)$$

where we note: $\Phi(x, \lambda)$-matrix of the decision of (31) that can be represented in the form (symbol τ means transponisation):

$$\Phi(x,\lambda) = (\varphi_1(x,\lambda), \varphi_2(x,\lambda), \varphi_3(x,\lambda))^\tau \qquad (30)$$

and we denote by : : the normal order of the operators. We shall define the transition matrix $T(\lambda)$ on the infinite interval. Let decision matrix $\Phi(x, \lambda)$ of the equation (29) satisfy the boundary condition:

$$\Phi(x, \lambda)|_{x=-\infty} = (1, 0, 1)^{\tau}. \tag{31}$$

Then the transition matrix will be defined by the value of the decision matrix in the point $x = +\infty$

$$T(\lambda) \equiv (A(\lambda),\ B(\lambda),\ C(\lambda))^{\tau} = \Phi(x, \lambda)|_{x=+\infty}. \tag{32}$$

We can construct the following form of the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ using the expressions (28)—(32):

$$A(\lambda) = i\int_{-\infty}^{+\infty} dx \exp(i\,k_1 x)\frac{1}{\lambda}\,\psi_1^{+}(x)\,g_1(x, \lambda)\,\psi_1(x) +$$

$$+ i\int_{-\infty}^{+\infty} dx \exp(i\,k_2 x)\left[\sqrt{\varkappa_1}\,\varepsilon_1^{+}(x)\,g_2(x, \lambda) + \frac{\varkappa_1}{\lambda}\,\psi_1^{+}(x)\,g_2(x, \lambda)\,\psi_0(x)\right] +$$

$$+ i\int_{-\infty}^{+\infty} dx \exp(i\,k_3 x)\left[\sqrt{\varkappa_2}\,\varepsilon_2(x)\,g_3(x, \lambda) + \frac{\varkappa_2}{\lambda}\,\psi_1^{+}(x)\,g_3(x, \lambda)\,\psi_2(x)\right],$$

$$B(\lambda) = 1 + i\int_{-\infty}^{+\infty} dx \exp(i\,k_2 x)\frac{1}{\lambda}\,\psi_0^{+}(x)\,g_2(x, \lambda)\,\psi_0(x) +$$

$$+ i\int_{-\infty}^{+\infty} dx \exp(i\,k_1 x)\left[\sqrt{\varkappa_1}\,\varepsilon_1(x)\,g_1(x, \lambda) + \frac{\varkappa_1}{\lambda}\,\psi_0^{+}(x)\,g_1(x, \lambda)\,\varphi_1(x)\right] +$$

$$+ i\int_{-\infty}^{+\infty} dx \exp(i\,k_3 x)\left[\sqrt{\varkappa_3}\,\varepsilon_3(x)\,g_3(x_j\, \lambda) + \frac{\varkappa_3}{\lambda}\,\psi_0^{+}(x)\,g_3(x, \lambda)\,\psi_2(x)\right], \tag{33}$$

$$C(\lambda) = i\int_{-\infty}^{+\infty} dx \exp(i\,k_3 x)\frac{1}{\lambda}\,\psi_2^{+}(x)\,g_3(x, \lambda)\,\psi_2(x) +$$

$$+ i\int_{-\infty}^{+\infty} dx \exp(i\,k_2 x)\left[\sqrt{\varkappa_3}\,\varepsilon_3^{+}(x)\,g_2(x, \lambda) + \frac{\varkappa_3}{\lambda}\,\psi_2^{+}(x)\,g_2(x, \lambda)\,\psi_0(x)\right] +$$

$$+ i\int_{-\infty}^{+\infty} dx \exp(i\,k_1 x)\left[\sqrt{\varkappa_2}\,\varepsilon_2^{+}(x)\,g_1(x, \lambda) + \frac{\varkappa_2}{\lambda}\,\psi_2^{+}(x)\,g_1(x, \lambda)\,\psi_1(x)\right],$$

where we denote:

$$g_j(x, \lambda) = \psi_j(x, \lambda)\exp(i\,k_j x), \quad k_j = -\lambda - (\varkappa_j/\lambda), \qquad j = \overline{1, 3}. \tag{34}$$

The aim of the application of the auxiliary quantum eigenvalue problem is to transit from the description of the model in terms of the local operator fields $\varepsilon_j(x)$, $\psi_k(x)$, $j = \overline{1, 3}$, $k = \overline{0, 2}$ to it description in the terms of the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ which are the quantum analogues of the action-angle variables, and, as we shall see later, they create the one-particle excitation state of the 3-level supperradiance Dicke model.

5. Exactly excitation eigenstate of the 3-level supperradiance Dicke model. Let us define a vacuum state of the model as the state in which all of the 3-level atoms are in the ground states and there are no photons in the system. This state (we shall denote it by $|0\rangle$) possesses the following properties

$$\begin{aligned}
&\varepsilon_1(x)|0\rangle = 0, \quad \varepsilon_1^{+}(x)|0\rangle \neq 0, \quad \psi_0(x)|0\rangle \neq 0, \quad \psi_0^{+}(x)|0\rangle = 0,\\
&\varepsilon_2(x)|0\rangle = 0, \quad \varepsilon_2^{+}(x)|0\rangle = 0, \quad \psi_1(x)|0\rangle = 0, \quad \psi_1^{+}(x)|0\rangle \neq 0,\\
&\varepsilon_3(x)|0\rangle = 0, \quad \varepsilon_3^{+}(x)|0\rangle \neq 0, \quad \psi_2(x)|0\rangle = 0, \quad \psi_2^{+}(x)|0\rangle \neq 0.
\end{aligned} \tag{35}$$

Evidently, the vacuum state is the eigenstate of the operator $B(\lambda)$; action of the operators $A(\lambda)$, $C(\lambda)$ on the vacuum state will create the following states

$$A(\lambda)|0\rangle = |\Psi_1(\lambda)\rangle = i\sqrt{\varkappa_1}\int_{-\infty}^{+\infty} dx \exp(i k_1 x)\,[\varepsilon_1^+(x) + (\varkappa_1/\lambda]\,\psi_1^+(x)\,\psi_0(x)]\,|0\rangle, \tag{36}$$

$$C(\mu)|0\rangle = |\Psi_2(\mu)\rangle = i\sqrt{\varkappa_3}\int_{-\infty}^{+\infty} dx \exp(i k_2 x)\,[\varepsilon_3^+(x) + (\varkappa_3/\mu)\,\psi_2^+(x)\,\psi_0(x)]\,|0\rangle, \tag{37}$$

where

$$k_1 = -\lambda - (\varkappa_1/\lambda), \quad k_2 = -\mu - (\varkappa_3/\mu). \tag{38}$$

We can specify the expressions:

$$N|\Psi_1(\lambda)\rangle = |\Psi_1(\lambda)\rangle, \quad H|\Psi_1(\lambda)\rangle = -\lambda|\Psi_1(\lambda)\rangle, \tag{39}$$

$$N|\Psi_2(\mu)\rangle = |\Psi_2(\mu)\rangle, \quad H|\Psi_2(\mu)\rangle = -\lambda|\Psi_2(\mu)\rangle. \tag{40}$$

It is evident that the states $|\Psi_1(\lambda)>$ and $|\Psi_2(\mu)>$ are oneparticle eigenstates of the model; they describe the systems which contains one excitation atom (its electron is on the first or the second excitation level respectively) and one photon of respective energy; spectral parameters represent the energy of this excitations (with inverse sign).

Multiparticles states of the system can be constructed by expressions

$$|\Phi(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m)\rangle = A(\lambda_1)\ldots A(\lambda_n)\,C(\mu_1)\ldots C(\mu_m)|0\rangle, \tag{41}$$

$$N|\Phi(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m)\rangle = (n+m)|\Phi(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m)\rangle, \tag{42}$$

$$H|\Phi(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m)\rangle = -(\lambda_1 + \ldots + \lambda_n + \mu_1 + \ldots + \mu_m)\times \\ \times|\Phi(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m)\rangle. \tag{43}$$

If all λ_j, μ_k in (41)—(43) are real, then state (41) describe $(n+m)$ free quasiparticles of the model with energies Ω_1 and Ω_2, relationships between the quasiparticles energy $\Omega_j = -\lambda_j$ and its impulses are defined by the formula (38).

As well as in the nonlinear Schrodinger model [6] and in the two-level supperradiance model [3, 5] the quasiparticles of the model can create the constrained states with a complex λ_j, μ_k. One can find the expressions:

$$B(\lambda)|\Phi(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m)\rangle = \frac{\lambda-\lambda_1 - i\varkappa_1/2}{\lambda-\lambda_1 + i\varkappa_1/2}\ldots$$

$$\ldots\frac{\lambda-\lambda_n - i\varkappa_1/2}{\lambda-\lambda_n + i\varkappa_1/2}\,\frac{\lambda-\mu_1 - i\varkappa_3/2}{\lambda-\mu_1 + i\varkappa_3/2}\ldots\frac{\lambda-\mu_m - i\varkappa_3/2}{\lambda-\mu_m + i\varkappa_3/2} =$$

$$= b_{n+m}(\lambda, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m)\,|\Phi(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m)\rangle. \tag{44}$$

According to the quantum inverse scattering problem method [6, 7] we can find the spectra of the constraind states of the model by analytical continuation of the eigenvalues of the operator $B(\lambda)$.

This operation is defined by two conditions:

1) $b_{n+m}(\ldots)$ is analytical on λ if $\mathrm{Im}\,\lambda > 0$ and has one zero in the top half of the co-ordinate plane;

2) $|b_{n+m}(\ldots)| = 1$ if $\mathrm{Im}\,\lambda = 0$.

Conditions 1), 2) are satistied if λ_j, μ_k obey the next relationships:

$$\lambda_j = \frac{\lambda}{n} + i\varkappa_1\left(\frac{n+1}{2} - j\right), \quad \mu_k = \frac{\mu}{m} + i\varkappa_3\left(\frac{m+1}{2} - k\right), \tag{45}$$

where $j = 1, \ldots, n$, $k = 1, \ldots, m$ and the constrained state energy is:

$$\Omega = -\lambda_1 - \ldots - \lambda_n - \mu_1 - \ldots - \mu_m. \tag{46}$$

As well as in the classical theory of the Maxwell-Bloch equations system [3] and in the theory of the quantum 2-level supperradiance Dicke model [4], we can interprete the transitions from this quasiparticle states as the quantum solitons describing the supperradianse pulses. From the other side, the transition from the multiparticle states of the free quasiparticles describe the spontaneous atoms radiation. We would like to emphasize here, that existence of the many different sets of the constrained quasiparticles as eigenstates of the model completely explain the experimentally observed oscillator-type of the supperradiance phenomena.

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