UDC 515.12+512.58

O. B. Pikhurko, stu...
M. M. Zarichnyi, cand. phys.-math. sci. (Lviv. Univ.)

On lifting of functors to the Eilenberg-Moore category of the triple generated by the functor C_pC_p

Про підняття функторів на категорію Ейленберга — Мура монади, породженої функтором C_pC_p

The second iteration of contravariant functor of the space of continuous functions with topology of pointwise convergence is functorial part of a triple on the category of Tychonoff spaces. The problem of lifting of functors to the Eilenberg-Moore category of this triple is investigated.

Друга ітерація контраваріантного функтора просторів неперервних функцій в топології поточкової збіжності є функторіальною частиною монади (трійки) на категорії тихоновських просторів. Досліджується задача підняття функторів на категорію Ейленберга — Мура цієї монади.

1. Triple generated by the functor C_pC_p. A triple $\mathbf{T}=(T, \eta, \mu)$ on category C consists of an endofunctor $T: C \to C$ and natural transformations $\eta: 1_C \to T$ and $\mu: TT \to T$ satisfying the following conditions: $\mu \circ \eta T = \mu \circ T\eta = 1_T$, $\mu \circ T\mu = \mu \circ \mu T$ [1].

Denote by Tych the category of Tychonoff topological spaces and continuous maps. The contravariant functor C_p : Tych $\to$ Tych is defined as follows. The space C_pX of all continuous real-valued functions on X is equipped with topology of pointwise convergence [2]; for mapping $f: X \to Y$ (both X and Y are Tychonoff spaces) we define $C_pf: C_pY \to C_pX$ by the formula: $C_pf(\varphi) = \varphi \circ f$, $\varphi \in C_pY$. For every $x \in X$ denote by $ev_x: C_pX \to \mathbb{R}$ a map defined by the formula $ev_x(\varphi) = \varphi(x)$, $\varphi \in C_pX$. It is well known that the map $\eta X: X \to C_pC_pX$, $\eta X(x) = ev_x$, $x \in X$, is continuous [2]. It is easy to see that $\eta = (\eta X)$ is natural transformation from 1_{Tych} into C_pC_p. We construct the natural transformation $\mu : C_pC_pC_pC_p \to C_pC_p$ using the following equality $\mu X(\Phi)(\varphi) = \Phi(ev_\varphi)$, $\Phi \in C_pC_pC_pC_pX$, $\varphi \in C_pX$. The map $\mu X(\Phi): C_pX \to \mathbb{R}$ is continuous as the composition of $ev: C_pX \to C_pC_pC_pX$ and $\Phi: C_pC_pC_pX \to \mathbb{R}$, so μX is correctly defined. In order to see that μX is continuous we need to verify a simple inclusion $\mu X(\Phi, ev_\varphi, \varepsilon) \subseteq (\mu X(\Phi), \varphi, \varepsilon)$ and to use the fact that the sets $(\Gamma, \psi, \varepsilon) = \{E \in C_pC_pX \mid |E(\psi) - \Gamma(\psi)| < \varepsilon\}$ form a subbase in C_pC_pX (for $\Gamma \in C_pC_pX$, $\psi \in C_pX$, $\varepsilon \in \mathbb{R}$).

Proposition 1. *$\mathbb{C}_p^2 = (C_pC_p, \eta, \mu)$ is a triple on the category Tych.*

Proof. Since

$$\mu X \circ C_pC_pX(\varphi)(\psi) = C_pC_p\eta X(\varphi)(ev_\psi) = \varphi(C_p(\eta X(ev_\psi))) = \varphi(\psi)$$

and $\mu X \circ \eta C_pC_pX(\varphi)(\psi) = \mu X(ev_\varphi)(\psi) = ev_\varphi(e\psi_\psi) = \varphi(\psi)$ for arbitrary $\varphi \in C_pC_pX$, $\psi \in C_pX$, we obtain $\mu \circ C_pC_p\eta = \mu \circ \eta C_pC_p = 1$. For every $\psi \in C_pC_pC_pC_pX$, $\varphi \in C_pX$ we have: $C_p\mu X(ev_\varphi)(\psi) = ev_\varphi \circ \mu X(\psi) = \mu X(\psi) \times (\varphi) = \psi(ev_\varphi)$, and therefore $C_p\mu X(ev_\varphi) = ev$. Putting this equalities together we can obtain

$$\mu X \circ C_pC_p X(\Phi)(\varphi) = C_pC_p\mu X(\Phi)(ev_\varphi) = \Phi \circ C_p\mu X(ev_\varphi) = \Phi(ev_\varphi) =$$

$$= \mu C_pC_pX(\Phi)(ev_\varphi) = (\mu X \circ \mu C_pC_pX(\Phi))(\varphi) \text{ (i. e. } \mu \circ C_pC_p\mu = \mu \circ \mu C_pC_p).$$

The proposition is proved.

Remark that the space C_pC_pX has obvious algebra structure with respect to pointwise addition and multiplication of functions and multiplication on scalars. Let L_pX be the linear subspace of C_pX generated by the image of X under the mapping ηX and A_pX be the least subalgebra of C_pX including the set $\eta X(X)$. It is clear that both L_p and A_p are subfunctors of C_p.

The condition $\mu X \circ \eta C_pC_pX = 1_x$ implies that $\mu X(L_pL_pX) \subseteq L_pX$ and $\mu X(A_pA_pX) \subseteq A_pX$. Therefore, we obtain two new triples $\mathbf{L}_p = (L_p, \eta, \mu | L_pL_p)$, $\mathbf{A}_p = (A_p, \eta, \mu | A_pA_p)$ on the category Tych.

2. Normal functors on the category Tych. Functor F : Tych $\to$ Tych is called normal if it is continuous, preserves weight, monomorphisms, intersections, inverse images, empty space, one point space and transforms k-covering maps into surjective (see [3]). (Note that a map $f : X \to Y$ is k-covering iff for arbitrary compact subspace $K \subseteq Y$ there exists a compact subspace $L \subseteq X$ such that $f(L) = K$.

A normal functor F : Tych $\to$ Tych is said to be of degree $\leqslant n$ (briefly $\deg(F) \leqslant n$) if for every $a \in FX$ there exists $b \in Fn$ and mapping $f : n \to X$ such that $a = Ff(b)$. A normal functor is called finite if it preserves the class of finite spaces and is called multiplicative if it preserves products.

Proposition 2. *Normal multiplicative functor F is isomorphic to power functor $(-)^i$ for sowe $i < \infty$, if either*

a) $\deg(F) = n$ *(and then $i = n$); or*

b) *F is finite*

Proof. See [4, 5].

Theorem 1. *Let F be a normal multiplicative functor F : Tych $\to$ Tych such that F* (Comp)$\subseteq$ Comp. *Then F is a subfunctor of $(-)^\omega$.*

Proof. Without loss of generality we can assume that $\deg(F) = \infty$. One can find in [6] the following result : there exists a functor isomorphism $h : F|\,\mathrm{Comp} \to (-)^\omega\,|\,\mathrm{Comp}$. First we prove that for every Tychonoff space X and its compactification $i_x : X \to bX$ the following inclusions hold: $hbX \circ F|i_x(FX) \subseteq X^\omega \subseteq (bX)^\omega$. Assuming the contrary, without loss of generality we can consider X to be the discrete countable space and bX to be αX (the one point compactification of X). Let $a \in h\,\alpha X \circ Fi_x(FX) \setminus X^\omega$. Then there exists a sequence $(b_i)_{i=1,2,\dots}$ in $F\alpha X$ converging to a and such that the supports of b_i are finite and lie in X.

Suppose that $a = (x_i)_{i=1,2,\dots} \in (\alpha X)^\omega$. Without loss of generality we can assume that $x_0 \in \alpha X \setminus X$. Suppose that $h\alpha X \circ Fi_x(b_i) = (y_{ij})_{j=1,2,\dots}$. There exists a mapping $f: X \to Y$ such that the sequence $(f(y_{i0})_{i=1,2,\dots})$ is not convergent in αX. Therefore the sequence $(Fi_x \circ Ff(b_i))_{i=1,2,\dots}$ is not convergent in $F(\alpha X)$, and we get a contradiction.

Now we define the natural transformation $j : F \to (-)^\omega$ by the formula: $jX = h\beta X \circ Fi_x$ (where $i_x : X \to \beta X$ is the canonical embedding X into Stone-Cech compactification βX of space X). Theorem is proved.

3. Lifting normal functors to the Eilenberg-Moore category. A couple (X, ξ), where $\xi : TX \to X$ is C-morphism, is called $\mathbf{T}$-algebra iff $\xi \circ \mu X = 1_x$ and $\xi \circ \mu X \xi \circ T\xi$. A morphism $f : X \to Y$ is called morphism of $\mathbf{T}$-algebra (X, ξ) into $\mathbf{T}$-algebra (Y, ζ) if $f \circ \xi = \zeta \circ Tf$. $\mathbf{T}$-algebras and their morphisms form a category which is usually denoted by $C^{\mathbf{T}}$ (Eilenberg-Moore category). We can define the forgetful functor $U^{\mathbf{T}} : C^{\mathbf{T}} \to C$ by $U^{\mathbf{T}}(X, \xi) = X$, $U^{\mathbf{T}}(f) = f$. (For details see [1].)

A lifting of functor $F : C \to C$ on the category $C^{\mathbf{T}}$ is a functor $G : C^{\mathbf{T}} \to C^{\mathbf{T}}$ such that $U^{\mathbf{T}} \circ G = F \circ U^{\mathbf{T}}$. The following proposition gives a criterion of existing of a lifting: it is dual to a result of J. Vinárek [7] (see also [8]).

P r o p o s i t i o n 3 (s e e [9]). *There exists a bijective correspondence between the liftings of functor F to $C^{\mathbf{T}}$ and the such natural transformations $\delta : TF \to FT$ that $\delta \circ \eta F = F\eta$ and $\delta \circ \mu F = F\mu \circ \delta T \circ T\delta$.*

Let $\mathbf{T}$ denote one of the triples $\mathbb{C}_p^2$, $\mathbf{A}_p$, $\mathbf{L}_p$. We use below the method used in [5] for describing functors which admit a lifting to the category of compact groups.

T h e o r e m 2. *If a normal functor F can be lifted to the category $Tych^{\mathbf{T}}$, then F is multiplicative.*

P r o o f. Let T be one of the functors C_pC_p, L_p, A_p. We consider the free $\mathbf{T}$-algebra $(TQ, \mu Q)$ denoting TQ by X (Q is the Hilbert cube). Supposing that F admit a lifting to $\mathrm{Tych}^{\mathbf{T}}$ we obtain that the mapping $f = (Fpr_1, Fpr_2) : F(X \times X) \to FX \times FX$ is bijective. Indeed, from the conditions of preserving inverse images and intersections by F we obtain $\ker(f) = 0$ (here we use fact that f is linear mapping of topological linear spaces). Besides, since the set $\ker(Fpr_1) = F(\ker(pr_1))$ is homeomorphically mapped onto FX by the mapping Fpr_2 we obtain that f is surjective (see [5]).

Since Q can be topologically embedded into X, we obtain that F is multiplicative (see [4]).

C o r o l l a r y. *If F is a normal functor admitting a lifting to the category $Tych^{\mathbf{T}}$ and F is either finite or* $\deg(F) < \infty$, *then F is isomorphic to a power functor.*

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Received 06.03.92