DOI: 10.37863/umzh.v72i7.830

UDC 517.98

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ESSENTIAL AMENABILITY OF FRÉCHET ALGEBRAS

СУТТЄВА АМЕНАБЕЛЬНІСТЬ АЛГЕБР ФРЕШЕ

Essential amenability of Banach algebras have been defined and investigated. Here, this concept will be introduced for Fréchet algebras. Then a number of well-known results of essential amenability of Banach algebras are generalized for Fréchet algebras. Moreover, related results about Segal – Fréchet algebras are provided. As the main result, it is proved that if $(\mathcal{A}, p_\ell)$ is an amenable Fréchet algebra with a uniformly bounded approximate identity, then every symmetric Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$ is essentially amenable.

Суттєву аменабельність банахових алгебр було визначено та вивчено раніше. Тут цю концепцію визначено для алгебр Фреше. Після цього ряд відомих результатів із аменабельності банахових алгебр узагальнено на випадок алгебр Фреше. Також наведено результати, які стосуються алгебр Сігала – Фреше. Основним є твердження про те, що у випадку, коли $(\mathcal{A}, p_\ell)$ — аменабельна алгебра Фреше з рівномірно обмеженою наближеною тотожністю, всі симетричні алгебри Сігала – Фреше у $(\mathcal{A}, p_\ell)$ є суттєво аменабельними.

1. Introduction. Amenability of Fréchet algebras was introduced by Helemskii in [7] (Definition 2.16) and was studied by A. Yu. Pirkovskii in [10]. Also in [8], P. Lawson and C. J. Read introduced and studied approximate amenability and approximate contractibility of Fréchet algebras. In [1], we studied the notion of weak amenability of Fréchet algebras and obtained some results on weak amenability of Fréchet algebras. Also according to the basic definition of Segal algebras and abstract Segal algebras, we introduced the concept of a Segal – Fréchet algebra in the Fréchet algebra $(\mathcal{A}, p_\ell)$. We then showed that every continuous linear left multiplier on a Fréchet algebra $(\mathcal{A}, p_\ell)$ is a continuous linear left multiplier on any Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$. Moreover, we showed that if $\mathcal{A}$ is a commutative Fréchet Q-algebra, then the space of all modular maximal closed ideals of $\mathcal{A}$ and any Segal – Fréchet algebra $(\mathcal{B}, q_m)$ in $(\mathcal{A}, p_\ell)$ are homeomorphic. In particular, we proved that $(\mathcal{A}, p_\ell)$ is semisimple if and only if $(\mathcal{B}, q_m)$ is semisimple (see [2]). Recently, we introduced the concept of character contractibility of Fréchet algebras, according to its definition in the Banach case [3]. We then verified available results about right φ-contractibility and right character contractibility of Banach algebras for Fréchet algebras. Finally, we provided related results about Segal – Fréchet algebras.

In [5], Ghahramani and Loy introduced and investigated the notion of essential amenability of Banach algebra. Then Samea [12] continued this verification and as a main result, he generalized [5] (Theorem 7.1). In fact he proved that any symmetric abstract Segal algebra with respect to an amenable Banach algebra is essentially amenable [12] (Theorem 4.4).

In the present work, we first introduce the concept of essential amenability of Fréchet algebras. Our definition of essential amenability coincides with the Banach algebra case, whenever Banach algebra $\mathcal{A}$ is considered as a Fréchet algebra. Then we verify most of the available results in the Banach algebra case, for Fréchet algebras. The last section contains the main results of this paper. We first recall from [2], the concept of Segal – Fréchet algebra in a Fréchet algebra $(\mathcal{A}, p_\ell)$. We show that a proper Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$ cannot contain a right (left, two-sided) locally bounded approximate identity. This result is a stronger result in comparison with the known result about abstract Segal algebras [4] (Theorem 1.2). Then we study the results of [12] for Segal – Fréchet algebras. As the main result we verify [12] (Theorem 4.4) for the Fréchet algebra case and prove that if $(\mathcal{A}, p_\ell)$ is an amenable Fréchet algebra with a uniformly bounded approximate identity, then every symmetric Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$ is essentially amenable. At the end, we provide some examples of essentially amenable Fréchet algebras, which are in fact Segal – Fréchet algebras in some Banach algebras.

2. Preliminaries. A topological algebra $\mathcal{A}$ is an algebra, that is also a topological vector space and the multiplication $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, defined by $(a, b) \mapsto ab$ is separately continuous. Moreover, a Fréchet algebra is a complete topological algebra, such that its topology is defined by the countable family of increasing submultiplicative seminorms; see, for example, [6]. Note that our definition of a Fréchet algebra is presented hear, with a slight modification of that given in [10]. Indeed by [10], a Fréchet algebra is a complete topological algebra, such that its topology is given by a countable family of increasing (not necessarily submultiplicative) seminorms. In fact the definition of a Fréchet algebra in this paper coincides with the concept of Fréchet – Arens – Michael (or m-convex) algebras, given in [10].

Note that a closed subalgebra F of a Fréchet algebra $(\mathcal{A}, p_\ell)$, is always a Fréchet algebra under the restricted seminorms p_ℓ on F. Moreover, if I is a proper closed ideal of $\mathcal{A}$, then $\mathcal{A}/I$ is a Fréchet space and its topology is defined by the seminorms

$$\hat{p}_\ell(a + I) = \inf \{p_\ell(a + b) : b \in I\}.$$

In fact $(\mathcal{A}/I, \hat{p}_\ell)$ is a Fréchet algebra (see [6], 3.2.10).

A net $(e_\alpha)_\alpha$ in a Fréchet algebra $(\mathcal{A}, p_\ell)$ is called right approximate identity if $a = \lim_\alpha a e_\alpha$, for all $a \in \mathcal{A}$. Left approximate identities are defined similarly. A net $(e_\alpha)_\alpha$ is called two-sided approximate identity (or just an approximate identity) if it is both left and right approximate identity. An approximate identity $(e_\alpha)_\alpha$ (right, left, or two-sided) is bounded if the set $\{e_\alpha\}$ is a bounded set in $\mathcal{A}$. Furthermore, an approximate identity (e_α) in a Fréchet algebra is called uniformly bounded if

$$\sup_{\ell \in \mathbb{N}} \sup_\alpha p_\ell(e_\alpha) < \infty.$$

Note that most of the Fréchet algebras with a bounded uniformly approximate identity are unital. However, $C^\infty(\Omega)$ is a non-unital Fréchet algebra which possesses a bounded uniformly approximate identity. It is obvious that both concepts of bounded approximate identity and uniformly bounded approximate identity are the same, in Banach algebras.

Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra. A Fréchet $\mathcal{A}$-bimodule is a Fréchet space X together with the structure of an $\mathcal{A}$-bimodule, such that the corresponding mappings are separately continuous. We call X a Banach $\mathcal{A}$-bimodule, in the case where X is a Banach space. A Banach $\mathcal{A}$-bimodule X is called neo-unital if

$$X = \mathcal{A}.X.\mathcal{A} = \{a.x.b : a, b \in \mathcal{A}, x \in X\}.$$

Note that if I is a closed ideal of $\mathcal{A}$, then $(\mathcal{A}/I, \hat{p}_\ell)$ is a Fréchet $\mathcal{A}$-bimodule, with the following module actions:

$$b.(a + I) = ba + I \qquad \text{and} \qquad (a + I).b = ab + I$$

for all $a, b \in \mathcal{A}$ (see [16] for more details). Also the quotient map $q\colon \mathcal{A} \to \mathcal{A}/I$, defined by $a \mapsto a + I$, is a continuous $\mathcal{A}$-bimodule epimorphism.

Now consider X^*, the dual space of X, with the module actions given by

$$\langle a.f, x\rangle = \langle f, x.a\rangle, \qquad \langle f.a, x\rangle = \langle f, a.x\rangle$$

for all $a \in \mathcal{A}$, $x \in X$ and $f \in X^*$. As it is mentioned in [10], by [16] (3.1) if X is a Banach $\mathcal{A}$-bimodule, then so is X^*.

3. Essential amenability of Fréchet algebras. Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra and X be a Banach $\mathcal{A}$-bimodule. According to [7] a continuous derivation of $\mathcal{A}$ into X is a continuous linear mapping D from $\mathcal{A}$ into X such that

$$D(ab) = a.D(b) + D(a).b$$

for all $a, b \in \mathcal{A}$. For each $x \in X$ the mapping $D_x : \mathcal{A} \to X$ defined by

$$D_x(a) = a.x - x.a, \quad a \in \mathcal{A},$$

is a continuous derivation and is called the inner derivation associated with x.

Many concepts related to Banach algebras, have been introduced and studied for Fréchet algebras. In all of these generalizations, it has been observed that these definitions become compatible, whenever a Banach algebra is considered as a Fréchet algebra. Here, we introduce the concept of essential amenability for Fréchet algebras, according to its definition for the Banach algebra case. Recall that a Banach algebra $\mathcal{A}$ is called essentially amenable if for every neo-unital Banach $\mathcal{A}$-bimodule X, each continuous derivation of $\mathcal{A}$ into X^* is inner.

Definition 3.1. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra. We call $\mathcal{A}$, essentially amenable if for every neo-unital Banach $\mathcal{A}$-bimodule X, each continuous derivation of $\mathcal{A}$ into X^* is inner.*

It is obvious that if $\mathcal{A}$ is an essentially amenable Banach algebra then $\mathcal{A}$ is essentially amenable, when considered as a Fréchet algebra. Moreover by [10] (Theorem 9.8), $(\mathcal{A}, p_\ell)$ is amenable if and only if for each Banach $\mathcal{A}$-bimodule X, every continuous derivation from $\mathcal{A}$ to X^* is inner. It follows that every amenable Fréchet algebra is essentially amenable.

By [11] (Proposition 2.1.5), if a Banach algebra $\mathcal{A}$ has a bounded approximate identity, then $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is essentially amenable. In fact both concepts of amenability and essential amenability are coincided. The main key in the proof of [11] (Proposition 2.1.5) is Cohen factorization theorem. It is worth noting that Cohen factorization theorem is also valid for the Fréchet algebras, having a uniformly bounded approximate identity (see [17]). Thus one can prove the following result, with the same arguments as in the proof of [11] (Proposition 2.1.5), and so the proof is left to the reader.

Theorem 3.1. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra with a uniformly bounded approximate identity. Then $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is essentially amenable.*

The following result is a generalization of [5] (Proposition 2.2), for the Fréchet algebra case. The proof is similar and is left to the reader.

Proposition 3.1. *Let $(\mathcal{A}, p_\ell)$ and $(\mathcal{B}, q_m)$ be Fréchet algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ be a continuous epimorphism. If $\mathcal{A}$ is essentially amenable then $\mathcal{B}$ is essentially amenable.*

Since for every closed two-sided ideal I of a Fréchet algebra $(\mathcal{A}, p_\ell)$, the quotient map $q\colon \mathcal{A} \to \mathcal{A}/I$ is a continuous epimorphism, thus the following result is obtained.

***Corollary* 3.1.** *Let $(\mathcal{A}, p_\ell)$ be an essentially amenable Fréchet algebra and I be a closed two-sided ideal of $\mathcal{A}$. Then $(\mathcal{A}/I, \hat{p}_\ell)$ is essentially amenable.*

4. Essential amenability of Segal – Fréchet algebras. Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra. According to [2], a Fréchet algebra $(\mathcal{B}, q_m)$ is a Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$, if the following conditions are satisfied:

(i) $\mathcal{B}$ is a dense left ideal in $\mathcal{A}$;

(ii) the map

$$i\colon (\mathcal{B}, q_m) \longrightarrow (\mathcal{A}, p_\ell), \qquad a \mapsto a \quad a \in \mathcal{B},$$

is continuous;

(iii) the map

$$(\mathcal{B}, p_\ell) \times (\mathcal{B}, q_m) \longrightarrow (\mathcal{B}, q_m), \qquad (a, b) \mapsto ab, \quad a, b \in \mathcal{B}, \tag{4.1}$$

is jointly continuous.

It is not hard to see that the implication (4.1) implies that the map

$$(\mathcal{A}, p_\ell) \times (\mathcal{B}, q_m) \longrightarrow (\mathcal{B}, q_m), \qquad (a, b) \mapsto ab, \quad a \in \mathcal{A}, \quad b \in \mathcal{B},$$

is also jointly continuous. Moreover $(\mathcal{B}, q_m)$ is a symmetric Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$ if $\mathcal{B}$ is a dense two-sided ideal in $\mathcal{A}$, and the map

$$(\mathcal{B}, q_m) \times (\mathcal{B}, p_\ell) \longrightarrow (\mathcal{B}, q_m), \qquad (a, b) \mapsto ab, \quad a, b \in \mathcal{B},$$

is jointly continuous.

Note that the concept of Segal – Fréchet algebra corresponds to the concept of abstract Segal algebra, in the case where $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras.

We investigate the results of [12], for the Segal – Fréchet algebras. Recall that for Fréchet algebra $(\mathcal{A}, p_\ell)$,

$$\mathcal{A}.\mathcal{A} = \{a.b\colon a, b \in \mathcal{A}\}.$$

Moreover, $\mathcal{A}^2$ is the linear span of $\mathcal{A}.\mathcal{A}$. We also denote by $\overline{\mathcal{A}^2}^{\mathcal{A}}$, the closure of $\mathcal{A}^2$ in $\mathcal{A}$. We commence with the following lemma which is a generalization of [12] (Lemma 3.1). The proof is similar and is left to the reader.

Lemma 4.1. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra such that $\mathcal{A}.\mathcal{A}$ is dense in $\mathcal{A}$, and $(\mathcal{B}, q_m)$ be a Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$. Then $\overline{B^2}^{\mathcal{B}}$ is a Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$.*

We recall from [10], the concept of locally bounded approximate identity. A topological algebra $\mathcal{A}$ has a right (respectively, left) locally bounded approximate identity if for each neighborhood U of the zero element of $\mathcal{A}$, there exists $C_U > 0$ such that for each finite subset F of $\mathcal{A}$, there exists $b_F \in C_U U$ with $a - ab_F \in U$ (respectively, $a - b_F a \in U$) for all $a \in F$. We say that $\mathcal{A}$ has a locally bounded approximate identity, if for each neighborhood U of the zero element of $\mathcal{A}$, there exists $C_U > 0$ such that for each finite subset F of $\mathcal{A}$, there exists $b_F \in C_U U$ with $a - ab_F \in U$ and $a - b_F a \in U$ for all $a \in F$.

It is not hard to see that if $\mathcal{A}$ has a locally bounded approximate identity, then $\mathcal{A}.\mathcal{A}$ is dense in $\mathcal{A}$. Thus, we have the following result from Lemma 4.1, immediately.

***Corollary* 4.1.** *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra with a locally bounded approximate identity, and $(\mathcal{B}, q_m)$ be a Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$. Then $\overline{B^2}^{\mathcal{B}}$ is a Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$.*

***Remark* 4.1.** Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra.

1. By [10] (Proposition 5.2), $\mathcal{A}$ has a right (respectively, left) approximate identity if and only if, for each finite subset $F \subseteq \mathcal{A}$ and each zero neighborhood $U \subseteq \mathcal{A}$, there exists $b \in \mathcal{A}$ such that $a - ab \in U$ $(a - ba \in U)$ for all $a \in F$. Moreover, $\mathcal{A}$ has a right (respectively, left) bounded approximate identity if and only if there exists a bounded subset $B \subseteq \mathcal{A}$ such that for each finite subset $F \subseteq \mathcal{A}$ and each zero neighborhood $U \subseteq \mathcal{A}$, there exists $b \in B$ such that $a - ab \in U$ $(a - ba \in U)$ for all $a \in F$.

2. By [10] (Remark 6.4), the existence of a bounded right (respectively, left, two-sided) approximate identity implies the existence of a right (respectively, left, two-sided) locally bounded approximate identity, which, in turn, implies the existence of a right (respectively, left, two-sided) approximate identity. But the existence of a right (respectively, left, two-sided) locally bounded approximate identity does not imply, in general, the existence of a bounded right (respectively, left, two-sided) approximate identity (see [10], Proposition 10.4).

The following lemma is useful in application. One can easily prove it, similar to [10] (Proposition 5.2) and also [10] (Remark 5.3).

Lemma 4.2. *Let* $(\mathcal{A}, p_\ell)$ *be a Fréchet algebra and* $C \subseteq \mathcal{A}$. *Then* $\mathcal{A}$ *has a right* (*respectively*, *left*) *approximate identity with the elements in* C *if and only if for each finite subset* $F \subseteq \mathcal{A}$ *and each zero neighborhood* $U \subseteq \mathcal{A}$ *there exists* $c \in C$ *such that* $a - ac \in U$ $(a - ca \in U)$ *for all* $a \in F$. *Moreover*, $\mathcal{A}$ *has a two-sided approximate identity with the elements in* C *if and only if for each finite subset* $F \subseteq \mathcal{A}$ *and each zero neighborhood* $U \subseteq \mathcal{A}$ *there exists* $c \in C$ *such that* $a - ac \in U$ *and* $a - ca \in U$ *for all* $a \in F$.

The following result is known for Banach algebras, that we generalize it for Fréchet algebras.

Proposition 4.1. *Let* $(\mathcal{A}, p_\ell)$ *be a Fréchet algebra with a right* (*respectively*, *left*, *two-sided*) *approximate identity and* $(\mathcal{B}, q_m)$ *be a Segal – Fréchet algebra in* $(\mathcal{A}, p_\ell)$. *Then* $(\mathcal{A}, p_\ell)$ *has a right* (*respectively*, *left*, *two-sided*) *approximate identity with elements in* $\mathcal{B}$.

Proof. We only prove the right version. By Lemma 4.2, it is sufficient to show that for each finite subset $F \subseteq \mathcal{A}$ and each zero neighborhood $U \subseteq \mathcal{A}$ there exists $b \in \mathcal{B}$ such that $a - ab \in U$ for all $a \in F$. Let $F = \{a_1, \ldots, a_n\}$ be a finite subset of $\mathcal{A}$ and $U = p_\ell^{-1}([0, \varepsilon))$, where $\varepsilon > 0$ is arbitrary. By the hypothesis and also Remark 4.1, part 1, for $V = p_\ell^{-1}\left(\left[0, \frac{\varepsilon}{2}\right)\right)$ there is $a_F \in \mathcal{A}$ such that $a_i - a_i a_F \in V$ for all $i = 1, \ldots, n$. Suppose that

$$K = \max\left\{p_\ell(a_i) \colon i = 1, \ldots, n\right\}.$$

By the density of $\mathcal{B}$ in $\mathcal{A}$, there is $b_F \in \mathcal{B}$ such that

$$p_\ell(a_F - b_F) \leq \frac{\varepsilon}{2K}.$$

Now, for each $i = 1, \ldots, n$, we have

$$p_\ell(a_i b_F - a_i) \leq p_\ell(a_i b_F - a_i a_F) + p_\ell(a_i a_F - a_i) \leq$$

$$\leq p_\ell(a_i) p_\ell(b_F - a_F) + \frac{\varepsilon}{2} \leq$$

$$\leq K \frac{\varepsilon}{2K} + \frac{\varepsilon}{2} = \varepsilon.$$

It follows that $a_i b_F - a_i \in U$ for all $i = 1, \ldots, n$.

Proposition 4.1 is proved.

The following result is obtained immediately from Remark 4.1 and Proposition 4.1.

***Corollary* 4.2.** *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra with a locally bounded approximate identity, and $(\mathcal{B}, q_m)$ be a Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$. Then $(\mathcal{A}, p_\ell)$ has an approximate identity with elements in $\mathcal{B}$.*

In the next proposition, we extend [12] (Lemma 3.2) to the Fréchet algebra case.

Proposition 4.2. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra with a uniformly bounded left approximate identity, and $(\mathcal{B}, q_m)$ be a Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$. Then there is a left approximate identity for $(\mathcal{A}, p_\ell)$ with elements in $\overline{B^2}^{\mathcal{B}}$, which is also a left approximate identity for the Segal–Fréchet algebra $\left(\overline{B^2}^{\mathcal{B}}, q_m\right)$.*

Proof. Let (e_α) be a uniformly bounded left approximate identity for $\mathcal{A}$. By the Cohen's factorization theorem [17] (Theorem 3), $\mathcal{A}.\mathcal{A} = \mathcal{A}$. It is easily verified that $\mathcal{B}.\mathcal{B}$ is dense in $\mathcal{A}$. So, for each $\ell \in \mathbb{N}$, $\varepsilon > 0$ and α, there exists $e_{(\alpha,\ell,\varepsilon)} \in \mathcal{B}.\mathcal{B}$ such that

$$p_\ell(e_\alpha - e_{(\alpha,\ell,\varepsilon)}) < \min\{1, \varepsilon\}. \tag{4.2}$$

Since $(p_\ell)_{\ell\in\mathbb{N}}$ is an increasing sequence of seminorms, thus, for all $k \leq \ell$,

$$p_k(e_\alpha - e_{(\alpha,\ell,\varepsilon)}) \leq p_\ell(e_\alpha - e_{(\alpha,\ell,\varepsilon)}) < \min\{1, \varepsilon\} \tag{4.3}$$

and so for each $k \leq \ell$

$$p_k(e_{(\alpha,\ell,\varepsilon)}) \leq p_\ell(e_\alpha) + \varepsilon. \tag{4.4}$$

Then $(e_{(\alpha,\ell,\varepsilon)})_{(\alpha,\ell,\varepsilon)}$ is a directed net in $\mathcal{B}.\mathcal{B}$ with $(\alpha_1, \ell_1, \varepsilon_1) \preceq (\alpha_2, \ell_2, \varepsilon_2)$ if and only if $\alpha_1 \preceq \alpha_2$, $\ell_1 \leq \ell_2$ and $\varepsilon_2 \leq \varepsilon_1$. By the hypothesis, there is $K > 0$ such that

$$\sup_\ell \sup_\alpha p_\ell(e_\alpha) \leq K. \tag{4.5}$$

Let $n_0 \in \mathbb{N}$ and $a \in \mathcal{A}$ be fixed. There is α_0 such that, for all $\alpha \geq \alpha_0$,

$$p_{n_0}(e_\alpha a - a) < \varepsilon.$$

By using (4.2) and (4.3), for all $\alpha \geq \alpha_0$ and $\ell \geq n_0$, we have

$$p_{n_0}(e_{(\alpha,\ell,\varepsilon)}a - a) \leq p_{n_0}(e_{(\alpha,\ell,\varepsilon)}a - e_\alpha a) + p_{n_0}(e_\alpha a - a) <$$

$$< \varepsilon + p_{n_0}(a) p_{n_0}(e_{(\alpha,\ell,\varepsilon)} - e_\alpha) < \varepsilon + p_{n_0}(a)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $(e_{(\alpha,\ell,\varepsilon)})_{(\alpha,\ell,\varepsilon)}$ is a left approximate identity for $\mathcal{A}$. Now let $m \in \mathbb{N}$ be fixed. Since $\mathcal{B}$ is a Segal–Fréchet algebra, by (4.1), there exist $C_m > 0$ and $\ell_0, m_0 \in \mathbb{N}$ such that

$$q_m(b_1 b_2) \leq C_m p_{\ell_0}(b_1) q_{m_0}(b_2) \tag{4.6}$$

for all $b_1, b_2 \in \mathcal{B}$. Thus,

$$q_m(e_{(\alpha,\ell,\varepsilon)} b_1 b_2 - b_1 b_2) \leq C_m p_{\ell_0}(e_{(\alpha,\ell,\varepsilon)} b_1 - b_1) q_{m_0}(b_2) \xrightarrow[(\alpha,\ell,\varepsilon)]{} 0.$$

Consequently, for each $b \in \mathcal{B}^2$,

$$q_m(e_{(\alpha,\ell,\varepsilon)} b - b) \xrightarrow[(\alpha,\ell,\varepsilon)]{} 0. \tag{4.7}$$

Now suppose that $b \in \overline{\mathcal{B}^2}^{\mathcal{B}}$ and $\delta > 0$. By the density of $\mathcal{B}^2$ in $\overline{\mathcal{B}^2}^{\mathcal{B}}$, there is $b_\delta \in \mathcal{B}^2$ such that

$$q_{\max\{m,m_0\}}(b - b_\delta) < \delta.$$

Therefore, by using (4.6), we obtain

$$q_m(b - e_{(\alpha,\ell,\varepsilon)}b) \leq q_m(b - b_\delta) + q_m(b_\delta - e_{(\alpha,\ell,\varepsilon)}b_\delta) + q_m(e_{(\alpha,\ell,\varepsilon)}b_\delta - e_{(\alpha,\ell,\varepsilon)}b) \leq$$

$$\leq \delta + q_m(b_\delta - e_{(\alpha,\ell,\varepsilon)}b_\delta) + C_m p_{\ell_0}(e_{(\alpha,\ell,\varepsilon)})\delta.$$

Consequently, by (4.3)–(4.5) and (4.7) we get

$$\limsup_{(\alpha,\ell,\varepsilon)} q_m(b - e_{(\alpha,\ell,\varepsilon)}b) \leq \delta(1 + C_m \sup_{(\alpha,\ell,\varepsilon)} (p_{\ell_0}(e_\alpha) + 1)) \leq$$

$$\leq \delta(1 + C_m(K + 1)).$$

Since $\delta > 0$ is arbitrary, it follows that, for each $m \in \mathbb{N}$,

$$\lim_{(\alpha,\ell,\varepsilon)} q_m(b - e_{(\alpha,\ell,\varepsilon)}b) = 0,$$

which implies that $(e_{(\alpha,\ell,\varepsilon)})$ is a left approximate identity for $\overline{\mathcal{B}^2}$.

Proposition 4.2 is proved.

Theorem 4.1. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra with a uniformly bounded approximate identity, and $(\mathcal{B}, q_m)$ be a symmetric Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$. Then $\overline{B^2}^{\mathcal{B}}$ is a symmetric Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$ such that there exists an approximate identity for $\overline{B^2}^{\mathcal{B}}$, which is also an approximate identity for $(\mathcal{A}, p_\ell)$.*

Proof. By similar methods, one can show that if in Corollary 4.1, $(\mathcal{B}, q_m)$ is a symmetric Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$, then $\overline{B^2}^{\mathcal{B}}$ is also symmetric Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$. Moreover, $(e_{(\alpha,\ell,\varepsilon)})_{(\alpha,\ell,\varepsilon)}$, constructed in the proof of Proposition 4.2, is also a right approximate identity for both $(\mathcal{A}, p_\ell)$ and also $\left(\overline{B^2}^{\mathcal{B}}, q_m\right)$.

Theorem 4.1 is proved.

A known result due to Burnham asserts that a proper abstract Segal algebra cannot contain a bounded (right, left, two-sided) approximate identity [4] (Theorem 1.2). In the following result, we prove a stronger result about Segal–Fréchet algebras. In fact we show that a proper Segal–Fréchet algebra does not contain even a right (left, two-sided) locally bounded approximate identity.

Proposition 4.3. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra and $(\mathcal{B}, q_m)$ be a proper Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$. Then $(\mathcal{B}, q_m)$ does not contain a right* (*left*, *two-sided*) *locally bounded approximate identity.*

Proof. Suppose on the contrary that $\mathcal{B}$ contains a right locally bounded approximate identity. For $a \in \mathcal{A}$, there is a sequence (b_n) in $\mathcal{B}$ such that $b_n \to a$, in the topology of $\mathcal{A}$. We show that (b_n) is a Cauchy sequence in $\mathcal{B}$. For $m \in \mathbb{N}$ by (4.1), there exist $\ell_0, m_0 \in \mathbb{N}$ and $K_m > 0$ such that

$$q_m(bc) \leq K_m p_{\ell_0}(b) q_{m_0}(c) \tag{4.8}$$

for all $b, c \in \mathcal{B}$. For $\varepsilon > 0$, set

$$U = q_m^{-1}\big([0,\varepsilon)\big) \cap q_{m_0}^{-1}\big([0,\varepsilon)\big).$$

Thus by the hypothesis, there is $C_{m,\varepsilon} > 0$ such that for each $F = \{b\} \subseteq \mathcal{B}$, there is $b_F \in C_{m,\varepsilon}U$ with

$$b - bb_F \in U.$$

Thus,

$$q_m(b - bb_F) < \varepsilon \qquad \text{and} \qquad q_{m_0}(b_F) < \varepsilon C_{m,\varepsilon}. \tag{4.9}$$

By (4.8) and (4.9), we obtain

$$q_m(b) < q_m(bb_F) + \varepsilon \leq K_m p_{\ell_0}(b) q_{m_0}(b_F) + \varepsilon \leq \big(K_m C_{m,\varepsilon} p_{\ell_0}(b) + 1\big)\varepsilon.$$

Note that the above inequalities are independent from b. It follows that for each $b \in \mathcal{B}$

$$q_m(b) \leq \big(K_m C_{m,\varepsilon} p_{\ell_0}(b) + 1\big)\varepsilon.$$

Since (b_n) is convergent in $\mathcal{A}$, thus for ℓ_0 and $\eta = \dfrac{1}{K_m C_{m,\varepsilon}}$, there is $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$p_{\ell_0}(b - b_n) < \eta.$$

Thus, for each $n \geq N$, we have

$$q_m(b - b_n) \leq (K_m C_{m,\varepsilon}\eta + 1)\varepsilon = 2\varepsilon.$$

It follows that (b_n) is a Cauchy sequence with respect to the seminorm q_m. Thus, (b_n) is Cauchy with respect to all the seminorm q_m, which implies that (b_n) is Cauchy in the topology of $\mathcal{B}$. Thus, (b_n) is convergent in the topology of $\mathcal{B}$. Since this sequence is convergent to a, in the topology of $\mathcal{A}$, it follows that $a \in \mathcal{B}$. Therefore, $\mathcal{B} = \mathcal{A}$, which is a contradiction. Similarly, $\mathcal{B}$ can not contain left or two-sided locally bounded approximate identity.

Proposition 4.3 is proved.

Lemma 4.3. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra with a locally bounded approximate identity, and $(\mathcal{B}, q_m)$ be a Segal – Fréchet algebra in $(\mathcal{A}, p_\ell)$. Then the following assertions are equivalent:*

(i) *$(\mathcal{B}, q_m)$ has a locally bounded approximate identity;*

(ii) *$\mathcal{B} = \mathcal{A}$, as sets;*

(iii) *$(\mathcal{B}, q_m)$ is a Fréchet algebra isomorphic to $(\mathcal{A}, p_\ell)$.*

Proof. (i) $\Rightarrow$ (ii). It is obtained by Proposition 4.3.

(ii) $\Rightarrow$ (iii). Consider the identity map $\iota\colon \mathcal{B} \to \mathcal{A}$. By the definition of a Segal – Fréchet algebra, ι is a continuous bijection. Now open mapping theorem for Fréchet spaces implies that ι is an isomorphism.

(iii) $\Rightarrow$ (i). It is trivial.

The following proposition is a generalization of [12] (Proposition 4.1). The proof is similar and is left to the reader.

Proposition 4.4. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra and I be a closed subalgebra of $\mathcal{A}$ that contains $\mathcal{A}.\mathcal{A}$. If I is essentially amenable, then so is $\mathcal{A}$.*

We state here the main result of the present work. In fact we generalize [12] (Theorem 4.4) to the Fréchet algebra case.

Theorem 4.2. *Let $(\mathcal{A}, p_\ell)$ be an amenable Fréchet algebra with a uniformly bounded approximate identity and $(\mathcal{B}, q_m)$ be a symmetric Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$. Then $\mathcal{B}$ is essentially amenable. Moreover, $(\mathcal{B}, q_m)$ is amenable if and only if $\mathcal{A} = \mathcal{B}$, as two sets.*

Proof. Suppose that $(\mathcal{B}, q_m)$ is amenable. Then by [10] (Theorem 9.7), $\mathcal{B}$ has a locally bounded approximate identity. By Proposition 4.3 and Lemma 4.3, $\mathcal{B} = \mathcal{A}$. Conversely suppose that $\mathcal{A} = \mathcal{B}$, as two sets. Then by Lemma 4.3, $(\mathcal{B}, q_m)$ is a Fréchet algebra isomorphic to $(\mathcal{A}, p_\ell)$. It follows that $\mathcal{B}$ is amenable. Now suppose that $\mathcal{B}$ is not amenable. We show that $\mathcal{B}$ is essentially amenable. By Theorem 4.1, $J = \overline{B^2}^{\mathcal{B}}$ is a symmetric Segal–Fréchet algebra in $\mathcal{A}$. Moreover, J has an approximate identity $(e_\alpha)_\alpha$, which is an approximate identity for $\mathcal{A}$. Now let X be a neo-unital Banach J-bimodule and also let $D\colon J \to X^*$ be a continuous derivation. Some arguments similar to the proof of [5] (Theorem 7.1), show that X is an $\mathcal{A}$-bimodule. In fact for each $a \in \mathcal{A}$ and $x \in X$, we define $a.x = (ab_1).y_1$ and also $x.a = y_2.(b_2 a)$, where $x = b_1.y_1$ and $x = y_2.b_2$, for some $b_1, b_2 \in J$ and $y_1, y_2 \in X$. Then one can use the closed graph theorem [9] (Theorem 8.8), to show that the module actions are continuous. Take $x \in X$ and $a \in \mathcal{A}$ and write $x = b.y$, for some $b \in J$ and $y \in X$. Then as the proof of [5] (Theorem 7.1) we obtain

$$\big\langle D(ab) - a.D(b), y\big\rangle = \lim_{\alpha} \big\langle D(e_\alpha a), x\big\rangle.$$

Define $\widetilde{D}\colon \mathcal{A} \to X^*$ by

$$\big\langle \widetilde{D}(a), x\big\rangle = \big\langle D(ab) - a.D(b), y\big\rangle,$$

where $x = b.y$, for some $b \in J$ and $y \in X$. With the reasons exactly similar to those given in [5] (Theorem 7.1), such as using the uniform boundedness theorem and closed graph theorem [9] (Theorem 8.8), we obtain that $\widetilde{D}$ is a well defined continuous linear map, which is in fact an extension of D to $\mathcal{A}$. Since J is dense in $\mathcal{A}$ it follows that $\widetilde{D}$ must also be a derivation. But then by the amenability of $\mathcal{A}$, $\widetilde{D}$ is inner. It follows that D is inner. Therefore, J is essentially amenable. Since J is a closed two-sided ideal of $\mathcal{B}$ that contains $\mathcal{B}.\mathcal{B}$, thus $\mathcal{B}$ is essentially amenable, by Proposition 4.4.

Theorem 4.2 is proved.

We end this paper with some examples of essentially amenable Fréchet algebras.

***Examples* 4.1.** 1. We explain here part (b) of [13] (Example 3.3), which is a nice example in the field of Fréchet algebras. Let X be a countable set. A function $\sigma\colon X \to [1, \infty)$ is called a scale on X. If $\sigma = \{\sigma_n\}_{n=0}^{\infty}$ is a family of scales on X, define the Fréchet space

$$\mathcal{S}_\sigma^\infty(X) = \big\{\varphi\colon X \to \mathbb{C},\ \ \|\varphi\|_n^\infty < \infty, n \in \mathbb{N}\big\},$$

where

$$\|\varphi\|_n^\infty = \sup_{x\in X} \big\{\sigma_n(x)|\varphi(x)|\big\}.$$

Then $\mathcal{S}_\sigma^\infty(X)$ is called the sup-norm σ-rapidly vanishing functions on X. The family σ will satisfy $\sigma_0 \le \sigma_1 \le \ldots \le \sigma_n \le \ldots$, so that the families of norms $\big\{\|.\|_n^\infty\big\}_{n=0}^\infty$ are increasing. Moreover, it is easy to see that all of them are submultiplicative under pointwise multiplication. Consequently, $\mathcal{S}_\sigma^\infty(X)$ is a Fréchet algebra. Now let $\mathcal{A} = c_0(X)$ be the commutative Banach algebra of complex-valued sequences which vanish at infinity, with pointwise multiplication and sup-norm $\|.\|_{\mathcal{A}} = \|.\|_\infty$.

Also let $\mathcal{B}$ be $\mathcal{S}_\sigma^\infty(X)$, topologized by the sup-norms

$$\|f\|_n = \|\sigma^n f\|_\infty.$$

Then it is easy to see that the inequalities $\|fg\|_n \leq \|f\|_n \|g\|_\infty$ are satisfied for all $n \in \mathbb{N}$. Moreover, $\mathcal{B}$ is a dense Fréchet ideal in $\mathcal{A}$. In fact $\mathcal{B}$ is a Segal – Fréchet algebra in $\mathcal{A}$. It is known that $c_0(X)$ is a commutative C^*-algebra. All commutative C^*-algebras are amenable and admit an approximate identity bounded by 1 (see [11]). So $c_0(X)$ is amenable in the sense of a Fréchet algebra and has uniformly bounded approximate identity. Now Theorem 4.2 implies that $\mathcal{S}_\sigma^\infty(X)$ is an essentially amenable Segal – Fréchet algebra.

2. Let G be a compact connected Lie group. Also let $C^\infty(G)$ be the space consisting of all infinitely differentiable functions on G. We refer to [15] for a full information about the construction of $C^\infty(G)$. Consider the general group algebra $L^1(G)$ under convolution product. It is known that $L^1(G)$ has a bounded approximate identity and since G is compact, $L^1(G)$ is an amenable Banach algebra. Thus $L^1(G)$ is amenable as a Fréchet algebra [10] (Theorem 9.6), which has a uniformly bounded approximate identity. By [13] (Example 8.4) and [14], $C^\infty(G)$ is a Fréchet algebra under convolution product, which satisfies all the conditions of a Segal – Fréchet algebra in $\mathcal{A} = L^1(G)$. Now Theorem 4.2 implies that $C^\infty(G)$ is an essentially amenable Segal – Fréchet algebra.

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Received 07.06.17,
after revision — 15.10.19