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CHARACTERIZATIONS OF ADDITIVE ξ-LIE DERIVATIONS ON UNITAL ALGEBRAS

ХАРАКТЕРИЗАЦІЯ АДИТИВНОГО ξ-ДИФЕРЕНЦІЮВАННЯ ЛІ НА УНІТАЛЬНИХ АЛГЕБРАХ

Let $\mathcal{R}$ be a commutative ring with unity and $\mathcal{U}$ be a unital algebra over $\mathcal{R}$ (or field $\mathbb{F}$). An $\mathcal{R}$-linear map $L:\mathcal{U}\rightarrow\mathcal{U}$ is called a Lie derivation on $\mathcal{U}$ if $L([u,v])=[L(u),v]+[u,L(v)]$ holds for all $u,\ v\in\mathcal{U}$. For scalar $\xi\in\mathbb{F}$, an additive map $L:\mathcal{U}\rightarrow\mathcal{U}$ is called an additive ξ-Lie derivation on $\mathcal{U}$ if $L([u,v]_\xi)=[L(u),v]_\xi+[u,L(v)]_\xi$, where $[u,v]_\xi=uv-\xi vu$ holds for all $u,v\in\mathcal{U}$. In the present paper, under certain assumptions on $\mathcal{U}$ it is shown that every Lie derivation (resp., additive ξ-Lie derivation) L on $\mathcal{U}$ is of standard form, i.e., $L=\delta+\phi$, where δ is an additive derivation on $\mathcal{U}$ and ϕ is a mapping $\phi:\mathcal{U}\rightarrow Z(\mathcal{U})$ vanishing at $[u,v]$ with $uv=0$ in $\mathcal{U}$. Moreover, we also characterize the additive ξ-Lie derivation for $\xi\neq 1$ by its action at zero product in a unital algebra over $\mathbb{F}$.

Нехай $\mathcal{R}$ — комутативне кільце з одиницею, а $\mathcal{U}$ — унітальна алгебра над $\mathcal{R}$ (або полем $\mathbb{F}$). $\mathcal{R}$-лінійне відображення $L:\mathcal{U}\rightarrow\mathcal{U}$ називається диференціюванням Лі на $\mathcal{U}$, якщо $L([u,v])=[L(u),v]+[u,L(v)]$ виконується для всіх u, $v\in\mathcal{U}$. Для скаляра $\xi\in\mathbb{F}$ адитивне відображення $L:\mathcal{U}\rightarrow\mathcal{U}$ називається адитивним ξ-диференціюванням Лі на $\mathcal{U}$, якщо $L([u,v]_\xi)=[L(u),v]_\xi+[u,L(v)]_\xi$, де $[u,v]_\xi=uv-\xi vu$ виконується для всіх $u,v\in\mathcal{U}$. У цій роботі при деяких припущеннях на $\mathcal{U}$ доведено, що кожне диференціювання Лі (відповідно, адитивне ξ-диференціювання Лі) L на $\mathcal{U}$ має стандартний вигляд, тобто $L=\delta+\phi$, де δ — адитивне диференціювання на $\mathcal{U}$, а ϕ — відображення $\phi:\mathcal{U}\rightarrow Z(\mathcal{U})$, що зникає на $[u,v]$, якщо $uv=0$ у $\mathcal{U}$. Більш того, охарактеризовано адитивне ξ-диференціювання Лі для $\xi\neq 1$ через його дію на нульовий добуток в унітальній алгебрі над $\mathbb{F}$.

1. Introduction. Throughout, let $\mathcal{R}$ be a commutative ring with unity and $\mathcal{U}$ be a unital algebra over $\mathcal{R}$ with the center $Z(\mathcal{U})$. For any $u,\ v\in\mathcal{U}$, $[u,v]$ will denote the commutator $uv-vu$, while $u\circ v$ will represent the anticommutator $uv+vu$. An $\mathcal{R}$-linear map $L\colon\mathcal{U}\rightarrow\mathcal{U}$ is called a derivation (resp., Jordan derivation) on $\mathcal{U}$ if $L(uv)=L(u)v+uL(v)$ (resp., $L(uv+vu)=L(u)v+uL(v)+L(v)u+vL(u)$) holds for all $u,\ v\in\mathcal{U}$. An $\mathcal{R}$-linear map $L\colon\mathcal{U}\rightarrow\mathcal{U}$ is called a Lie derivation on $\mathcal{U}$ if $L([u,v])=[L(u),v]+[u,L(v)]$ holds for all $u,\ v\in\mathcal{U}$. Obviously, every derivation is a Jordan derivation and Lie derivation but not conversely (see [1, 2]).

During the recent past there has been a great deal of work concerning characterization of different linear mappings viz., Lie derivation, additive ξ-Lie derivation, generalized Lie derivation on various algebras (see [5, 6, 8 – 14] and references therein). In most of the cases, the object of the studies is to obtain the conditions under which derivations (Lie derivations) can be completely determined by the action on some subsets of the algebras. There are several papers on the study of local actions of Lie derivations of operator algebras. Lu and Jing [8] proved that if X is Banach space of dimension greater then two and a linear map $L\colon\mathcal{B}(X)\rightarrow\mathcal{B}(X)$ such that $L([u,v])=[L(u),v]+[u,L(v)]$ for all $u,\ v\in\mathcal{B}(X)$ with $uv=0,$ than there exists an operator $r\in\mathcal{B}(X)$ and a linear map $\phi\colon\mathcal{B}(X)\rightarrow\mathbb{C}I$ vanishes at all the commutators $[u,v]$ with $uv=0$ such that $L(u)=ru-ur+\phi(u)$ for all $u\in\mathcal{B}(X)$. Inspired by this result, Ji and Qi [5] proved that under certain restrictions on triangular algebra $\mathcal{T}$ over commutative ring $\mathcal{R}$, if $L\colon\mathcal{T}\rightarrow\mathcal{T}$ is an $\mathcal{R}$-linear map such that $L([u,v])=[L(u),v]+[u,L(v)]$ for all $u,\ v\in\mathcal{T}$ with $uv=0$, then there exists a derivation $\delta\colon\mathcal{T}\rightarrow\mathcal{T}$

and an $\mathcal{R}$-linear map $\phi\colon \mathcal{T} \to Z(\mathcal{T})$ vanishes at all the commutators $[u,v]$ with $uv=0$ such that $L=\delta+\phi$. Also, Ji et al. [6] studied the same result on factor von Neumann algebra with dimension greater than 4. Qi and Hou [12] gave the characterization of Lie derivation on any von Neumann algebra $\mathfrak{U}$ without central summands of type I_1 and obtained that if $L\colon \mathfrak{U}\to\mathfrak{U}$ is a linear map such that $L([u,v]) = [L(u),v]+[u,L(v)]$ for all $u,\ v\in\mathfrak{U}$ with $uv=0$, then there exists a derivation $\delta\colon \mathfrak{U}\to\mathfrak{U}$ and a linear map $\phi\colon \mathfrak{U}\to Z(\mathfrak{U})$ vanishes at all the commutators $[u,v]$ with $uv=0$ such that $L=\delta+\phi$. In particular, $L\colon \mathfrak{U}\to\mathfrak{U}$ is a linear map such that $L([u,v]) = [L(u),v]+[u,L(v)]$ for all $u,\ v\in\mathfrak{U}$ with $uv=0$, if and only if there exists an operator $r\in\mathfrak{U}$ and a linear map $\phi\colon \mathfrak{U}\to Z(\mathfrak{U})$ vanishes at all the commutators $[u,v]$ with $uv=0$ such that $L(u)=ur-ru+\phi(u)$ for all $u\in\mathfrak{U}$. Furthermore, Qi [13] characterized Lie derivation on $\mathcal{J}$-subspace lattice algebras and proved the same result due to Lu and Jing [8] on $\mathcal{J}$-subspace lattice algebra Alg$\mathcal{L}$, where $\mathcal{L}$ is $\mathcal{J}$-subspace lattice on a Banach space X over the real or complex field with dimension greater than 2.

Let $\mathfrak{U}$ be a unital algebra over real or complex field $\mathbb{F}$. If any pair $u,v\in\mathfrak{U}$ commute, then their Lie product is zero. For scalar $\xi\in\mathbb{F}$ and $u,v\in\mathfrak{U}$, u commutes with v up to a factor ξ if $uv=\xi vu$. In the theme of quantum groups and operator algebras [3, 7], the notion of commutativity up to a factor for pairs of operators has been studied. Qi and Hou [9] introduced the concept of ξ-Lie derivation. For any $u,v\in\mathfrak{U}$, $[u,v]_\xi = uv-\xi vu$ will denote the ξ-Lie product. A linear map $L\colon \mathfrak{U}\to\mathfrak{U}$ is said to be a ξ-Lie derivation if $L([u,v]_\xi) = [L(u),v]_\xi + [u,L(v)]_\xi$. If L is additive, then ξ-Lie derivation is called an additive ξ-Lie derivation. It can be easily seen that if $\xi=0,1,-1$, then ξ-Lie derivation is called derivation, Lie derivation and Jordan derivation, respectively. Note that an additive map $L\colon \mathfrak{U}\to\mathfrak{U}$ is called a generalized derivation if $L(uv)=L(u)v+uL(v)-uL(I)v$ for all $u,v\in\mathfrak{U}$. During the recent years many authors characterized ξ-Lie derivation on several rings and operator algebras (see [10, 12, 14]). Characterization of ξ-Lie derivation on prime algebras, von Neumann algebra and triangular algebra can be found in [10, 14, 15].

Motivated by the above observations, in Section 3, we characterize a Lie derivation on unital algebras over a commutative ring $\mathcal{R}$ at zero product and prove that if $L\colon \mathfrak{U}\to\mathfrak{U}$ is an $\mathcal{R}$-linear map such that $L([u,v]) = [L(u),v]+[u,L(v)]$ for all $u,\ v\in\mathfrak{U}$ with $uv=0$, then under certain appropriate restrictions on $\mathfrak{U}$ there exists a derivation $\delta\colon \mathfrak{U}\to\mathfrak{U}$ and an $\mathcal{R}$-linear map $\phi\colon \mathfrak{U}\to Z(\mathfrak{U})$ vanishes at all the commutators $[u,v]$ with $uv=0$ such that $L=\delta+\phi$. In Section 4, we study the characterization of ξ-Lie derivation at zero product on unital algebras over a field $\mathbb{F}$ with certain limitations and find that an additive map satisfies $L([u,v]_\xi) = [L(u),v]_\xi + [u,L(v)]_\xi$ for all $u,v\in\mathfrak{U}$ with $uv=0$ if and only if $L(I)\in Z(\mathfrak{U})$ and (i) for $\xi\neq 0,-1$, $L(\xi uv)=\xi L(u)v+\xi uL(v)$ and there exists an additive derivation δ satisfying $\delta(\xi I)=\xi L(I)$ such that $L(u)=\delta(u)+L(I)u$ for all $u\in\mathfrak{U}$; (ii) for $\xi=-1$, L is an additive derivation; (iii) for $\xi=0$, there exists an additive derivation δ such that $L(u)=\delta(u)+L(I)u$ for all $u\in\mathfrak{U}$. In the last section, we discuss some applications of these results on few important examples of unital algebras.

2. Preliminaries. Let $\mathfrak{U}$ be an unital algebra over a commutative ring $\mathcal{R}$ with an idempotent $p\neq 0$ and let $q=1-p$. Then according to the well known Peirce decomposition formula, $\mathfrak{U}$ can be represented as $\mathfrak{U}=p\mathfrak{U}p+p\mathfrak{U}q+q\mathfrak{U}p+q\mathfrak{U}q$, where $p\mathfrak{U}p$ and $q\mathfrak{U}q$ are subalgebras with unital elements p and q, respectively, $p\mathfrak{U}q$ is an $(p\mathfrak{U}p, q\mathfrak{U}q)$-bimodule and $q\mathfrak{U}p$ is an $(q\mathfrak{U}q, p\mathfrak{U}p)$-bimodule. We will assume that $\mathfrak{U}$ satisfies

$$pup.p\mathcal{U}q = \{0\} = q\mathcal{U}p.pup \quad \text{implies} \quad pup = 0,$$
$$p\mathcal{U}q.quq = \{0\} = quq.q\mathcal{U}p \quad \text{implies} \quad quq = 0 \tag{2.1}$$

for all $u \in \mathcal{U}$. Some specific examples of unital algebras with nontrivial idempotents having the property (2.1) are triangular algebras, matrix algebras, algebras of all bounded linear operators of Banach space and the unital prime algebras with nontrivial idempotents.

Throughout, this paper we shall use the following notions: Let $\mathcal{U} = p\mathcal{U}p + p\mathcal{U}q + q\mathcal{U}p + q\mathcal{U}q$ be unital algebra with nontrivial idempotents p and $q = 1 - p$ satisfying (2.1). Let $\mathcal{U}_{11} = p\mathcal{U}p$, $\mathcal{U}_{12} = p\mathcal{U}q$, $\mathcal{U}_{21} = q\mathcal{U}p$ and $\mathcal{U}_{22} = q\mathcal{U}p$. Then $\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}$. The center of $\mathcal{U}$ is

$$Z(\mathcal{U}) = \{u_{11} + u_{22} \in \mathcal{U}_{11} + \mathcal{U}_{22} \mid u_{11}u_{12} = u_{12}u_{22}, \quad u_{21}u_{11} = u_{22}u_{21} \; \forall u_{12} \in \mathcal{U}_{12}, \; u_{21} \in \mathcal{U}_{21}\}.$$

Define two natural projections $\pi_{\mathcal{U}_{11}} : \mathcal{U} \to \mathcal{U}_{11}$ and $\pi_{\mathcal{U}_{22}} : \mathcal{U} \to \mathcal{U}_{22}$ by

$$\pi_{\mathcal{U}_{11}}(u_{11} + u_{12} + u_{21} + u_{22}) = u_{11} \quad \text{and} \quad \pi_{\mathcal{U}_{22}}(u_{11} + u_{12} + u_{21} + u_{22}) = u_{22}.$$

Moreover, $\pi_{\mathcal{U}_{11}}(Z(\mathcal{U})) \subseteq Z(\mathcal{U}_{11})$ and $\pi_{\mathcal{U}_{22}}(Z(\mathcal{U})) \subseteq Z(\mathcal{U}_{22})$ and there exists a unique algebra isomorphism $\tau : \pi_{\mathcal{U}_{11}}(Z(\mathcal{U})) \to \pi_{\mathcal{U}_{22}}(Z(\mathcal{U}))$ such that $u_{11}u_{12} = u_{12}\tau(u_{11})$ and $u_{21}u_{11} = \tau(u_{11})u_{21}$ for all $u_{11} \in \pi_{\mathcal{U}_{11}}(Z(\mathcal{U}))$, $u_{12} \in \mathcal{U}_{12}$, $u_{21} \in \mathcal{U}_{21}$.

3. Characterization of Lie derivations. In this section, we characterize Lie derivation by action at zero product on a unital algebra with a nontrivial idempotent. Actually, we prove the following result.

Theorem 3.1. *Let $\mathcal{U}$ be a 2-torsion free unital algebra over a commutative ring $\mathcal{R}$ with a nontrivial idempotent p satisfying* (2.1), $\pi_{\mathcal{U}_{11}}(Z(\mathcal{U})) = Z(\mathcal{U}_{11})$ *and* $\pi_{\mathcal{U}_{22}}(Z(\mathcal{U})) = Z(\mathcal{U}_{22})$. *If $L\colon \mathcal{U} \to \mathcal{U}$ is an $\mathcal{R}$-linear mapping satisfying $L([u,v]) = [L(u), v] + [u, L(v)]$ for all u, $v \in \mathcal{U}$ with $uv = 0$, there exists a derivation $\delta\colon \mathcal{U} \to \mathcal{U}$ and an $\mathcal{R}$-linear map $\phi\colon \mathcal{U} \to Z(\mathcal{U})$ vanishing at all the commutators $[u,v]$ with $uv = 0$ such that $L = \delta + \phi$.*

Throughout this section we assume that $\mathcal{U}$ is 2-torsion free unital algebra over a commutative ring $\mathcal{R}$ satisfying the hypotheses of Theorem 3.1. In order to prove the above result, we start with the following sequence of lemmas.

Lemma 3.1. *For any p, q*:

(i) $pL(p)p + qL(p)q \in Z(\mathcal{U})$,

(ii) $pL(q)p + qL(q)q \in Z(\mathcal{U})$,

(iii) $pL(I)p + qL(I)q \in Z(\mathcal{U})$.

Proof. (i) Since $u_{12}p = 0$ for any $u_{12} \in \mathcal{U}_{12}$, it follows that

$$-L(u_{12}) = [L(u_{12}), p] + [u_{12}, L(p)] = L(u_{12})p - pL(u_{12}) + u_{12}L(p) - L(p)u_{12}. \tag{3.1}$$

Multiplying the last equality by p and q from left and right, respectively, we find $u_{12}qL(p)q = pL(p)pu_{12}$. Also, $pu_{21} = 0$ implies that $qL(p)qu_{21} = u_{21}pL(p)p$. From the last two expressions we arrive at $pL(p)p + qL(p)q \in Z(\mathcal{U})$.

(ii) Since $u_{21}q = 0 = qu_{12}$, using similar steps as used in (i), we obtain that $pL(q)p + qL(q)q \in Z(\mathcal{U})$.

(iii) Since $p(I - p) = 0$, we have

$$0 = [L(p), I - p] + [p, L(I - p)] = pL(I) - L(I)p.$$

This implies that $pL(I)q = 0 = qL(I)p$. Now, from (i) and (ii), we get $L(I) = pL(I)p + qL(I)q \in Z(\mathcal{U})$.

***Remark* 3.1.** Define a map $L'\colon \mathcal{U} \to \mathcal{U}$ by $L'(u) = L(u) + [u_0, u]$ for all $u \in \mathcal{U}$, where $u_0 = pL'(p)q - qL'(p)p$.

Accordingly, consider only those Lie derivation $L\colon \mathcal{U} \to \mathcal{U}$ which satisfies $pL(p)q = 0 = qL(p)p$. Now, it can be easily seen from the following lemma.

Lemma 3.2. *For any* $u,\ v \in \mathcal{U}$:

(i) $L([u,v]) = [L(u), v] + [u, L(v)]$,

(ii) $L(I),\ L(p),\ L(q) \in Z(\mathcal{U})$.

Lemma 3.3. *For any* $u_{ij} \in \mathcal{U}_{ij},\ L(u_{ij}) \in \mathcal{U}_{ij}$, *where* $i \neq j \in \{1, 2\}$.

Proof. Consider the case for $i = 1$ and $j = 2$. As $u_{12}p = 0$, using (3.1) we find that $pL(u_{12})p = qL(u_{12})p = qL(u_{12})q = 0$. Hence, $L(u_{12}) \in \mathcal{U}_{12}$. Similarly, we can calculate for $i = 2$ and $j = 1$.

Lemma 3.4. $L(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{11} \oplus \mathcal{U}_{22}$, *and there exists a map* $\phi_i : \mathcal{U}_{ii} \to Z(\mathcal{U})$ *such that* $L(u_{ii}) - \phi_i(u_{ii}) \in \mathcal{U}_{ii}$ *for all* $u_{ii} \in \mathcal{U}_{ii},\ i = 1, 2$.

Proof. Since $u_{11}q = 0$ for any $u_{11} \in \mathcal{U}_{11}$, it follows that

$$0 = [L(u_{11}), q] + [u_{11}, L(q)] = L(u_{11})q - qL(u_{11}).$$

This yields that $L(u_{11}) \in \mathcal{U}_{11} + \mathcal{U}_{22}$. Now for any $v_{ii} \in \mathcal{U}_{ii},\ i = 1, 2$, we can write $L(u_{11}) = v_{11} + v_{22}$. Since $u_{11}u_{22} = 0$, we obtain

$$0 = [L(u_{11}), u_{22}] + [u_{11}, L(u_{22})] = v_{22}u_{22} - u_{22}v_{22} + [u_{11}, L(u_{22})].$$

Now multiplying both sides by q in the above expression, it follows that $v_{22} \in Z(\mathcal{U}_{22})$. Thus, for any $z \in Z(\mathcal{U})$,

$$L(u_{11}) = v_{11} + zq = v_{11} + z - zp = (v_{11} - zp) + z \in \mathcal{U}_{11} + Z(\mathcal{U}).$$

Hence we conclude that there exists a map $\phi_1 : \mathcal{U}_{11} \to Z(\mathcal{U})$ such that $L(u_{11}) - \phi_1(u_{11}) \in \mathcal{U}_{11}$ for all $u_{11} \in \mathcal{U}_{11}$. Since L is $\mathcal{R}$-linear map, we can easily see that ϕ_1 is $\mathcal{R}$-linear map. Similarly, we can show the result for $i = 2$.

***Remark* 3.2.** Now we define two $\mathcal{R}$-linear maps $\phi\colon \mathcal{U} \to Z(\mathcal{U})$ and $\delta\colon \mathcal{U} \to \mathcal{U}$ by $\phi(u) = \phi_1(pup) + \phi_2(quq)$ and $\delta(u) = L(u) - \phi(u)$ for all $u \in \mathcal{U}$. It is clear that $\delta(u_{ij}) \in \mathcal{U}_{ij}$ for $1 \leq i,\ j \leq 2$ and $\delta(u_{ij}) = L(u_{ij})$ for $1 \leq i \neq j \leq 2$.

Lemma 3.5. *For any* $u_{ij} \in \mathcal{U}_{ij},\ 1 \leq i,\ j \leq 2$:

(i) $\delta(u_{ii}u_{ij}) = \delta(u_{ii})u_{ij} + u_{ii}\delta(u_{ij})$, *where* $i \neq j \in \{1, 2\}$,

(ii) $\delta(u_{ji}u_{ii}) = \delta(u_{ji})u_{ii} + u_{ji}\delta(u_{ii})$, *where* $i \neq j \in \{1, 2\}$.

Proof. (i) Since $u_{12}u_{11} = 0$, for any $u_{11} \in \mathcal{U}_{11}$ and $u_{12} \in \mathcal{U}_{12}$, we have

$$-\delta(u_{11}u_{12}) = -L(u_{11}u_{12}) =$$

$$= [L(u_{12}), u_{11}] + [u_{12}, L(u_{11})] =$$

$$= L(u_{12})u_{11} - u_{11}L(u_{12}) + u_{12}L(u_{11}) - L(u_{11})u_{12} =$$

$$= -\delta(u_{11})u_{12} - u_{11}\delta(u_{12}).$$

This gives that $\delta(u_{11}u_{12}) = \delta(u_{11})u_{12} + u_{11}\delta(u_{12})$ for all $u_{11} \in \mathcal{U}_{11}$ and $u_{12} \in \mathcal{U}_{12}$.

In a similar manner, we can obtain rest of the cases.

Lemma 3.6. *For any* $u_{ii},\ v_{ii} \in \mathfrak{U}_{ii},\ i = 1, 2:$

(i) $\delta(u_{ii}v_{ii}) = \delta(u_{ii})v_{ii} + u_{ii}\delta(v_{ii})$,

(ii) $\delta(p) = 0 = \delta(q)$.

Proof. Consider the case for $i = 1$. For any $t_{12} \in \mathfrak{U}_{12}$, using Lemma 3.5 we find $\delta(u_{11}v_{11}t_{12}) = \delta(u_{11}v_{11})t_{12} + u_{11}v_{11}\delta(t_{12})$.

On the other hand, $\delta(u_{11}v_{11}t_{12} = \delta(u_{11})v_{11}t_{12} + u_{11}v_{11}\delta(t_{12}) + u_{11}\delta(v_{11})t_{12}$. From the above two expressions, we have $\{\delta(u_{11}v_{11}) - \delta(u_{11})v_{11} - u_{11}\delta(v_{11})\}t_{12} = 0$. Similarly, by using Lemma 3.6, we obtain $t_{21}\{\delta(u_{11}v_{11}) - \delta(u_{11})v_{11} - u_{11}\delta(v_{11})\} = 0$. Now by (2.1) we arrive at $\delta(u_{11}v_{11}) = \delta(u_{11})v_{11} + u_{11}\delta(v_{11})$ for all $u_{11}, v_{11} \in \mathfrak{U}_{11}$. Similarly for $i = 2$.

(ii) From (i) and using Remark 3.2, we find $\delta(p) = 0 = \delta(q)$.

Lemma 3.7. *For any* $u_{ij}, v_{ij} \in \mathfrak{U}_{ij},\ \delta(u_{ij}v_{ji}) = \delta(u_{ij})v_{ji} + u_{ij}\delta(v_{ji})$, *where* $i \neq j \in \{1, 2\}$.

Proof. First consider $i = 1$ and $j = 2$. Since $(u_{12}v_{21} - u_{12} - v_{21} + q)(p + v_{21}) = 0$ for any $u_{12} \in \mathfrak{U}_{12}$ and $v_{21} \in \mathfrak{U}_{21}$, we have

$$-\delta(u_{12}v_{21}) + \delta(u_{12}) - \delta(v_{21}u_{12}v_{21}) + \delta(v_{21}u_{12}) = -\delta(u_{12}v_{21} - u_{12} + v_{21}u_{12}v_{21} - v_{21}u_{12}) =$$

$$= \delta([u_{12}v_{21} - u_{12} - v_{21} + q, p + v_{21}]) =$$

$$= [L(u_{12}v_{21}) - L(u_{12}) - L(v_{21}) + L(q), p + v_{21}]+$$

$$+[u_{12}v_{21} - u_{12} - v_{21} + q, L(p) + L(v_{21})] =$$

$$= \delta(u_{12}) - \delta(u_{12})v_{21} - v_{21}\delta(u_{12}v_{21}) + v_{21}\delta(u_{12})-$$

$$-u_{12}\delta(v_{21}) - \delta(v_{21})u_{12}v_{21} + \delta(v_{21})u_{12}.$$

By using Lemma 3.5, it follows that

$$-\delta(u_{12})v_{21} + v_{21}\delta(u_{12}) - u_{12}\delta(v_{21}) + \delta(v_{21})u_{12} + \delta(u_{12}v_{21}) - \delta(v_{21}u_{12}) = 0. \tag{3.2}$$

Multiplying the above expression by u_{12} from right and u_{21} from left, respectively, we obtain $\{-\delta(u_{12})v_{21} - u_{12}\delta(v_{21}) + \delta(u_{12}v_{21})\}u_{12} = 0$ and $u_{21}\{-\delta(u_{12})v_{21} - u_{12}\delta(v_{21}) + \delta(u_{12}v_{21})\} = 0$. By assumption, we have $\delta(u_{12}v_{21}) = \delta(u_{12})v_{21} + u_{12}\delta(v_{21})$ for all $u_{12} \in \mathfrak{U}_{12}$ and $v_{12} \in \mathfrak{U}_{21}$.

In the similar manner, multiplying (3.2) by u_{21} from the right and u_{12} from the left, respectively, we obtain for $i = 1$ and $j = 2$.

***Proof of Theorem* 3.1.** In view of Remark 3.2 it remains to show that $\delta(uv) = \delta(u)v + u\delta(v)$ for all $u,\ v \in \mathfrak{U}$. Suppose that $u = u_{11} + u_{12} + u_{21} + u_{22}$ and $v = v_{11} + v_{12} + v_{21} + v_{22}$ for all u_{11}, $v_{11} \in \mathfrak{U}_{11}$; $u_{12},\ v_{12} \in \mathfrak{U}_{12}$; $u_{21},\ v_{21} \in \mathfrak{U}_{21}$ and $u_{22},\ v_{22} \in \mathfrak{U}_{22}$. Now, using Lemmas 3.6 and 3.7 it follows that

$$\delta(uv) = \delta((u_{11} + u_{12} + u_{21} + u_{22})(v_{11} + v_{12} + v_{21} + v_{22})) =$$

$$= \delta(u_{11}v_{11}) + \delta(u_{11}v_{12}) + \delta(u_{12}v_{21}) + \delta(u_{12}v_{22})+$$

$$+\delta(u_{21}v_{11}) + \delta(u_{21}v_{12}) + \delta(u_{22}v_{21}) + \delta(u_{22}v_{22}) =$$

$$= \delta(u_{11})v_{11} + \delta(u_{11})v_{12} + \delta(u_{12})v_{21} + \delta(u_{12})v_{22} + \delta(u_{21})v_{11} +$$

$$+\delta(u_{21})v_{12} + \delta(u_{22})v_{21} + \delta(u_{22})v_{22} + u_{11}\delta(v_{11}) + u_{11}\delta(v_{12}) +$$

$$+u_{12}\delta(v_{21}) + u_{12}\delta(v_{22}) + u_{21}\delta(v_{11}) + u_{21}\delta(v_{12}) + u_{22}\delta(v_{21}) + u_{22}\delta(v_{22}) =$$

$$= \delta(u_{11} + u_{12} + u_{21} + u_{22})(v_{11} + v_{12} + v_{21} + v_{22}) +$$

$$+(u_{11} + u_{12} + u_{21} + u_{22})\delta(v_{11} + v_{12} + v_{21} + v_{22}) =$$

$$= \delta(u)v + u\delta(v).$$

That is, δ is a derivation on $\mathfrak{U}$. Lastly, we have to show that $\phi[u, v] = 0$ with $uv = 0$:

$$\phi([u, v]) = [L(u), v] + [u, L(v)] - \delta([u, v]) = [\delta(u), v] + [u, \delta(v)] - \delta([u, v]) = 0.$$

Therefore, Lie derivation L has standard form, i.e., L can be written as a sum of derivation and a linear map vanishing at commutator by the action at zero product.

Theorem 3.1 is proved.

4. Characterization of ξ-Lie derivations. In this section, we characterize ξ-Lie derivations for $\xi \neq 1$ by its action at zero product on a unital algebra containing nontrivial idempotents.

Theorem 4.1. *Suppose that $\mathfrak{U}$ is a 2-torsion free unital algebra over a field $\mathbb{F}$ with a nontrivial idempotent p satisfying* (2.1) *and $\pi_{\mathfrak{U}_{11}}(Z(\mathfrak{U})) = Z(\mathfrak{U}_{11})$, $\pi_{\mathfrak{U}_{22}}(Z(\mathfrak{U})) = Z(\mathfrak{U}_{22})$. Let $L\colon \mathfrak{U} \to \mathfrak{U}$ be an additive mapping satisfying $L([u, v]_\xi) = [L(u), v]_\xi + [u, L(v)]_\xi$ for all $u, v \in \mathfrak{U}$ with $uv = 0$. Then we have the following cases:*

(i) *If $\xi \neq 0, -1$, then $L(\xi uv) = \xi L(u)v + \xi uL(v)$ and there exists an additive derivation δ satisfying $\delta(\xi I) = \xi L(I)$ such that $L(u) = \delta(u) + L(I)u$ if only if $L(I) \in Z(\mathfrak{U})$ for all $u \in \mathfrak{U}$.*

(ii) *If $\xi = -1$, then L is an additive derivation.*

(iii) *If $\xi = 0$, then there exists an additive derivation δ such that $L(u) = \delta(u) + L(I)u$ if and only if $L(I) \in Z(\mathfrak{U})$ for all $u \in \mathfrak{U}$.*

Assume that $\mathfrak{U}$ is 2-torsion free unital algebra over a field $\mathbb{F}$ satisfying the hypotheses of Theorem 4.1. The direct part is obvious. To prove only if part, we need the following lemmas.

Lemma 4.1. *$pL(I)q = 0 = qL(I)p$ and $qL(p)q = 0 = pL(q)p$.*

Proof. Since $pq = 0$, we find

$$0 = L([p, q]_\xi) = L(p)q - \xi qL(p) + pL(q) - \xi L(q)p. \tag{4.1}$$

Now multiplying by q on both side of (4.1), we get $qL(p)q = 0$. Again, multiplying by p and q from left and right, respectively, in (4.1), we obtain $pL(p)q + pL(q)q = pL(I)q = 0$. As $qp = 0$, on the similar steps we can find that $qL(I)p = 0$ and $pL(q)p = 0$.

***Remark* 4.1.** Define a map $L'\colon \mathfrak{U} \to \mathfrak{U}$ by $L'(u) = L(u) + [u_0, u]$ for all $u \in \mathfrak{U}$, where $u_0 = pL'(p)q - qL'(p)p$.

Accordingly, consider only those ξ-Lie derivation $L\colon \mathfrak{U} \to \mathfrak{U}$ which satisfies $pL(p)q = 0 = qL(p)p$. Now, it can be easily seen from the following lemma.

Lemma 4.2. *$L(p) = pL(p)p$ and $L(q) = qL(q)q$.*

Lemma 4.3. *For any* $u_{ij} \in \mathfrak{U}_{ij}$, $i \neq j$ *and* $i, j \in \{1, 2\}$:

(i) $L(I) \in Z(\mathfrak{U})$,

(ii) $L(u_{ij}) \in \mathfrak{U}_{ij}$.

Proof. (i) Since $u_{12}p = 0$ for all $u_{12} \in \mathfrak{U}_{12}$, we have

$$L(-\xi u_{12}) = L([u_{12}, p]_\xi) = L(u_{12})p - \xi pL(u_{12}) - \xi L(p)u_{12}. \tag{4.2}$$

Since $qu_{12} = 0$, using Lemma 4.2, we obtain

$$L(-\xi u_{12}) = -\xi u_{12}L(q) + qL(u_{12}) - \xi L(u_{12})q. \tag{4.3}$$

Combining (4.2) and (4.3), we arrive at

$$L(u_{12})p - \xi pL(u_{12}) - \xi L(p)u_{12} = -\xi u_{12}L(q) + qL(u_{12}) - \xi L(u_{12})q. \tag{4.4}$$

Now, if $\xi \neq 0$, then by multiplying p and q on left and right, respectively, in (4.4) and using Lemma 4.2, we have $L(p)u_{12} = pL(p)pu_{12} = u_{12}qL(q)q = u_{12}L(q)$.

Now, in case of $\xi = 0$, using $(p + u_{12})(u_{12} - q) = 0 = (u_{12} - q)(p + u_{12})$, we find

$$0 = L(p + u_{12})(u_{12} - q) + (p + u_{12})L(u_{12} - q) =$$

$$= L(p)u_{12} + L(u_{12})u_{12} - L(u_{12})q + pL(u_{12}) + u_{12}L(u_{12}) - u_{12}L(q)$$

and

$$0 = L(u_{12} - q)(p + u_{12}) + (u_{12} - q)L(p + u_{12}) =$$

$$= L(u_{12})p + L(u_{12})u_{12} + u_{12}L(u_{12}) - qL(u_{12}).$$

From the above two expressions, we arrive at $L(p)u_{12} = u_{12}L(q)$. We have

$$L(I)u_{12} = L(p + q)u_{12} = L(p)u_{12} = u_{12}L(q) + u_{12}L(p) = u_{12}L(I).$$

Also, we can show that $L(q)u_{21} = u_{21}L(p)$ and hence $L(I)u_{21} = u_{21}L(I)$. This implies that $L(I) \in Z(\mathfrak{U})$.

(ii) Now multiplying by p and q, respectively, on both the sides of (4.4), we find $pL(u_{12})p = 0 = qL(u_{12})q$.

If $\xi = 0$, then from (4.2), we get $qL(u_{12})p = 0$ which leads to $L(u_{12}) \in \mathfrak{U}_{12}$.

If $\xi = -1$, then on using $(p + u_{12})(u_{12} - q) = 0$ and by (4.4), we obtain

$$0 = L(p + u_{12})(u_{12} - q) + (p + u_{12})L(u_{12} - q) + L(u_{12} - q)(p + u_{12}) + (u_{12} - q)L(p + u_{12}) =$$

$$= L(p)u_{12} + 2L(u_{12})u_{12} - L(u_{12})q + 2u_{12}L(u_{12}) =$$

$$= -qL(u_{12}) + L(u_{12})p + pL(u_{12}) - u_{12}L(q) =$$

$$= 2L(u_{12})u_{12} + 2u_{12}L(u_{12}).$$

This yields that $qL(u_{12})pu_{12} = 0 = u_{12}qL(u_{12})p$ and hence $qL(u_{12})p = 0$.

If $\xi \neq 0, -1$, then for any $u_{12}, v_{12} \in \mathfrak{U}_{12}$ using the fact $(v_{12} - q)(p + u_{12}) = 0$, we obtain

$$L(\xi u_{12} - \xi v_{12}) = L([v_{12} - q, p + u_{12}]_\xi) =$$

$$= L(v_{12})p + L(v_{12})u_{12} - \xi pL(v_{12}) - \xi u_{12}L(v_{12}) + \xi u_{12}L(q)+$$

$$+v_{12}L(u_{12}) - qL(u_{12}) - \xi L(p)v_{12} - \xi L(u_{12})v_{12} + \xi L(u_{12})q. \tag{4.5}$$

Now multiplying (4.5) by p on both sides and using the fact $pL(u_{12})p = 0$, we get $\xi u_{12}qL(v_{12})p = v_{12}qL(u_{12})p$. Multiplying (4.2) by q on the left-hand side and p on the right-hand side, we have $qL(-\xi u_{12})p = qL(u_{12})p$. Combining the above two equations, we obtain

$$v_{12}qL(u_{12})p = v_{12}qL(-\xi u_{12})p = -\xi^2 u_{12}qL(v_{12})p = -\xi v_{12}qL(u_{12})p.$$

This implies that $qL(u_{12})p = 0$ and hence $L(u_{12}) \in \mathfrak{U}_{12}$ for all $u_{12} \in \mathfrak{U}_{12}$. In the similar manner, we can prove that $L(u_{21}) \in \mathfrak{U}_{21}$ for all $u_{21} \in \mathfrak{U}_{21}$.

Lemma 4.4. *For any $u_{ii} \in \mathfrak{U}_{ii}$, $L(u_{ii}) \in \mathfrak{U}_{ii}$, where $i = 1, 2$.*

Proof. Since $u_{11}q = 0$ for any $u_{11} \in \mathfrak{U}_{11}$, we have

$$0 = [L(u_{11}), q]_\xi + [u_{11}, L(q)]_\xi =$$

$$= L(u_{11})q - \xi qL(u_{11}) + u_{11}L(q) - \xi L(q)u_{11} =$$

$$= L(u_{11})q - \xi qL(u_{11}). \tag{4.6}$$

Now using the fact $\xi \neq 1$ and multiplying (4.6) by q on both sides, by p on left- and right-hand side, respectively, we get $qL(u_{11})q = pL(u_{11})q = qL(u_{11})p = 0$. This implies that $L(u_{11}) = pL(u_{11})p$ for all $u_{11} \in \mathfrak{U}_{11}$.

If $\xi = 0$, then from (4.6) we obtain $pL(u_{11})q = 0 = qL(u_{11})q$. Note that $qu_{11} = 0$ which gives that

$$0 = [q, L(u_{11})]_\xi + [L(q), u_{11}]_\xi =$$

$$= qL(u_{11}) - \xi L(u_{11})q + L(q)u_{11} - \xi u_{11}L(q) =$$

$$= qL(u_{11}) - \xi L(u_{11})q.$$

This implies that $qL(u_{11})p = 0$ and hence $L(u_{11}) \in \mathfrak{U}_{11}$ for all $u_{11} \in \mathfrak{U}_{11}$. Similarly, we can show for $i = 2$.

***Proof of Theorem* 4.1.** The proof is divide in following two steps:

Step 1. The following statements are true:

(i) If $\xi \neq 0, -1$, then $L(\xi xy) = \xi L(x)y + \xi xL(y)$ for all $x, y \in \mathcal{A}$.

(ii) If $\xi = -1$, then L is an additive derivation.

(iii) If $\xi = 0$, then there exists an additive derivation δ such that $L(x) = \delta(x) + L(I)x$ for all $x \in \mathcal{A}$.

(i) Since $y_{ij}x_{ii} = 0$ for all $x_{ii} \in \mathcal{A}_{ii}, y_{ij} \in \mathcal{A}_{ij}$ and $1 \leq i \neq j \leq 2$, we obtain

$$-L(\xi x_{ii}y_{ij}) = [L(y_{ij}), x_{ii}]_\xi + [y_{ij}, L(x_{ii})]_\xi = -\xi x_{ii}L(y_{ij}) - \xi L(x_{ii})y_{ij}.$$

This implies that

$$L(x_{ii}y_{ij}) = \xi L(x_{ii})y_{ij} + \xi x_{ii}L(y_{ij}). \tag{4.7}$$

Similarly, we can find

$$L(x_{ij}y_{jj}) = \xi L(x_{ij})y_{jj} + \xi x_{ij}L(y_{jj}). \tag{4.8}$$

Now, for any $x_{ii}, y_{ii} \in \mathcal{A}_{ii}$, we have $L(x_{ii}y_{ii}y_{ij}) = \xi L(x_{ii}y_{ii})y_{ij} + \xi x_{ii}y_{ii}L(y_{ij})$.

On the other hand, $L(x_{ii}y_{ii}y_{ij}) = \xi L(x_{ii})y_{ii}y_{ij} + \xi^2 x_{ii}L(\xi^{-1}y_{ii})y_{ij} + \xi x_{ii}y_{ii}L(y_{ij})$. Combining the above two expressions, we get

$$0 = (L(x_{ii}y_{ii}) - L(x_{ii})y_{ii} - \xi x_{ii}L(\xi^{-1}y_{ii}))y_{ij} =$$

$$= (L(\xi x_{ii}y_{ii}) - \xi L(x_{ii})y_{ii} - \xi x_{ii}L(y_{ii}))y_{ij}.$$

Similarly, using (4.8) we find $y_{ji}(L(\xi x_{ii}y_{ii}) - \xi L(x_{ii})y_{ii} - \xi x_{ii}L(y_{ii})) = 0$. Now by assumption, the last two expressions leads to

$$L(\xi x_{ii}y_{ii}) = \xi L(x_{ii})y_{ii} + \xi x_{ii}L(y_{ii}). \tag{4.9}$$

Again for all $x_{ij} \in \mathcal{A}_{ij}, y_{ji} \in \mathcal{A}_{ji}$ and $i \neq j$, $(x_{ij}y_{ji} - x_{ij} - y_{ji} + q)(p + y_{ji}) = 0$. Now applying similar steps as used in proof of Theorem 3.1, we arrive at

$$L(\xi x_{ij}y_{ji}) = \xi L(x_{ij})y_{ji} + \xi x_{ij}L(y_{ji}). \tag{4.10}$$

Applying (4.7) – (4.10) and using the similar calculation as used in proof of Theorem 3.1, we obtain $L(\xi xy) = \xi L(x)y + \xi xL(y)$ for all $x, y \in \mathcal{A}$.

(ii) If we take $\xi = -1$ in (i), then we find that L is an additive derivation.

(iii) Note that if $\xi = 0$ and $xy = 0$, then by definition we have $L(x)y + xL(y) = 0$ for all $x, y \in \mathcal{A}$. Also, here we will use the fact $L(p)x_{12} = x_{12}L(q)$ and $L(q)x_{21} = x_{21}L(p)$ for all $x_{12} \in \mathcal{A}_{12}$ and $x_{21} \in \mathcal{A}_{21}$. Since $(x_{11} + x_{11}y_{12})(q - y_{12}) = 0$ for all $x_{11} \in \mathcal{A}_{11}$ and $y_{12} \in \mathcal{A}_{12}$, we have

$$0 = L(x_{11} + x_{11}y_{12})(q - y_{12}) + (x_{11} + x_{11}y_{12})L(q - y_{12}) =$$

$$= -L(x_{11})y_{12} + L(x_{11}y_{12}) - x_{11}L(y_{12}) + x_{11}y_{12}L(q).$$

This implies that

$$L(x_{11}y_{12}) = L(x_{11})y_{12} + x_{11}L(y_{12}) - x_{11}L(p)y_{12}. \tag{4.11}$$

Also, note that $(x_{22} + x_{22}y_{21})(p - y_{21}) = (p - x_{12})(y_{22} + x_{12}y_{22}) = (q - x_{21})(y_{22} + x_{12}y_{22}) = 0$. By using these relations, we obtain

$$L(x_{22}y_{21}) = L(x_{22})y_{21} + x_{22}L(y_{21}) - x_{22}L(q)y_{21}, \tag{4.12}$$

$$L(x_{12}y_{22}) = L(x_{12})y_{22} + x_{12}L(y_{22}) - x_{12}L(p)y_{22}, \tag{4.13}$$

$$L(x_{21}y_{11}) = L(x_{21})y_{11} + x_{21}L(y_{11}) - x_{21}L(q)y_{11}. \tag{4.14}$$

Now, using (4.11), (4.14) and applying similar steps after (4.8), we find

$$L(x_{ii}y_{ii}) = L(x_{ii})y_{ii} + x_{ii}L(y_{ii}) - x_{ii}L(p)y_{ii}. \tag{4.15}$$

Also, since $(x_{12} + x_{12}y_{21})(p - y_{21}) = 0 = (x_{21} + x_{21}y_{12})(q - y_{12})$ and using Lemma 4.3, we have

$$L(x_{12}y_{21}) = L(x_{12})y_{21} + x_{12}L(y_{21}) - x_{12}L(p)y_{21}, \tag{4.16}$$

$$L(x_{21}y_{12}) = L(x_{21})y_{12} + x_{21}L(y_{12}) - x_{21}L(q)y_{12}. \tag{4.17}$$

Applying (4.11) – (4.17) and using the similar calculation as used in the proof of Theorem 3.1, we obtain that $L(xy) = L(x)y + xL(y) - xL(I)y$ for all $x, y \in \mathcal{A}$.

Step 2. If $\xi \neq 0, -1$, then there exists an additive derivation δ satisfying $\delta(\xi I) = \xi L(I)$ such that $L(x) = \delta(x) + L(I)x$ for all $x \in \mathcal{A}$.

By Step 1, we have $L(\xi xy) = \xi L(x)y + \xi xL(y)$. As $xy = 0$ we find $\xi L(x)y + \xi xL(y) = 0$ and hence $L(x)y + xL(y) = 0$. Again, by Lemma 4.4 (Step 2), it is similar to the case $\xi = 0$ and hence there exists an additive derivation δ satisfying $L(x) = \delta(x) + L(I)x$ for all $x \in \mathcal{A}$. Also, $L(\xi I) = \xi L(I)I + \xi IL(I) = 2\xi L(I)$. Therefore, $L(\xi I) \in Z(\mathcal{A})$. Now, since $x_{12}p = 0$,

$$\delta(-\xi x_{12}) - \xi L(I)x_{12} = L(-\xi x_{12}) = [L(x_{12}), p]_\xi + [x_{12}, L(p)]_\xi =$$

$$= [\delta(x_{12}) + L(I)x_{12}, p]_\xi + [x_{12}, \delta(p) + L(I)p]_\xi = -\xi\delta(x_{12}) - 2\xi L(I)x_{12}.$$

This gives that $\delta(\xi x_{12}) = \xi\delta(x_{12}) + \xi L(I)x_{12}$. On the other hand, $\delta(\xi x_{12}) = \delta(\xi I)x_{12} + \xi\delta(x_{12})$. It follows that $\big(\delta(\xi I) - \xi L(I)\big)x_{12} = 0$. In the similar way, we can obtain that $x_{21}\big(\delta(\xi I) - \xi L(I)\big) = 0$. Hence by (2.1), we have $\delta(\xi I) = \xi L(I)$.

Theorem 4.1 is proved.

5. Applications. As a direct consequence of our Theorem 3.1, we have the following results.

***Corollary* 5.1** ([8], Theorem 2.1). *Let X be a Banach space of dimension greater than 2, and $L\colon \mathcal{B}(X) \to \mathcal{B}(X)$ be a linear map satisfying $L([u, v]) = [L(u), v] + [u, L(v)]$ for any $u, v \in \mathcal{B}(X)$ with $uv = 0$. Then there exists an operator $r \in \mathcal{B}(X)$ and a linear map $\phi\colon \mathcal{B}(X) \to \mathbb{C}I$ vanishing at commutators $[u, v]$ when $uv = 0$ such that $L(u) = ru - ur + \phi(u)$ for all $u \in \mathcal{B}(X)$.*

***Corollary* 5.2** ([5], Theorem 2.1). *Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over a commutative ring $\mathcal{R}$ with unity I_1 and I_2, respectively. Let $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule and $\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{B}$. If $\pi_{\mathcal{A}}(Z(\mathcal{T})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{T})) = Z(\mathcal{B})$ and $L\colon \mathcal{T} \to \mathcal{T}$ is an $\mathcal{R}$-linear map such that $L([u, v]) = [L(u), v] + [u, L(v)]$ for any u, $v \in \mathcal{T}$ with $uv = 0$, then there exists a derivation δ of $\mathcal{T}$ and an $\mathcal{R}$-linear map $\phi\colon \mathcal{T} \to Z(\mathcal{T})$ vanishing at commutators $[u, v]$ with $uv = 0$ such that $L(u) = \delta(u) + \phi(u)$ for all $u \in \mathcal{T}$.*

***Corollary* 5.3** ([5], Corollary 2.1). *Let $\mathcal{N}$ be an arbitrary nest on a Hilbert space H of dimension greater than 2 and $\mathrm{Alg}\,\mathcal{N}$ be the associated nest algebra. Let $L\colon \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ be a linear map satisfying $L([u, v]) = [L(u), v] + [u, L(v)]$ for any $u, v \in \mathrm{Alg}\,\mathcal{N}$ with $uv = 0$. Then there exists an operator $r \in \mathrm{Alg}\,\mathcal{N}$ and a linear map $\phi\colon \mathrm{Alg}\,\mathcal{N} \to \mathbb{F}I$ vanishing at commutators $[u, v]$ when $uv = 0$ such that $L(u) = ru - ur + \phi(u)$ for all $u \in \mathrm{Alg}\,\mathcal{N}$.*

For the finite dimensional case, it is clear that every nest algebra on a finite dimensional space is isomorphic to an upper triangular block matrix algebra [4].

***Corollary* 5.4.** *Let $\mathbb{F}$ be the real or complex field and $n > 2$ be a positive integer. Let $\mathcal{B}_n(\mathcal{R})$ be a proper block upper triangular matrix algebra over $\mathbb{F}$ and $L\colon \mathcal{B}_n(\mathcal{R}) \to \mathcal{B}_n(\mathcal{R})$ be a linear map satisfying $L([u,v]) = [L(u),v] + [u,L(v)]$ for any $u,v \in \mathcal{B}_n(\mathcal{R})$ with $uv = 0$. Then there exists an operator $r \in \mathcal{B}_n(\mathcal{R})$ and a linear map $\phi\colon \mathcal{B}_n(\mathcal{R}) \to \mathbb{F}I$ vanishing at commutators $[u,v]$ when $uv = 0$ such that $L(u) = ru - ur + \phi(u)$ for all $u \in \mathcal{B}_n(\mathcal{R})$.*

***Corollary* 5.5.** *Let $\mathcal{U}$ be a factor von Neumann algebra with $deg(\mathcal{U}) > 1$ and $L\colon \mathcal{U} \to \mathcal{U}$ be a linear map. Then L satisfies $L([u,v]) = [L(u),v] + [u,L(v)]$ for any $u,v \in \mathcal{U}$ with $uv = 0$ if and only if it has the form $L(u) = ru - ur + \tau(u)$ for all $u \in \mathcal{U}$, where $r \in \mathcal{U}$ and $\tau\colon \mathcal{U} \to \mathbb{F}I$ is a linear functional vanishing on each commutator $[u,v]$ whenever $uv = 0$.*

Proof. Since factor von Neumann algebra $\mathcal{U}$ satisfies (2.1) and all linear derivations of von Neumann algebras are inner, L is the sum of inner derivation and a linear functional vanishing on each commutator $[u,v]$ whenever $uv = 0$.

Since every triangular algebra is the example of algebra that satisfies (2.1), the following result is an immediate consequence of the Theorem 4.1.

***Corollary* 5.6** ([11], Theorem 4.1)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a field $\mathbb{F}$, and $\mathcal{M}$ be an $(\mathcal{A},\mathcal{B})$ bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{T} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ be the triangular algebra consisting of $\mathcal{A},\mathcal{B},\mathcal{M}$ and $\xi \in \mathbb{F}$ with $\xi \neq 0,1$. Assume that $L\colon \mathcal{T} \to \mathcal{T}$ is an additive map, $\pi_{\mathcal{A}}(Z(\mathcal{T})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{T})) = Z(\mathcal{B})$. Then L satisfies $L([u,v]_\xi) = [L(u),v]_\xi + [u,L(v)]_\xi$ for all $u,\ v \in \mathcal{T}$ with $uv = 0$ if and only if $L(I) \in Z(\mathcal{T})$ and there exists an additive derivation $\delta\colon \mathcal{T} \to \mathcal{T}$ with $\delta(\xi I) = \xi L(I)$ such that $L(u) = \delta(u) + L(I)u$ for all $u \in \mathcal{T}$.*

***Corollary* 5.7** ([11], Theorem 4.2)**.** *Let $\mathcal{N}$ be a nest on an infinite dimensional Banach space X over the real or complex field $\mathbb{F}$, and let $\mathrm{Alg}\,\mathcal{N}$ be the associated nest algebra. Assume that $\xi \in \mathbb{F}$ with $\xi \neq 0,1$ and $L\colon \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is an additive map and there exists a nontrivial element in $\mathcal{N}$ which is complemented in X. Then L satisfies $L([u,v]_\xi) = [L(u),v]_\xi + [u,L(v)]_\xi$ for any $u,v \in \mathrm{Alg}\,\mathcal{N}$ with $uv = 0$ if and only if there exists an operator $r \in \mathrm{Alg}\,\mathcal{N}$ such that $L(u) = ur - ru$ for all $u \in \mathrm{Alg}\,\mathcal{N}$.*

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