DOI: 10.37863/umzh.v72i8.847

UDC 515.1

T. Körpinar, R. C. Demirkol (Muş Alparslan Univ., Turkey)

CURVATURE AND TORSION DEPENDENT ENERGY OF ELASTICA AND NONELASTICA FOR A LIGHTLIKE CURVE IN THE MINKOWSKI SPACE

ЗАЛЕЖНА ВІД КРИВИНИ ТА КРУЧЕННЯ ЕНЕРГІЯ ПРУЖНОСТІ ТА НЕПРУЖНОСТІ ДЛЯ СВІТЛОПОДІБНИХ КРИВИХ У ПРОСТОРІ МІНКОВСЬКОГО

We firstly describe conditions for being elastica or nonelastica for a lightlike elastic Cartan curve in the Minkowski space $\mathbb{E}_1^4$ by using the Bishop orthonormal vector frame and associated Bishop components. Then we compute the energy of the lightlike elastic and nonelastic Cartan curve in the Minkowski space $\mathbb{E}_1^4$ and investigate its relationship with the energy of the same curve in Bishop vector fields in $\mathbb{E}_1^4$. Here, energy functionals are computed in terms of Bishop curvatures of the lightlike Cartan curve lying in the Minkowski space $\mathbb{E}_1^4$.

Спочатку описано умови пружності та непружності світлоподібних пружних кривих Картана у просторі Мінковського $\mathbb{E}_1^4$ за допомогою ортонормального векторного репера та відповідних компонент Бішопа. Потім обчислено енергію пружних та непружних світлоподібних кривих Картана у просторі Мінковського $\mathbb{E}_1^4$ та вивчено їхній зв'язок із енергією тієї ж кривої у векторних полях Бішопа у $\mathbb{E}_1^4$. Тут функціонали енергії обчислюються у термінах кривини Бішопа для світлоподібних кривих Картана, що належать простору Мінковського $\mathbb{E}_1^4$.

1. Introduction. Minkowski space-time is an important structure to define many well-known physical and geometrical concepts such as black holes, gravitational dilation of time, cosmology, length string theory, contraction, etc. In this Minkowski space, mass-energy equivalence states the relationship between mass and energy and special relativity estimates this equivalence by the formula $E = mc^2$, where c is the light's speed in a vacuum [1, 2].

Some traditional geometric topics such as local and global features of different types of curves are used to many physical subjects. For instance, Altin [3] calculated the energy of Frenet orthonormal vector fields by using nonlightlike curves. Körpınar [4] considered a timelike biharmonic particle and computed its energy functional in Heisenberg spacetime. Lightlike curves are a thoroughly complicated field to study since their tangent vectors cannot be normalized in an ordinary manner in contrast to nonlightlike curves. Körpinar characterized the energy of different types of lightlike curves in Minkowski space $\mathbb{E}_1^4$ [5].

The fundamental ingredient for the study of the geodesic lightlike congruences, gravitational radiation, Killing horizon are determined via tetrad formalism of Newman – Penrose, which is deduced by a lightlike curve. Furthermore, it is known that relativistic string can be defined as a surface in Minkowski space such that it is a Lorentzian analogue of the equations of minimal surface. Wave equations can also be simplified by string equations and solving 2-dimensional wave equations leads that strings and pair of lightlike curves are equivalents [6 – 10].

Materials having the feature of deformable property such as flexible metals, paper, cloth, rubber are the main instances and study fields of the elasticity theory. However, elastic theory enlightens a broad range of other physical and mathematical studies such as the study of variational problems, the solution of the elliptic integral, mechanical equilibrium, equilibrium of moments, which constitutes the elementary principle of statics. Further, it is seen that elastica gives a natural solution for the variational problem, which deals with the minimizing of bending energy of the elastic curve. Later, the equivalence between the motion of the simple pendulum and fundamental differential equation of elastica was investigated. Recently, numerical computation implemented on the elastica is used to develop mathematical spline theory [11].

In this study, we firstly determine differential equations satisfied by non-rigid deformable lightlike Cartan curves in order to model the behavior of lightlike elastic Cartan curves in 4-dimensional Minkowski space $\mathbb{E}_1^4$. We compute energy on the lightlike elastic Cartan curves by using the variational method. Finally, we define lightlike nonelastic Cartan curves and compute their energy to characterize their structure and investigate the relationship between the elastic and nonelastic cases.

2. Lightlike curves in Minkowski space $\mathbb{E}_1^4$. Minkowski space $\mathbb{E}_1^4$ is the 4-dimensional standard real vector space equipped with the usual indefinite metric $(\cdot,\cdot)$ described by

$$(x,y) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4,$$

where $x, y \in \mathbb{E}_1^4$. Owing to the intrinsic features of the indefinite metric $(\cdot,\cdot)$, any vector $x \in \mathbb{E}_1^4$ possesses three causal characters, which is determined by the norm of the given vector $\|x\| = \sqrt{|(x,x)|}$. To be more specific, any vector $x \in \mathbb{E}_1^4$ can be characterized as follows:

x is spacelike, if (x,x) is positive,

x is timelike, if (x,x) is negative,

x is lightlike, if (x,x) is zero.

One can give a similar characterization for any space curve defined in $\mathbb{E}_1^4$. Namely, an arbitrary space curve $\alpha\colon I \to \mathbb{E}_1^4$ is called a spacelike, timelike or lightlike curve provided that all tangent vectors of α are spacelike, timelike or lightlike along with the curve. In this study, we mainly focus on a special class of lightlike curves which is known as a Cartan lightlike curve. A curve α is called a lightlike Cartan curve supposed that its parametrization is determined by the pseudo-arc function defined by

$$s(t) = \int_0^t \sqrt{\|\alpha''(u)\|du}.$$

As we recall, some of the geometric and physical features of a space curve are defined with the help of the geometric quantities known as the torsion and curvature. To handle with the intrinsic characterization for a lightlike Cartan curve in $\mathbb{E}_1^4$, these quantities are also defined for lightlike Cartan curves with the name of the first, second, and third Cartan curvatures and denoted respectively by $k_1,\ k_2,\ k_3$. A unique orthonormal vectors $\{\mathbf{T},\mathbf{N},\mathbf{B}_1,\mathbf{B}_2\}$ satisfy the following set of equation systems along the nongeodesic lightlike Cartan curve α and it is called by the Frenet–Serret frame of lightlike Cartan curve [12]:

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{N},\\
\nabla_{\mathbf{T}}\mathbf{N} &= -k_2\mathbf{T} + k_1\mathbf{B}_1, \qquad (1)\\
\nabla_{\mathbf{T}}\mathbf{B}_1 &= -k_2\mathbf{N} + k_3\mathbf{B}_2,
\end{aligned}$$

$$\nabla_{\mathbf{T}}\mathbf{B}_2 = k_3\mathbf{T},$$

where $k_1 = 1,\ k_2$, and k_3 are arbitrary functions. Here, we also have

$$0 = (\mathbf{T},\mathbf{T}) = (\mathbf{B}_1,\mathbf{B}_1) = (\mathbf{N},\mathbf{B}_1) = (\mathbf{B}_1,\mathbf{B}_2) = (\mathbf{T},\mathbf{N}) = (\mathbf{T},\mathbf{B}_2),$$

$$1 = (\mathbf{N},\mathbf{N}) = (\mathbf{B}_2,\mathbf{B}_2), \qquad (\mathbf{T},\mathbf{B}_1) = -1.$$

Bishop frame is known as the relatively parallel frame and it is obtained by conducting some modification on the Frenet – Serret frame. Using the Bishop frame provides some advantages and effectiveness compared to the classical Frenet – Serret frame. For this reason, we will state Bishop's frame of the lightlike Cartan curve and consider it as the main frame for the rest of the paper.

***Case* 1:** Let α be a lightlike Cartan curve in $\mathbb{E}_1^4$ and $k_1 = 1,\ k_3 = 0$, and k_2 be an arbitrary function, then Bishop frame equations for orthonormal vectors $\{\mathbf{T},\mathbf{N}_1,\mathbf{N}_2,\mathbf{N}_3\}$ are stated as the following [12]:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \pi_2\mathbf{T} + \pi_1\mathbf{N}_1,\\ \nabla_{\mathbf{T}}\mathbf{N}_1 &= \pi_1\mathbf{N}_2,\\ \nabla_{\mathbf{T}}\mathbf{N}_2 &= -\pi_2\mathbf{N}_2,\\ \nabla_{\mathbf{T}}\mathbf{N}_3 &= 0,\end{aligned} \tag{2}$$

where the first, second and third Bishop curvature satisfies respectively that $\pi_1 = 1,\ \pi_2' = -\dfrac{1}{2}\pi_2^2 - k_2$, $\pi_3 = 0$. Here, we also have

$$0 = (\mathbf{T},\mathbf{T}) = (\mathbf{N}_2,\mathbf{N}_2) = (\mathbf{T},\mathbf{N}_1) = (\mathbf{N}_1,\mathbf{N}_2) = (\mathbf{T},\mathbf{N}_3) = (\mathbf{N}_2,\mathbf{N}_3),$$

$$(\mathbf{N}_1,\mathbf{N}_1) = (\mathbf{N}_3,\mathbf{N}_3) = 1, \qquad (\mathbf{T},\mathbf{N}_2) = -1.$$

***Case* 2:** Let α be a lightlike Cartan curve in $\mathbb{E}_1^4$ and $k_1 = 1,\ k_2,\ k_3$ be arbitrary functions, then Bishop frame equations for orthonormal vectors $\{\mathbf{T},\mathbf{N}_1,\mathbf{N}_2,\mathbf{N}_3\}$ are stated as the following [12]:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \pi_2\mathbf{T} + \pi_1\mathbf{N}_1 - \pi_3\mathbf{N}_3,\\ \nabla_{\mathbf{T}}\mathbf{N}_1 &= \pi_1\mathbf{N}_2,\\ \nabla_{\mathbf{T}}\mathbf{N}_2 &= -\pi_2\mathbf{N}_2,\\ \nabla_{\mathbf{T}}\mathbf{N}_3 &= -\pi_3\mathbf{N}_2,\end{aligned} \tag{3}$$

where the first, second and third Bishop curvature satisfies respectively that $\pi_1 = \sin\theta,\ \pi_2 = -\dfrac{k_3 - \theta''}{\theta'},\ \pi_3 = \cos\theta\ (\theta' \neq 0)$ and

$$2\theta'(\theta''' - k_3') + 2\theta''(k_3 - \theta'') + \theta'4 - (k_3 - \theta'')^2 - 2k_2\theta'2 = 0.$$

Here, we also have

$$0 = (\mathbf{T},\mathbf{T}) = (\mathbf{N}_2,\mathbf{N}_2) = (\mathbf{T},\mathbf{N}_1) = (\mathbf{N}_1,\mathbf{N}_2) = (\mathbf{T},\mathbf{N}_3) = (\mathbf{N}_2,\mathbf{N}_3),$$

$$(\mathbf{N}_1,\mathbf{N}_1) = (\mathbf{N}_3,\mathbf{N}_3) = 1, \qquad (\mathbf{T},\mathbf{N}_2) = -1.$$

3. Energy on the classical Bernoulli – Euler elastica. ***3.1. Energy on the Frenet vector field.*** We first give fundamental definitions and propositions, which are used to compute the energy of the vector field.

Definition 3.1. *Let (M,ρ) and (N,h) betwo Riemannian manifolds, then the energy of a differentiable map $f\colon (M,\rho)\to(N,h)$ can be defined as*

$$\varepsilon nergy(f)=\frac{1}{2}\int\limits_M\sum_{a=1}^{n}h\big(df(e_a),df(e_a)\big)v, \tag{4}$$

where $\{e_a\}$ is a local basis of the tangent space and v is the canonical volume form in M [3, 13].

Proposition 3.1. *Let $Q\colon T\left(T^1M\right)\to T^1M$ be the connection map. Then the following two conditions hold*:

i) $\omega\circ Q=\omega\circ d\omega$ *and* $\omega\circ Q=\omega\circ\tilde{\omega}$, *where* $\tilde{\omega}\colon T(T^1M)\to T^1M$ *is the tangent bundle projection*;

ii) *for* $\varrho\in T_xM$ *and a section* $\xi\colon M\to T^1M$, *we have*

$$Q\big(d\xi(\varrho)\big)=\nabla_\varrho\xi, \tag{5}$$

where ∇ is the Levi – Civita covariant derivative [3].

Definition 3.2. *Let $\varsigma_1,\varsigma_2\in T_\xi(T^1M)$, then we define*

$$\rho_S(\varsigma_1,\varsigma_2)=\big(d\omega(\varsigma_1),d\omega(\varsigma_2)\big)+\big(Q(\varsigma_1),Q(\varsigma_2)\big). \tag{6}$$

This yields a Riemannian metric on TM. As known ρ_S is called the Sasaki metric that also makes the projection $\omega\colon T^1M\to M$ a Riemannian submersion.

3.2. Energy on the lightlike elastic Cartan curves in $\mathbb{E}_1^4$. The research on the curvature-based energies for space curves began with Bernoulli and Euler. They focus on elastic thin beams and rods. This type of energy is both essential in the mechanical context and it is also significant in computer vision, image processing and computer vision besides mathematical and physical importance [14 – 17].

Let $\alpha\in\mathbb{E}_1^4$ be a regular curve defined on any fixed interval $[y_1,y_2]$ so that

$$\alpha\colon[y_1,y_2]\to\mathbb{E}_1^4$$

is parameterized by the pseudo arc-length $\rho\big(\alpha''(s),\alpha''(s)\big)=1$.

Elastica of bending energy is defined for the curve α in $\mathbb{E}_1^4$ over each point on a fixed interval $[y_1,y_2]$ as

$$\mathcal{G}=\frac{1}{2}\int\limits_{y_1}^{y_2}\|\alpha''\|^2dt$$

with some boundary conditions [18, 19].

For any two points $p_1,p_2\in\mathbb{R}^4$ and any two non-zero vectors $p_1',\ p_2'$ space of smooth curves is defined as

$$\varphi=\big\{\alpha\colon\alpha(y_i)=p_i,\alpha'(y_i)=p_i'\big\}.$$

It is also defined as the smooth curves of unit speed as a subspace of φ in the form

$$\varphi_a = \{\alpha \in \varphi\colon \ \|\alpha'\| = 1\}.$$

Then $\mathcal{G}^{\pi} : \varphi \to \mathbb{R}$ can be defined by

$$\mathcal{G}^{\pi}(\alpha) = \frac{1}{2}\int\limits_{\alpha} \|\alpha''\|^2 + \Gamma(t)\big(\|\alpha'\|^2 - 1\big)dt, \tag{7}$$

where $\Gamma(t)$ is a pointwise multiplier. A stationary point of $\mathcal{G}^{\pi}$ is the minimum of $\mathcal{G}$ on φ_a for some $\Gamma(t)$ according to the multiplier principle of Lagrange.

Let α be an extremum of $\mathcal{G}^{\pi}$ and V be a vector field along α, which is a curve's infinitesimal variation, then we get [20]

$$\partial\mathcal{G}^{\pi}(V) = \frac{\partial}{\partial\Upsilon}\mathcal{G}^{\pi}(\alpha + \Upsilon V)\,\big|_{\Upsilon=0} = 0. \tag{8}$$

We obtain significant differences in conditions that have to be satisfied by lightlike elastic Cartan curves by using the Bishop equations formulae given by Eqs. (2), (3). Further, we also compute the energy of lightlike elastic Cartan curves by using the Lorentzian and Sasaki metrics.

***Case* 1:** Let $\alpha \in \mathbb{E}_1^4$ be a lightlike elastic Cartan curve defined on any fixed interval $[y_1, y_2]$ so that

$$\alpha\colon [y_1, y_2] \to \mathbb{E}_1^4$$

and $k_1 = 1,\ k_3 = 0,$ and k_2 be an arbitrary function.

By considering the orthonormal frame given by (2), the energy of Bishop vectors $(\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$ for lightlike Cartan curve $\alpha \in \mathbb{E}_1^4$ can be computed if one follows similar steps as in following research completed by K. Demirkol [4]. This study is helpful to see a relation between the energy of Bishop vectors and bending energy functional which is defined as

$$\mathcal{G} = \frac{1}{2}\int\limits_{\alpha} \|\nabla_{\mathbf{T}}\mathbf{T}\|^2\, ds.$$

Let V be a vector field along α such that it is a curve's infinitesimal variation. By using equations (7) and (8), we get

$$0 = \frac{1}{2}\frac{\partial}{\partial\Upsilon}\int\limits_{y_1}^{y_2} \left\|(\alpha + \Upsilon V)''\right\|^2 + \Gamma\left(\left\|(\alpha + \Upsilon V)'\right\|^2 - 1\right) dt = \int\limits_{y_1}^{y_2} (\alpha'', V'')dt.$$

Applying integration by parts, we obtain

$$0 = (\alpha'', V')\,\Big|_{y_1}^{y_2} - (V, \alpha''')\,\Big|_{y_1}^{y_2} + \int\limits_{y_1}^{y_2} (V, \mathcal{E}(\alpha))dt,$$

where $\mathcal{E}(\alpha) = \alpha''''$. Being elastica implies that

$$\mathcal{E}(\alpha) = \alpha'''' \equiv 0. \tag{9}$$

Thanks to Noether's theorem we know that

$$\mathcal{J} = \alpha'''$$

is a constant vector field. For a parameterized curve α with the arc-length s, we have from Eq. (2)

$$\alpha' = \mathbf{T}, \qquad \alpha'' = \mathbf{T}' = \pi_2\mathbf{T} + \pi_1\mathbf{N}_1,$$

$$\alpha''' = \left(\pi_2^2 + \pi_2'\right)\mathbf{T} + (\pi_1\pi_2 + \pi_1')\mathbf{N}_1 + \pi_1^2\mathbf{N}_2.$$

Thus, we get

$$\mathcal{J} = (\pi_2^2 + \pi_2')\mathbf{T} + (\pi_1\pi_2 + \pi_1')\mathbf{N}_1 + \pi_1^2\mathbf{N}_2.$$

By the fact that $\mathcal{J}$ is a constant vector field, we find $\mathcal{J}_s = 0$. From this, we have the system

$$3\pi_2\pi_2' + \pi_2'' + \pi_2^3 = 0,$$

$$2\pi_1\pi_2' + \pi_1\pi_2^2 + \pi_1'\pi_2 + \pi_1'' = 0,$$

$$\pi_1\pi_1' = 0.$$

Thus, we have following sample solution family of the nonlinear ordinary differential equation system for certain values of π_1, π_2, and π_3.

Theorem 3.1. *Energy of lightlike elastic Cartan curves having the Bishop characterization in the Eq.* (2) *in constant vector field* $\mathcal{J}$ *in* $\mathbb{E}_1^4$ *is stated by using the Sasaki metric as follows:*

$$\varepsilon nergy_1(\mathcal{J}) = 0.$$

Proof. From (4) and (5) we know

$$\varepsilon nergy_1\left(\mathcal{J}\right) = \frac{1}{2}\int_0^s \rho_S\left(d\mathcal{J}(\mathbf{T}), d\mathcal{J}(\mathbf{T})\right) ds.$$

If one considers the Eq. (6) then it is obtained that

$$\rho_S\left(d\mathcal{J}(\mathbf{T}), d\mathcal{J}(\mathbf{T})\right) = \rho(d\omega(\mathcal{J}(\mathbf{T})), d\omega(\mathcal{J}(\mathbf{T}))) + \rho(Q(\mathcal{J}(\mathbf{T})), Q(\mathcal{J}(\mathbf{T}))).$$

Since $\mathcal{J}$ is a section, we get

$$d(\omega) \circ d(\mathcal{J}) = d(\omega \circ \mathcal{J}) = d(id_C) = id_{TC}.$$

We also know

$$Q(\mathcal{J}(\mathbf{T})) = \bigtriangledown_{\mathbf{T}}\mathcal{J} = 0.$$

Thus, we find from the former statements that

$$\rho_S\big(d\mathcal{J}(\mathbf{T}), d\mathcal{J}(\mathbf{T})\big) = (\mathbf{T}, \mathbf{T}) + (\nabla_{\mathbf{T}}\mathcal{J}, \nabla_{\mathbf{T}}\mathcal{J}) = 0.$$

So, we can easily obtain that $\varepsilon nergy_1(\mathcal{J}) = 0$.

Theorem 3.1 is proved.

Theorem 3.2. *Energy of lightlike Cartan curves of Bishop orthonormal vectors of Eq.* (2) *are defined by using the Sasaki metric and Bishop curvatures given by the Eq.* (2) *in the following manner*:

$$\varepsilon nergy_1(\mathbf{T}) = \frac{1}{2}\int_0^s \pi_1^2 ds,$$

$$\varepsilon nergy_1(\mathbf{N}_1) = 0,$$

$$\varepsilon nergy_1(\mathbf{N}_2) = 0,$$

$$\varepsilon nergy_1(\mathbf{N}_3) = 0.$$

Proof. It is obvious if one considers Eqs. (4) – (6).

Corollary **3.1.** $\mathbb{E}_1^4$ *be a lightlike elastic Cartan curve having the Bishop characterization given by the Eq.* (2), *then we have the following relation*:

$$\varepsilon nergy_1(\mathbf{T}) - \varepsilon nergy_1(\mathcal{J}) = \frac{s}{2},$$

$$\varepsilon nergy_1(\mathcal{J}) = \varepsilon nergy_1(\mathbf{N}_1) = \varepsilon nergy_1(\mathbf{N}_2) = \varepsilon nergy_1(\mathbf{N}_3).$$

Proof. If one uses Theorems 3.1 and 3.2 it gives the result immediately.

Case **2:** Let $\alpha \in \mathbb{E}_1^4$ be a lightlike elastic Cartan curve defined on any fixed interval $[y_1, y_2]$ so that $\alpha\colon [y_1, y_2] \to \mathbb{E}_1^4$ and $k_1 = 1,\ k_2,\ k_3$ be arbitrary functions.

Theorem 3.3. *Let* α *be a lightlike elastic Cartan curve with the Bishop characterization given by the Eq.* (3). *If* V *is a vector field*, *which is an infinitesimal variation of the curve* α, *then we have constant vector field* J *and some restrictions as the following*:

$$\mathcal{J}=(\pi_2' + \pi_2^2)\mathbf{T} + (\pi_1' + \pi_1\pi_2)\mathbf{N}_1+(-\pi_3' - \pi_2\pi_3)\mathbf{N}_3,$$

$$0 = \pi_2' + \pi_2^2,$$

$$0 = \pi_1' + \pi_1\pi_2,$$

$$0 = -\pi_3' - \pi_2\pi_3.$$

Proof. If we follow a similar procedure as in Case 1 and use the characterization given in the Eq. (3), it is obvious.

Thus, we have following sample solution family of the nonlinear ordinary differential equation system for certain values of π_1, π_2, and π_3.

Theorem 3.4. *Energy of lightlike Cartan curves of Bishop orthonormal vectors of Eq.* (3) *are defined by using the Sasaki metric and Bishop curvatures given by the Eq.* (3) *in the following manner*:

$$\varepsilon nergy_1(\mathbf{T}) = 0,$$

$$\varepsilon nergy_1(\mathbf{N}_1) = 0,$$

$$\varepsilon nergy_1(\mathbf{N}_2) = 0,$$

$$\varepsilon nergy_1(\mathbf{N}_3) = 0.$$

Proof. It is obvious if one considers Eqs. (4)–(6).

***Corollary* 3.2.** *Let $\alpha \in \mathbb{E}_1^4$ be a lightlike elastic Cartan curve having the Bishop characterization given by the Eq.* (3), *then we have the following relation*:

$$\varepsilon nergy_1(\mathbf{T}) = \varepsilon nergy_1(\mathcal{J}) = \varepsilon nergy_1(\mathbf{N}_1) =$$
$$= \varepsilon nergy_1(\mathbf{N}_2) = \varepsilon nergy_1(\mathbf{N}_3).$$

Proof is obvious.

4. Energy on nonelastic lightlike Cartan curve in $\mathbb{E}_1^4$. In this section, we deal with the concept of different types of nonelastic lightlike Cartan curves and their energies in $\mathbb{E}_1^4$.

***Case* 1:** Let $\alpha \in \mathbb{E}_1^4$ be a lightlike Cartan curve defined on any fixed interval $[y_1, y_2]$ so that it has the Bishop characterization same as in the equation (2). For a vector field V, which is an infinitesimal variation of the curve α, by using Eqs. (7), (8) we get

$$0 = \langle \alpha'', V' \rangle \big|_{y_1}^{y_2} - \langle V, \alpha''' \rangle \big|_{y_1}^{y_2} + \int_{y_1}^{y_2} \langle V, \mathcal{E}(\alpha) \rangle \, dt,$$

where $\mathcal{E}(\alpha) = \alpha''''$. As opposed to (9), if we assume that the curve is nonelastic, then we have

$$\mathcal{E}(\alpha) = (3\pi_2\pi_2' + \pi_2'' + \pi_2^3)\mathbf{T} + (2\pi_1\pi_2' + \pi_1\pi_2^2 + \pi_1'\pi_2 + \pi_1'')\mathbf{N}_1 + 3\pi_1\pi_1'\mathbf{N}_2.$$

Theorem 4.1. *Energy of lightlike nonelastic Cartan curve having the Bishop characterization same as in equation* (2) *can be computed by using the Sasaki metric as follows*:

$$\varepsilon nergy_1\big(\mathcal{E}(\alpha)\big) = \frac{1}{2}\int_0^s ((3\pi_1'\pi_2' + 3\pi_1\pi_2'' + 3\pi_1\pi_2\pi_2' + \pi_1'\pi_2^2 +$$
$$+ 2\pi_1\pi_2' + \pi_1''\pi_2 + \pi_1''' + \pi_1\pi_2^3)^2 - (3\pi_2'^2 + 4\pi_2\pi_2'' +$$
$$+ 6\pi_2'\pi_2^2 + \pi_2''' + \pi_2^4)(3\pi_1'^2 + 4\pi_1\pi_1'' + 2\pi_1^2\pi_2' + \pi_1^2\pi_2^2 - 2\pi_1\pi_1'\pi_2))ds.$$

Proof. It is obvious if we apply the Sasaki metric given by Eqs. (4)–(6) to the vector field of $\mathcal{E}(\alpha)$.

***Corollary* 4.1.** *As stated in the Eq.* (2), *we have the following case for the energy of lightlike nonelastic Cartan curve $\alpha \in \mathbb{E}_1^4$ depending on Bishop and Frenet curvatures*:

$$\varepsilon nergy_1\big(\mathcal{E}(\alpha)\big) = \frac{1}{2}\int_0^s \left(\left(-3\left(\frac{1}{2}\pi_2^2 + k_2\right)' - 3\pi_2\left(\frac{1}{2}\pi_2^2 + k_2\right) - \right.\right.$$
$$\left.-2\left(\frac{1}{2}\pi_2^2 + k_2\right) + \pi_2^3\right)^2 - \left(-4\pi_2\left(\frac{1}{2}\pi_2^2 + k_2\right)' - 6\left(\frac{1}{2}\pi_2^2 + k_2\right)\pi_2^2 - \right.$$
$$\left.\left. - \left(\frac{1}{2}\pi_2^2 + k_2\right)'' + \pi_2^4\right)\left(-2\left(\frac{1}{2}\pi_2^2 + k_2\right) + \pi_2^2\right)\right) ds.$$

***Case* 2:** Let $\alpha \in \mathbb{E}_1^4$ be a lightlike Cartan curve defined on any fixed interval $[y_1, y_2]$ so that it has the Bishop characterization same as in equation (3). For a vector field V, which is an infinitesimal variation of the curve α, by using Eqs. (7), (8), we get

$$0 = \langle \alpha'', V' \rangle + \langle V, \Gamma\alpha' - \alpha'' \rangle + \int_{y_1}^{y_2} \langle V, \mathcal{E}(\alpha) \rangle \, dt,$$

where $\mathcal{E}(\alpha) = \alpha''''$. As opposed to (9), if we assume that the curve is not elastica then we have

$$\mathcal{E}(\alpha) = (\pi_2' + \pi_2^2)\mathbf{T} + (\pi_1' + \pi_1\pi_2)\mathbf{N}_1 + (\pi_1^2 + \pi_2^2)\mathbf{N}_2 - (\pi_3' + \pi_2\pi_3)\mathbf{N}_3.$$

Theorem 4.2. *Energy of lightlike nonelastic Cartan curve having the Bishop characterization same as in equation* (3) *can be computed by using the Sasaki metric as follows*:

$$\varepsilon nergy_2(\mathcal{E}(\alpha)) = \frac{1}{2} \int_0^s \Big((\pi_1'' + \pi_1'\pi_2 + \pi_1\pi_2' + \pi_1\pi_2^2)^2 +$$

$$+ (\pi_3'' + 2\pi_2'\pi_3 + \pi_2\pi_3' + \pi_2^2\pi_3)^2 -$$

$$- (\pi_2'' + 3\pi_2\pi_2' + \pi_2^3)(3\pi_1\pi_1' + 2\pi_2\pi_2' - \pi_2^3 + \pi_3\pi_3' - \pi_2\pi_3^2) \Big) ds.$$

Proof. It is obvious, if we apply the Sasaki metric given by Eqs. (4) – (6) to the vector field of $\mathcal{E}(\alpha)$.

***Corollary* 4.2.** *As stated in the Eq.* (3), *we have the following case for the energy of lightlike nonelastic Cartan curve* $\alpha \in \mathbb{E}_1^4$ *depending on Bishop and Frenet curvatures*:

$$\varepsilon nergy_2(\mathcal{E}(\alpha)) = \frac{1}{2} \int_0^s \Bigg(\Bigg(- \sin\theta + \cos\theta \left(\frac{k_3 + \theta''}{\theta'}\right) - \sin\theta \left(\frac{k_3 + \theta''}{\theta'}\right)' +$$

$$+ \sin\theta \left(\frac{k_3 + \theta''}{\theta'}\right)^2 \Bigg)^2 + \Bigg(- \cos\theta - 2 \left(\frac{k_3 + \theta''}{\theta'}\right)' \cos\theta +$$

$$+ \sin\theta \left(\frac{k_3 + \theta''}{\theta'}\right) + \cos\theta \left(\frac{k_3 + \theta''}{\theta'}\right)^2 \Bigg)^2 -$$

$$- \Bigg(\left(\frac{-k_3 - \theta''}{\theta'}\right)'' + 3 \left(\frac{k_3 + \theta''}{\theta'}\right) \left(\frac{k_3 + \theta''}{\theta'}\right)' -$$

$$- \left(\frac{k_3 + \theta''}{\theta'}\right)^3 \Bigg) \Bigg(3\sin\theta\cos\theta + 2 \left(\frac{k_3 + \theta''}{\theta'}\right) \left(\frac{k_3 + \theta''}{\theta'}\right)' -$$

$$- \left(\frac{k_3 + \theta''}{\theta'}\right)^3 - \sin\theta\cos\theta + \left(\frac{k_3 + \theta''}{\theta'}\right) \cos^2\theta \Bigg) \Bigg) ds,$$

where $2\theta'(\theta''' - k_3') + 2\theta''(k_3 - \theta'') + \theta'^4 - (k_3 - \theta'')^2 - 2k_2\theta'^2 = 0$ *and* $\theta' \neq 0$.

5. Application of the argument. In this section, we examine some special cases of lightlike Cartan curves in the case of being elastic or nonelastic.

***Example* 5.1.** Let α be a lightlike Cartan curve in $\mathbb{E}_1^4$ and $k_1 = 1,\ k_2 = k_3 = 0$. Then modified Bishop frame equations for orthonormal vectors $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$ having the form in the Eq. (2) can be stated as the following [12]:

$$\nabla_{\mathbf{T}}\mathbf{T} = \pi_2\mathbf{T} + \pi_1\mathbf{N}_1,$$

$$\nabla_{\mathbf{T}}\mathbf{N}_1 = \pi_1\mathbf{N}_2,$$

$$\nabla_{\mathbf{T}}\mathbf{N}_2 = -\pi_2\mathbf{N}_2,$$

$$\nabla_{\mathbf{T}}\mathbf{N}_3 = 0,$$

where $\pi_1 = 1,\ \pi_2 = \dfrac{2}{s},\ \pi_3 = 0.$

If α is a lightlike elastic Cartan curve, then

$$\varepsilon nergy_1(\mathcal{J}) = 0.$$

If α is a lightlike nonelastic Cartan curve, then

$$\varepsilon nergy_1\big(\mathcal{E}(\alpha)\big) = \frac{1}{2}\int\left(\frac{12}{s^3} - \frac{12}{s^3} - \frac{4}{s^2} + \frac{8}{s^3}\right)^2 - \left(\frac{32}{s^4} - \frac{48}{s^4} - \frac{12}{s^4} + \frac{16}{s^4}\right)\left(-\frac{4}{s^4} + \frac{4}{s^2}\right)ds,$$

which is equal to

$$\varepsilon nergy_1\big(\mathcal{E}(\alpha)\big) = 16\left(\frac{3}{7s^7} - \frac{7}{5s^5} + \frac{1}{s^4} - \frac{1}{3s^3}\right) + \mathcal{C},$$

where $\mathcal{C}$ is a constant term and $s \neq 0$.

***Example* 5.2.** Let α be a lightlike Cartan curve in $\mathbb{E}_1^4$ and $k_1 = 1,\ k_2 = k_3 = 0$. Then modified Bishop frame equations for orthonormal vectors $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$ having the form in the Eq. (3) can be stated as the following [12]:

$$\nabla_{\mathbf{T}}\mathbf{T} = \pi_1\mathbf{N}_1,$$

$$\nabla_{\mathbf{T}}\mathbf{N}_1 = \pi_1\mathbf{N}_2,$$

$$\nabla_{\mathbf{T}}\mathbf{N}_2 = 0,$$

$$\nabla_{\mathbf{T}}\mathbf{N}_3 = 0,$$

where $\pi_1 = 1,\ \pi_2 = 0,\ \pi_3 = 0.$

If α is a lightlike elastic Cartan curve, then

$$\varepsilon nergy_1(\mathcal{J}) = 0.$$

If α is a lightlike nonelastic Cartan curve, then

$$\varepsilon nergy_2(\mathcal{E}(\alpha)) = 0.$$

References

1. A. Einstein, *Zur Elektrodynamik bewegter Körper*, Ann. Phys., **17**, 891 – 921 (1905).
2. A. Einstein, *Relativity. The special and general theory*, Henry Holt, New York (1920).
3. A. Altin, *On the energy and pseudoangle of Frenet vector fields in* R_v^n, Ukr. Math. J., **63**, № 6, 969 – 975 (2011).
4. T. Körpınar, *New characterization for minimizing energy of biharmonic particles in Heisenberg spacetime*, Int. J. Phys., **53**, 3208 – 3218 (2014).
5. T. Körpınar, R. C. Demirkol, *A new geometric model of the energy functional of lightlike elastic curves in Minkowski* 4-*space*, J. Adv. Phys., **7**, № 3, 376 – 381 (2018).
6. L. G. Hughston, W. T. Shaw, *Classical strings in ten dimensions*, Proc. Roy. Soc. London. Ser. A, **414**, 423 – 431 (1987).
7. L. G. Hughston, W. T. Shaw, *Constraint-free analysis of relativistic strings*, Classical Quantum Gravity, **5**, 69 – 72 (1988).
8. L. G. Hughston, W. T. Shaw, *Spinor parametrizations of minimal surfaces*, The Mathematics of Surfaces, III, Oxford Univ. Press, New York (1989).
9. W. T. Shaw, *Twistors and strings*, Mathematics and General Relativity, Amer. Math. Soc., 337 – 363 (1988).
10. H. Urbantke, *On Pinl's representation of lightlike curves in* n *dimensions*, Relativity Today, World Sci. Publ., Teaneck, New York (1988).
11. A. E. H. Love, *A treatise on the mathematical theory of elasticity*, Cambribge Univ. Press (2013).
12. İlarslan K., A. Uçum, E. Nesovic, *On generalized spacelike Manheim curves in Minkowski space-time*, Proc. Nat. Acad. Sci., Sect. A, Phys. Sci., **86**, № 2, 249 – 258 (2016).
13. P. M. Chacon, A. M. Naveira, *Corrected energy of distribution on Riemannian manifolds*, Osaka J. Math., **41**, 97 – 105 (2004).
14. E. Bretin, J.-O. Lachaud, E. Oudet, *Regularization of discrete contour by Willmore energy*, J. Math. Imaging and Vision, **40**, № 2, 214 – 229 (2011).
15. T. Schoenemann, F. Kahl, S. Masnou, D. Cremers, *A linear framework for region-based image segmentation and inpainting involving curvature penalization*, Int. J. Comput. Vision, **99**, № 1, 53 – 68 (2012).
16. D. Mumford, *Elastica and computer vision*, Algebraic Geometry and its Applications, Springer-Verlag, New York (1994).
17. G. Citti, A. Sarti, *Cortical based model of perceptual completion in the roto-translation space*, J. Math. Imaging and Vision, **24**, № 3, 307 – 326 (2006).
18. J. Guven, D. M. Valencia, J. Vazquez-Montejo, *Environmental bias and elastic curves on surfaces*, J. Phys. A, **47** (2014).
19. L. Euler, *Additamentum 'de curvis elasticis'*, Methodus Inveniendi Lineas Curvas Maximi Minimive Probprietate Gaudentes, Lausanne (1744).
20. D. A. Singer, *Lectures on elastic curves and rods*, Dept. Math. Case Western Reserve Univ. (2007).

Received 01.09.17