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THE n-VALENT CONVEXITY OF FRASIN INTEGRAL OPERATORS

n-ВАЛЕНТНА ОПУКЛІСТЬ ІНТЕГРАЛЬНИХ ОПЕРАТОРІВ ФРАЗІНА

Let $f_i,\ i \in \{1, 2, \ldots, k\}$, is an analytic function on the unit disk in the complex plane of the form $f_i(z) = z^n + a_{i,n+1}z^{n+1} + \ldots, n \in \mathbb{N} = \{1, 2, \ldots\}$. We consider the Frasin integral operator as follows:

$$G_n(z) = \int_0^z n\xi^{(n-1)}\left(\frac{f_1'(\xi)}{n\xi^{n-1}}\right)^{\alpha_1}\cdots\left(\frac{f_k'(\xi)}{n\xi^{n-1}}\right)^{\alpha_k}d\xi.$$

In this paper, we obtain a sufficient condition under which this integral operator is n-valent convex and get other interesting results.

Нехай $f_i,\ i \in \{1, 2, \ldots, k\}$, — аналітична функція на одиничному диску у комплексній площині, яка має вигляд $f_i(z) = z^n + a_{i,n+1}z^{n+1} + \ldots,\ n \in \mathbb{N} = \{1, 2, \ldots\}$. Розглядається інтегральний оператор Фразіна вигляду

$$G_n(z) = \int_0^z n\xi^{(n-1)}\left(\frac{f_1'(\xi)}{n\xi^{n-1}}\right)^{\alpha_1}\cdots\left(\frac{f_k'(\xi)}{n\xi^{n-1}}\right)^{\alpha_k}d\xi.$$

Отримано достатні умови, за яких цей інтегральний оператор є n-валентно опуклим, та інші цікаві результати.

1. Introduction. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and let $\mathcal{A}(n)$ be the class of all analytic functions in $\mathbb{D}$ of the form

$$f(z) = z^n + a_{n+1}z^{n+1} + \ldots, \quad n \in \mathbb{N}.$$

So $\mathcal{A} := \mathcal{A}(1)$.

For $f, g \in \mathcal{A}$, we say that the function $f(z)$ is subordinate to $g(z)$, written by $f(z) \prec g(z)$, if exists an analytic function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{D}$ such that $f(z) = g\big(w(z)\big)$. If $g(z)$ is univalent in $\mathbb{D}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

A function $f \in \mathcal{A}(n)$ is said to be n-valent starlike functions of order β in $\mathbb{D}$, if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \qquad z \in \mathbb{D}, \quad 0 \le \beta < n, \quad n \in \mathbb{N},$$

and we denote this class by $S_n^*(\beta)$. If a function $f \in \mathcal{A}(n)$ satisfies the following inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \qquad z \in \mathbb{D}, \quad 0 \le \beta < n, \quad n \in \mathbb{N},$$

then it is said to be n-valent convex functions of order β in $\mathbb{D}$ and we denote this class by $C_n(\beta)$. It is known that $S_1^*(\beta) = S^*(\beta)$ and $C_1(\beta) = C(\beta)$ (the class of starlike functions of order β and convex functions of order β, respectively). These classes are subclasses of the class of univalent functions and, moreover, $C \subseteq S^*$ (see [3]), where $C = C(0)$ and $S^* = S^*(0)$ (the class of convex functions and starlike functions, respectively).

For $f_i \in \mathcal{A}$ and $\alpha_i > 0,\ i \in \{1, 2, \ldots, k\}$, Breaz et al. in [1] introduced the following integral operator:

$$F_{\alpha_1,\ldots,\alpha_k}(z) = \int\limits_0^z \left(f_1'(\xi)\right)^{\alpha_1} \ldots \left(f_k'(\xi)\right)^{\alpha_k} d\xi. \tag{1.1}$$

The most recent, Frasin [4] introduced the following integral operator, for $\alpha_i > 0$ and $f_i \in \mathcal{A}_n$, $i \in \{1, 2, \ldots, k\}$:

$$G_n(z) = \int\limits_0^z n\xi^{(n-1)} \left(\frac{f_1'(\xi)}{n\xi^{n-1}}\right)^{\alpha_1} \ldots \left(\frac{f_k'(\xi)}{n\xi^{n-1}}\right)^{\alpha_k} d\xi. \tag{1.2}$$

2. Preliminaries. In order to give our results, we need the following corollary, which is due to E. Deniz [2].

***Corollary* 2.1.** *Let the function* $f(z) \in \mathcal{A}(n)$ *satisfies the inequality*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{2n-1}{2}.$$

Then

$$\Re\left(\frac{f'(z)}{z^{n-1}}\right) > \frac{n}{2}.$$

3. Main results. In this section, we formulate and prove main results.

Theorem 3.1. *Let* $\alpha_i > 0$ *for all* $i \in \{1, 2, \ldots, k\}$ *and* $f_i \in \mathcal{A}(n)$ *such that*

$$\Re\left[\frac{zf_i''(z)}{f_i'(z)} + 1\right] \geq \beta_i, \quad z \in \mathbb{D},$$

where $\beta_i \in \mathbb{R}$ *and* $\sum_{i=1}^k \beta_i\alpha_i < n$. *If* $\sum_{i=1}^k \alpha_i \leq 1$, *then* G_n *is* n*-valent convex function of order* $\sum_{i=1}^k \beta_i\alpha_i$. *Here* G_n *is the integral operator define as in* (1.2).

Proof. From (1.2), we observe that $G_n \in \mathcal{A}(n)$ and obtain

$$G_n'(z) = nz^{n-1}\left(\frac{f_1'(z)}{nz^{n-1}}\right)^{\alpha_1} \ldots \left(\frac{f_k'(z)}{nz^{n-1}}\right)^{\alpha_k}.$$

Differentiating the above expression logarithmically and multiply by z we get

$$\frac{zG_n''(z)}{G_n'(z)} = (n-1) + \sum_{i=1}^k \alpha_i \left[\frac{zf_i''(z)}{f_i'(z)} - (n-1)\right].$$

So, we have

$$\frac{zG_n''(z)}{G_n'(z)} + 1 = n + \sum_{i=1}^k \alpha_i \left[\frac{zf_i''(z)}{f_i'(z)} + 1 - n\right] =$$

$$= n + \sum_{i=1}^{k} \alpha_i \left[\frac{z f_i''(z)}{f_i'(z)} + 1 \right] - n \sum_{i=1}^{k} \alpha_i. \tag{3.1}$$

Since $\sum_{i=1}^{k} \alpha_i \le 1$, then by hypothesis we have

$$\Re \left[\frac{z G_n''(z)}{G_n'(z)} + 1 \right] \ge \sum_{i=1}^{k} \alpha_i \Re \left[\frac{z f_i''(z)}{f_i'(z)} + 1 \right] \ge \sum_{i=1}^{k} \beta_i \alpha_i. \tag{3.2}$$

Thus, $G_n(z)$ is n-valent convex of order $\sum_{i=1}^{k} \beta_i \alpha_i$.

Theorem 3.1 is proved.

***Corollary* 3.1.** *Let $\alpha_i > 0$ for all $i \in \{1, 2, \ldots, k\}$ such that $\sum_{i=1}^{k} \alpha_i \le 1$. If $f_i \in C_n(\beta_i)$, then G_n is n-valent convex function of order $\sum_{i=1}^{k} \beta_i \alpha_i$.*

Proof. Since $f_i \in C_n(\beta_i)$, then $0 \le \beta_i < n$ and so $0 \le \sum_{i=1}^{k} \beta_i \alpha_i < n$. Therefore by using the relation (3.2), the proof of this theorem is obvious.

If we put $n = 1$ in Theorem 3.1, then we get the following corollary.

***Corollary* 3.2.** *Let $\alpha_i > 0$ for all $i \in \{1, 2, \ldots, k\}$ and $f_i \in \mathcal{A}$ such that*

$$\Re \left[\frac{z f_i''(z)}{f_i'(z)} + 1 \right] \ge \beta_i, \quad z \in \mathbb{D},$$

where $\beta_i \in \mathbb{R}$ and $\sum_{i=1}^{k} \beta_i \alpha_i < 1$. If $\sum_{i=1}^{k} \alpha_i \le 1$, then $F_{\alpha_1, \ldots, \alpha_k}$ is the convex function of order $\sum_{i=1}^{k} \beta_i \alpha_i$. Here $F_{\alpha_1, \ldots, \alpha_k}$ is the integral operator define as in (1.1).

***Corollary* 3.3.** *Let $\alpha_i > 0$ for all $i \in \{1, 2, \ldots, k\}$ such that $\sum_{i=1}^{k} \alpha_i \le 1$. If $f_i \in C(\beta_i)$, then $F_{\alpha_1, \ldots, \alpha_k}$ is convex function of order $\sum_{i=1}^{k} \beta_i \alpha_i$.*

Theorem 3.2. *Let f_i be in the class $\mathcal{S}$. If $r > 0$ satisfies the inequality*

$$\frac{r^2 - 4r + 1}{1 - r^2} \sum_{i=1}^{k} \alpha_i > 0$$

such that $\sum_{i=1}^{k} \alpha_i \le 1$, then $F_{\alpha_1, \ldots, \alpha_k}$ is convex univalent function in the disk $|z| < r$.

Proof. It is known that $f_i \in \mathcal{S}$, then for $z = re^{i\theta}$

$$\Re \left[\frac{z f_i''(z)}{f_i'(z)} + 1 \right] > \frac{r^2 - 4r + 1}{1 - r^2}.$$

Since $\sum_{i=1}^{k} \alpha_i \le 1$, then we get

$$1 + \sum_{i=1}^{k} \alpha_i \Re \left[\frac{z f_i''(z)}{f_i'(z)} \right] \ge \sum_{i=1}^{k} \alpha_i \Re \left[\frac{z f_i''(z)}{f_i'(z)} + 1 \right]. \tag{3.3}$$

If we put $n = 1$, in equation (3.1) and use of the hypothesis of this theorem and applying relation (3.2), then we get that the integral operator $F_{\alpha_1, \ldots, \alpha_k}$ is the convex function.

Theorem 3.2 is proved.

Theorem 3.3. *Let* $\alpha_i > 0$ *for all* $i \in \{1, 2, \ldots, k\}$. *Also, let* $f_i \in \mathcal{A}$ *such that*

$$\left[\frac{zf_i''(z)}{f_i'(z)} + 1\right] \geq \beta_i, \quad z \in \mathbb{D},$$

where $\beta_i \in \mathbb{R}$, $\sum_{i=1}^{k} \beta_i \alpha_i < 1$ *and* $\sum_{i=1}^{k} \alpha_i \leq 1$. *Assume that* $g(z) = a + b_n z^n + b_{n+1} z^{n+1} + \ldots$ *is analytic in* $\mathbb{D}$. *If*

$$g(z) + \frac{zg'(z)}{c} \prec F_{\alpha_1,\ldots,\alpha_k}(z), \quad z \in \mathbb{D}, \tag{3.4}$$

for $\Re(c) \geq 0$, $c \neq 0$, *then*

$$g(z) \prec q_n(z) \prec F_{\alpha_1,\ldots,\alpha_k}(z), \quad z \in \mathbb{D}, \tag{3.5}$$

where $q_n(z) = \dfrac{c}{nz^{c/n}} \displaystyle\int_0^z t^{c/n-1} F_{\alpha_1,\ldots,\alpha_k}(t)dt$. *Moreover, the function* $q_n(z)$ *is convex univalent and is the best dominant of* (3.4) *in the sense that* $g \prec q_n$ *for all* g *satisfying* (3.4) *and if there exists* q *such that* $g \prec q$ *for all* g *satisfying* (3.4), *then* $q_n \prec q$.

Proof. It is known [5] that the subordination (3.4) with convex univalent right-hand side is sufficient for (3.5) with the best dominated $q_n(z)$. By Theorem 3.1 the function $F_{\alpha_1,\ldots,\alpha_k}$ is convex univalent in the unit disk and we get the result.

Theorem 3.4. *Let* $f_i \in \mathcal{A}(n)$, $\alpha_i > 0$ *for all* $i \in \{1, 2, \ldots, k\}$ *and* $\sum_{i=1}^{k} \alpha_i \leq 1$. *If*

$$\sum_{i=1}^{k} \alpha_i \Re\left[\frac{zf_i''(z)}{f_i'(z)} + 1\right] \geq \frac{2n-1}{2}, \tag{3.6}$$

then

$$\Re\left[\prod_{i=1}^{k}\left(\frac{f_i'(z)}{nz^{n-1}}\right)^{\alpha_k}\right] > \frac{1}{2}.$$

Proof. Since $\sum_{i=1}^{k} \alpha_i \leq 1$, then by relation (3.1) we have

$$\Re\left[\frac{zG_n''(z)}{G_n'(z)} + 1\right] \geq \sum_{i=1}^{k} \alpha_i \Re\left[\frac{zf_i''(z)}{f_i'(z)} + 1\right].$$

We know that the integral operator $G_n(z) \in \mathcal{A}(n)$. So, by using Corollary 2.1 and applying equation (3.6), we get

$$\Re\left(\frac{G_n'(z)}{z^{n-1}}\right) > \frac{n}{2}.$$

Therefore,

$$\Re\left[\prod_{i=1}^{k}\left(\frac{f_i'(z)}{nz^{n-1}}\right)^{\alpha_k}\right] > \frac{1}{2}.$$

Theorem 3.4 is proved.

We put $n = 1$ in Theorem 3.4, then we get the following corollary.

***Corollary* 3.4.** *Let* $f_i \in \mathcal{A}$, $\alpha_i > 0$ *for all* $i \in \{1, 2, \ldots, k\}$ *and* $\sum_{i=1}^{k} \alpha_i \leq 1$. *If*

$$\sum_{i=1}^{k} \alpha_i \Re \left[\frac{z f_i''(z)}{f_i'(z)} + 1 \right] \geq \frac{1}{2},$$

then

$$\Re \left[\prod_{i=1}^{k} \left(f_i'(z) \right)^{\alpha_k} \right] > \frac{1}{2}.$$

If we take $k = 1$ in Corollary 3.4, then we obtain the following result.

***Corollary* 3.5.** *If* $f \in C(1/2)$ *and* $0 < \alpha \leq 1$, *then* $\Re \left(f'(z) \right)^{\alpha} > \dfrac{1}{2}$.

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