DOI: 10.37863/umzh.v72i12.881

UDC 517.9

M. R. H. Tavani (Dep. Math., Ramhormoz Branch, Islamic Azad Univ., Ramhormoz, Iran)

EXISTENCE OF THREE WEAK SOLUTIONS FOR FOURTH-ORDER ELASTIC BEAM EQUATIONS ON THE WHOLE SPACE

ІСНУВАННЯ ТРЬОХ СЛАБКИХ РОЗВ'ЯЗКІВ РІВНЯНЬ ПРУЖНОЇ БАЛКИ ЧЕТВЕРТОГО ПОРЯДКУ В УСЬОМУ ПРОСТОРІ

Multiplicity results for a perturbed fourth-order problem on the real line with a perturbed nonlinear term depending on one real parameter is investigated. Our approach is based on variational methods and critical point theory which are obtained in [G. Bonanno, A critical point theorem via the Ekeland variational principle, Nonlinear Anal., **75**, 2992 – 3007 (2012)].

Вивчено результати кратності для збуреної задачі четвертого порядку на дійсній прямій із збуреним нелінійним доданком, що залежить від одного дійсного параметра. Підхід базується на методах варіацій та теорії критичних точок, що отримані в [G. Bonanno, A critical point theorem via the Ekeland variational principle, Nonlinear Anal., **75**, 2992 – 3007 (2012)].

1. Introduction. In this paper we consider the following problem:

$$u^{iv}(x) + Au''(x) + Bu(x) = \lambda\alpha(x).f(u(x)) \quad \text{a.e.} \quad x \in \mathbb{R}, \qquad (P_\lambda)$$

where A is a real negative constant and B is a real positive constant, λ is a positive parameter and $\alpha, f\colon \ \mathbb{R} \to \mathbb{R}$ are two functions such that $\alpha \in L^1(\mathbb{R})$, $\alpha(x) \geq 0$, for a.e. $x \in \mathbb{R}$, $\alpha \not\equiv 0$ and also f is continuous and nonnegative. It is known that fourth-order problems are important in describing a large class of elastic deflections. Hence, many researchers have studied the existence and multiplicity of solutions for fourth-order two-point boundary-value problems. We refer the reader to [4 – 6, 8 – 10]. In [4], while A and B are real constants, using variational methods and critical point theory, multiplicity results for the fourth-order elliptic problem,

$$\begin{gathered} u^{iv} + Au'' + Bu = \lambda f(t,u), \quad t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{gathered} \qquad (1.1)$$

by condition on the nonlinear term was established, while in [8], applying the Morse theory, the existence of three solutions to problem (1.1), with $A = B = 0$, were discussed. Problems such as (P_λ) that are discussed on the whole space, occur naturally in a variety of settings in physics and material scinces, as in, for example, the study of mathematical models of deflection of beams. These beams which appear in many structures, deflect under their own weight or under the influence of some external forces.

Due to the lack of compactness of the operators on whole space, the study of such problems is very important. Because, in such cases the operators which solve the problem are not regular enough in comparison to operators which arise in problems on bounded domains. Due to this, for example, we can not aplly [3] (Corollary 3.1) for the problem (P_λ). In the present paper, using one kind of critical point theorem obtained in [1] which we recall in the next section (Theorem 2.1), we establish the existence of at least three nonnegative weak solutions for the problem (P_λ). In fact, in presenting

Theorem 3.1, which one of the main results of this paper, we aplly the requirement (non-standard Palais – Smale condition for functional I_λ which is the functional related to the problem (P_λ)) based on Theorem 2.1.

2. Preliminaries. Let us recall some basic concepts.
We denote by $|.|_t$ the usual norm on $L^t(\mathbb{R})$, for all $t \in [1, +\infty]$ and it is known that $W^{2,2}(\mathbb{R})$ is continuously embedded in $L^t(\mathbb{R})$ for each $t \in [2, +\infty]$.

The Sobolev space $W^{2,2}(\mathbb{R})$ is equipped with the following norm:

$$\|u\|_{W^{2,2}(\mathbb{R})} = \left(\int_{\mathbb{R}} (|u''(x)|^2 + |u'(x)|^2 + |u(x)|^2) dx \right)^{1/2}$$

for all $u \in W^{2,2}(\mathbb{R})$. Also, we consider $W^{2,2}(\mathbb{R})$ with the norm

$$\|u\| = \left(\int_{\mathbb{R}} (|u''(x)|^2 - A|u'(x)|^2 + B|u(x)|^2) dx \right)^{1/2}$$

for all $u \in W^{2,2}(\mathbb{R})$. According to

$$(\min\{1, -A, B\})^{\frac{1}{2}} \|u\|_{W^{2,2}(\mathbb{R})} \leq \|u\| \leq (\max\{1, -A, B\})^{\frac{1}{2}} \|u\|_{W^{2,2}(\mathbb{R})},$$

the norm $\|.\|$ is equvalent to the norm $\|.\|_{W^{2,2}(\mathbb{R})}$. Since embedding $W^{2,2}(\mathbb{R}) \to L^\infty(\mathbb{R})$ is continuous hence there exists a constant $C_{A,B}$ (depending on A and B) such that

$$|u|_\infty \leq C_{A,B} \|u\| \quad \forall u \in W^{2,2}(\mathbb{R}).$$

In the following proposition, we provide an approximation for this constant.

Proposition 2.1. *We have*

$$|u|_\infty \leq C_{A,B} \|u\| \tag{2.1}$$

where $C_{A,B} = \left(\dfrac{-1}{4AB} \right)^{\frac{1}{4}}$.

Proof. Let $v \in W^{1,1}(\mathbb{R})$, then from [7, p. 138] (formula (4.64)), one has

$$|v(x)| \leq \frac{1}{2} \int_{\mathbb{R}} |v'(t)| dt. \tag{2.2}$$

Now if $u \in W^{2,2}(\mathbb{R})$, then $v(x) = (-AB)^{\frac{1}{2}} |u(x)|^2 \in W^{1,1}(\mathbb{R})$ and, thus, from (2.2) and Hölder's inequality one has

$$(-AB)^{\frac{1}{2}} |u(x)|^2 \leq \int_{\mathbb{R}} (-AB)^{\frac{1}{2}} |u'(t)||u(t)| dt \leq ((-A)^{\frac{1}{2}} |u'|_2)(B^{\frac{1}{2}} |u|_2),$$

that is,

$$|u(x)| \leq \left(\frac{-1}{AB}\right)^{\frac{1}{4}} ((-A)^{\frac{1}{2}}|u'|_2)^{\frac{1}{2}}(B^{\frac{1}{2}}|u|_2)^{\frac{1}{2}}. \tag{2.3}$$

Now according to $x^a y^{1-a} \leq a^a(1-a)^{1-a}(x+y),\ x,\ y \geq 0,\ 0 < a < 1$ [7, p. 130] (formula (4.47)) and classical inequality $a^{\frac{1}{p}} + b^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}}(a+b)^{\frac{1}{p}}$, from (2.3) one has

$$|u(x)| \leq \left(\frac{-1}{AB}\right)^{\frac{1}{4}} \left(\frac{1}{2}\right)^{\frac{1}{2}} \left[\left(\int_{\mathbb{R}} -A|u'(t)|^2 dt\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}} B|u(t)|^2 dt\right)^{\frac{1}{2}}\right] \leq$$

$$\leq \left(\frac{-1}{AB}\right)^{\frac{1}{4}} \left(\frac{1}{2}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right)^{\frac{1}{2}} (2)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (-A|u'(t)|^2 + B|u(t)|^2) dt\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\frac{-1}{4AB}\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} (|u''(t)|^2 - A|u'(t)|^2 + B|u(t)|^2) dt\right)^{\frac{1}{2}},$$

which means that $|u|_\infty \leq C_{A,B}\|u\|$.

Let $\Phi, \Psi: \ W^{2,2}(\mathbb{R}) \to \mathbb{R}$ be defined by

$$(u) = \frac{1}{2}\|u\|^2 = \frac{1}{2}\int_{\mathbb{R}} (|u''(x)|^2 - A|u'(x)|^2 + B|u(x)|^2) dx \tag{2.4}$$

and

$$\Psi(u) = \int_{\mathbb{R}} \alpha(x) F(u(x)) dx \tag{2.5}$$

for every $u \in W^{2,2}(\mathbb{R})$, where $F(t) = \int_0^t f(\xi) d\xi$ for all $t \in \mathbb{R}$. Since $F'(t) = f(t) \geq 0$ for all $t \in \mathbb{R}$ so F is an increasing function. The functional Ψ is well defined because for every $u \in W^{2,2}(\mathbb{R})$ we have

$$|\Psi(u)| \leq \int_{\mathbb{R}} \alpha(x) \max\{-F(-|u|_\infty), F(|u|_\infty)\} dx < +\infty.$$

It is known that Ψ is a differentiable functional whose differential at the point $u \in W^{2,2}(\mathbb{R})$ is

$$\Psi'(u)(v) = \lim_{s \to 0} \frac{\Psi(u+sv) - \Psi(u)}{s} = \frac{d}{ds}\Psi(u+sv)\Big|_{s=0} = \int_{\mathbb{R}} \alpha(x) f(u(x)) v(x) dx,$$

and Φ is a continuously Gâteaux differentiable functional and in a similar way, whose differential at the point $u \in W^{2,2}(\mathbb{R})$ is

$$\Phi'(u)(v) = \int_{\mathbb{R}} (u''(x)v''(x) - Au'(x)v'(x) + Bu(x)v(x)) dx$$

for every $v \in W^{2,2}(\mathbb{R})$.

Definition 2.1. *Let Φ and Ψ be defined as above. Put $I_\lambda = \Phi - \lambda\Psi$, $\lambda > 0$. We say that $u \in W^{2,2}(\mathbb{R})$ is a critical point of I_λ when $I'_\lambda(u) = 0_{\{W^{2,2}(\mathbb{R})^*\}}$, that is, $I'_\lambda(u)(v) = 0$ for all $v \in W^{2,2}(\mathbb{R})$.*

Definition 2.2. *A function $u\colon \mathbb{R} \to \mathbb{R}$ is a weak solution to the problem (P_λ) if $u \in W^{2,2}(\mathbb{R})$ and*

$$\int_{\mathbb{R}} (u''(x)v''(x) - Au'(x)v'(x) + Bu(x)v(x) - \lambda\alpha(x)f(u(x))v(x))dx = 0$$

for all $v \in W^{2,2}(\mathbb{R})$.

***Remark* 2.1.** We clearly observe that the weak solutions of the problem (P_λ) are exactly the solutions of the equation $I'_\lambda(u)(v) = \Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$. Also if α is, in addition, a continuous function on $\mathbb{R}$ then each weak solution of (P_λ) is a classical solution.

Lemma 2.1. *If $u_0 \not\equiv 0$ is a weak solution for problem (P_λ), then u_0 is nonnegative.*

Proof. From Remark 2.1 one has, $I'_\lambda(u_0)(v) = 0$ for all $v \in W^{2,2}(\mathbb{R})$. Choose $v(x) = \bar{u}_0 = \max\{-u_0(x), 0\}$ and let $M = \{x \in \mathbb{R}\colon\ u_0(x) < 0\}$. Then we have

$$\int_{M\cup M^c} (u_0''(x)\bar{u}_0''(x) - Au_0'(x)\bar{u}_0'(x) + Bu_0(x)\bar{u}_0(x))dx = \int_{M\cup M^c} \lambda\alpha(x)f(u_0(x))\bar{u}_0(x)dx,$$

that is,

$$-\int_M (|\bar{u}_0''(x)|^2 - A|\bar{u}_0'(x)|^2 + B|\bar{u}_0(x)|^2)dx = \int_M \lambda\alpha(x)f(u_0(x))\bar{u}_0(x)dx \geq 0$$

which means that $-\|\bar{u}_0\|^2 \geq 0$ and one has $\bar{u}_0 = 0$. Hence $-u_0 \leq 0$, that is, $u_0 \geq 0$ and the proof is complete.

Definition 2.3. *A Gâtuax differentiable function I from Banach space X to $\mathbb{R}$ satisfies the Palais – Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ such that*

(a) $\{I(u_n)\}$ *is bounded,*

(b) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0 \quad \forall n \in \mathbb{N}$,

has a convergent subsequence.

Below, we will present a non-standard state of the Palais – Smale condition that is introduced in [1].

Definition 2.4 (see [1])**.** *Fix $r \in]-\infty, +\infty]$. A Gâtuax differentiable function I from Banach space X to $\mathbb{R}$ satisfies the Palais – Smale condition cut off upper at r (in short $(PS)^{[r]}$-condition) if any sequence $\{u_n\}$ such that*

(a) $\{I(u_n)\}$ *is bounded,*

(b) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,

(c) $\Phi(u_n) < r \quad \forall n \in \mathbb{N}$,

has a convergent subsequence.

Our main tool is the following critical point theorem.

Theorem 2.1 ([1], Theorem 7.3)**.** *Let X be a real Banach space, and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below and convex such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{u} \in X$, with $2r_1 < \Phi(\overline{u}) < \dfrac{r_2}{2}$, such that

(b$_1$) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\overline{u})}{\Phi(\overline{u})}$;

(b$_2$) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\overline{u})}{\Phi(\overline{u})}$.

Assume also that, for each

$$\lambda \in \Lambda = \left] \frac{3}{2}\frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \ \min\left\{ \frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}, \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)} \right\} \right[,$$

the functional $\Phi - \lambda\Psi$ satisfies the $(PS)^{[r_2]}$-condition and

$$\inf_{t\in[0,1]} \Psi(tu_1 + (1-t)u_2) \geq 0$$

for each $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$.

Then, for each $\lambda \in \Lambda$, the functional $\Phi - \lambda\Psi$ admits at least three critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

Now we present one proposition that will be needed to prove the main theorem of this paper.

Proposition 2.2. *Take Φ and Ψ as in the Definition 2.1 and fix $\lambda > 0$. Then $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for any $r > 0$.*

Proof. Consider sequence $\{u_n\} \subseteq W^{2,2}(\mathbb{R})$ such that $\{I_\lambda(u_n)\}$ is bounded, $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{W^{2,2}(\mathbb{R})^*} = 0$ and$\Phi(u_n) < r$ $\forall n \in \mathbb{N}$. Since $\Phi(u_n) < r$, we have $\dfrac{1}{2}\|u_n\|^2 < r$ and so $\{u_n\}$ is bounded in $W^{2,2}(\mathbb{R})$. Therefore passing to a subsequence if necessary we can assume that $u_n(x) \to u(x)$, $x \in \mathbb{R}$ (from the compact embedding $W^{2,2}(\mathbb{R}) \to C([-T,T]), T > 0$) and $\{u_n\}$ weakly converges to u in $L^\infty(\mathbb{R})$ (from the continuous embedding $W^{2,2}(\mathbb{R}) \to L^\infty(\mathbb{R})$) and, hence, there is $s > 0$ such that $|u_n(x)| \leq s$ for a.e. $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Here, it is useful to note that the subsequence $\{u_n\}$ converges weakly to u in $W^{2,2}(\mathbb{R})$ and we want to show that this subsequence is strongly converging to u in $W^{2,2}(\mathbb{R})$. For this purpose, according to Lebesque's dominated convergence theorem since $\alpha f(u_n(x)) \leq \alpha.\max_{|\xi|\leq s} f(\xi) \in L^1(\mathbb{R})$ for all $n \in \mathbb{N}$ and $f(u_n(x)) \to f(u(x))$ for a.e. $x \in \mathbb{R}$ (f is continuous function), one has $\alpha f(u_n)$ is strongly converging to $\alpha f(u)$ in $L^1(\mathbb{R})$. Now since $u_n \rightharpoonup u$ in $L^\infty(\mathbb{R})$ and $\alpha f(u_n) \to \alpha f(u)$ in $L^1(\mathbb{R}) \subseteq (L^\infty(\mathbb{R}))^*$ then from [2] (Proposition 3.5(iv)), one has

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \alpha(x) f(u_n(x))(u_n(x) - u(x))dx = 0. \tag{2.6}$$

From $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{W^{2,2}(\mathbb{R})^*} = 0$, there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$, such that

$$\left|\int\limits_{\mathbb{R}}(u_n''(x)v''(x) - Au_n'(x)v'(x) + Bu_n(x)v(x) - \lambda\alpha(x)f(u_n(x))v(x))dx\right| \le \varepsilon_n \qquad (2.7)$$

for all $n \in \mathbb{N}$ and for all $v \in W^{2,2}(\mathbb{R})$ with $\|v\| \le 1$. Putting $v(x) = \dfrac{u_n(x) - u(x)}{\|u_n - u\|}$, from (2.7) one has

$$\left|\int\limits_{\mathbb{R}}(u_n''(x)(u_n''(x) - u''(x)) - Au_n'(x)(u_n'(x) - u'(x)) + Bu_n(x)(u_n(x) - u(x)) - \right.$$

$$\left. -\lambda\alpha(x)f(u_n(x))(u_n(x) - u(x)))dx\right| \le \varepsilon_n\|u_n - u\| \qquad (2.8)$$

for all $n \in \mathbb{N}$. Now according to inequality $|a||b| \le \dfrac{1}{2}|a|^2 + \dfrac{1}{2}|b|^2$ we have

$$\int\limits_{\mathbb{R}}(u_n''(x)(u_n''(x) - u''(x)) - Au_n'(x)(u_n'(x) - u'(x)) + Bu_n(x)(u_n(x) - u(x)))dx =$$

$$= \int\limits_{\mathbb{R}}(|u_n''(x)|^2 - A|u_n'(x)|^2 + B|u_n(x)|^2)dx -$$

$$- \int\limits_{\mathbb{R}}(u_n''(x)u''(x) - Au_n'(x)u'(x) + Bu_n(x)u(x))dx \ge$$

$$\ge \|u_n\|^2 - \int\limits_{\mathbb{R}}\left(\frac{1}{2}|u_n''(x)|^2 + \frac{1}{2}|u''(x)|^2 - \frac{1}{2}A|u_n'(x)|^2 - \frac{1}{2}A|u'(x)|^2 + \frac{1}{2}B|u_n(x)|^2 + \right.$$

$$\left. + \frac{1}{2}B|u(x)|^2\right)dx = \|u_n\|^2 - \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2 = \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2.$$

Hence from (2.8), we obtain

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2 \le \lambda\int\limits_{\mathbb{R}}\alpha(x)f(u_n(x))(u_n(x) - u(x))dx + \varepsilon_n\|u_n - u\|,$$

that is,

$$\frac{1}{2}\|u_n\|^2 \le \frac{1}{2}\|u\|^2 + \lambda\int\limits_{\mathbb{R}}\alpha(x)f(u_n(x))(u_n(x) - u(x))dx + \varepsilon_n\|u_n - u\|. \qquad (2.9)$$

Taking into account (2.6), from (2.9) when $\varepsilon_n \to 0^+$, we have

$$\limsup_{n\to+\infty}\|u_n\| \le \|u\|.$$

Thus, [2] (Proposition 3.32) ensures that $u_n \to u$ in $W^{2,2}(\mathbb{R})$.

Proposition 2.2 is proved.

3. Main results. Before presenting the main theorems of this section, we introduce notations that are related to some constants that will appear in the main results of this section. Put

$$k = \left(\frac{2048}{27} - \frac{32}{9} A + \frac{13}{40} B \right)^{-1},$$

$$\alpha_0 = \int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x) dx,$$

$$E = \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x) dx}{\int_{\mathbb{R}} \alpha(x) dx} = \frac{\alpha_0}{|\alpha|_1}, \quad \text{and, hence,} \quad E \le 1,$$

$$h = C_{A,B} \left(\frac{2}{k} \right)^{\frac{1}{2}} \quad \text{and} \quad I = \frac{E}{h^2},$$

where $C_{A,B}$ is given in Proposition 2.1. Now we express the main results.

Theorem 3.1. *Assume that there exist three positive constants η, θ_1 and θ_2 with $2\theta_1 < \sqrt{2}\eta h < \theta_2$ such that*

(i) $\dfrac{F(\theta_1)}{{\theta_1}^2} < \dfrac{2}{3} I \dfrac{F(\eta)}{\eta^2}$,

(ii) $\dfrac{F(\theta_2)}{{\theta_2}^2} < \dfrac{1}{3} I \frac{F(\eta)}{\eta^2}$.

Then, for each

$$\lambda \in \Lambda' = \left] \frac{3}{4} \frac{1}{|\alpha|_1 C_{A,B}^2} \frac{1}{I} \frac{\eta^2}{F(\eta)}, \quad \min \left\{ \frac{1}{2|\alpha|_1 C_{A,B}^2} \frac{{\theta_1}^2}{F(\theta_1)}, \frac{1}{4} \frac{1}{|\alpha|_1 C_{A,B}^2} \frac{{\theta_2}^2}{F(\theta_2)} \right\} \right[,$$

the problem (P_λ) admits at least three distinct nonnegative weak solutions $u_i \in W^{2,2}(\mathbb{R})$ such that $|u_i|_\infty < \theta_2$, $i = 1, 2, 3$.

Proof. Our aim is to apply Theorem 2.1, to problem (P_λ). Fix λ, as in the conclusion. Take $X = W^{2,2}(\mathbb{R})$ and Φ and Ψ as in the previous section. We observe that the regularity assumptions of Theorem 2.1 on Φ and Ψ are satisfied and also according to Proposition 2.2, the functional I_λ satisfies the $(PS)^{[r]}$-condition for all $r > 0$.

Put

$$r_1 := \frac{1}{2} \left(\frac{\theta_1}{C_{A,B}} \right)^2, \qquad r_2 := \frac{1}{2} \left(\frac{\theta_2}{C_{A,B}} \right)^2$$

and

$$w(x) := \begin{cases} -\dfrac{64\eta}{9}\left(x^2 - \dfrac{3}{4}x\right), & \text{if} \quad x \in \left[0, \dfrac{3}{8}\right], \\ \eta, & \text{if} \quad x \in \left]\dfrac{3}{8}, \dfrac{5}{8}\right], \\ -\dfrac{64\eta}{9}\left(x^2 - \dfrac{5}{4}x + \dfrac{1}{4}\right), & \text{if} \quad x \in \left]\dfrac{5}{8}, 1\right], \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

We clearly observe that $w \in X$ and, in particular,

$$\Phi(w) = \frac{1}{2}\|w\|^2 = \frac{1}{2}\int\limits_{\mathbb{R}} (|w''(x)|^2 - A|w'(x)|^2 + B|w(x)|^2)dx =$$

$$= \eta^2\left(\frac{2048}{27} - \frac{32}{9}A + \frac{13}{40}B\right) = \frac{\eta^2}{k} = \frac{1}{2}\left(\frac{\eta h}{C_{A,B}}\right)^2.$$

Therefore, using the condition $2\theta_1 < \sqrt{2}\eta h < \theta_2$, one has, $2r_1 < \Phi(w) < \dfrac{r_2}{2}$.

Now for each $u \in X$ and bearing (2.1) in mind, we see that

$$\Phi^{-1}(]-\infty, r_i[) = \{u \in X; \Phi(u) < r_i\} =$$

$$= \left\{u \in X; \frac{1}{2}\|u\|^2 < \frac{1}{2}\left(\frac{\theta_i}{C_{A,B}}\right)^2\right\} =$$

$$= \{u \in X; C_{A,B}\|u\| < \theta_i\} \subseteq \{u \in X;\ |u|_\infty < \theta_i\},$$

and it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r_i[)} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r_i[)} \int\limits_{\mathbb{R}} \alpha(x) \quad F(u(x))dx \le$$

$$\le \int\limits_{\mathbb{R}} \alpha(x) \quad \sup_{|\xi|<\theta_i} F(\xi)dx = |\alpha|_1 \quad F(\theta_i).$$

Hence, we have

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} \le \frac{|\alpha|_1 F(\theta_1)}{\dfrac{1}{2}\left(\dfrac{\theta_1}{C_{A,B}}\right)^2} = 2|\alpha|_1 C_{A,B}^2 \frac{F(\theta_1)}{{\theta_1}^2} < \frac{1}{\lambda}. \tag{3.2}$$

On the other hand, one has

$$\frac{2}{3}\frac{\Psi(w)}{\Phi(w)} = \frac{2}{3}\frac{\int_{\mathbb{R}} \alpha(x)F(w(x))dx}{\dfrac{1}{2}\left(\dfrac{\eta h}{C_{A,B}}\right)^2} \ge \frac{2}{3}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)F(\eta)dx}{\dfrac{1}{2}\left(\dfrac{\eta h}{C_{A,B}}\right)^2} = \frac{2}{3}\frac{\alpha_0 F(\eta)}{\dfrac{1}{2}\left(\dfrac{\eta h}{C_{A,B}}\right)^2} =$$

$$= \frac{4}{3}|\alpha|_1 C_{A,B}^2 \frac{E}{h^2}\frac{F(\eta)}{\eta^2} = \frac{4}{3}|\alpha|_1 C_{A,B}^2 I \frac{F(\eta)}{\eta^2} > \frac{1}{\lambda}. \tag{3.3}$$

Now from (3.2) and (3.3) we have

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.$$

Analogously, from (3.3) we get

$$\frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} \le \frac{2|\alpha|_1 F(\theta_2)}{\dfrac{1}{2}\left(\dfrac{\theta_2}{C_{A,B}}\right)^2} = 4|\alpha|_1 C_{A,B}^2 \frac{F(\theta_2)}{{\theta_2}^2} < \frac{1}{\lambda} < \frac{2}{3}\frac{\Psi(w)}{\Phi(w)} \tag{3.4}$$

which means that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(w)}{\Phi(w)}.$$

Hence, (b_1) and (b_2) of Theorem 2.1 are established.

Now, if $u_1, u_2 \in W^{2,2}(\mathbb{R})$ be two local minima of the functional $I_\lambda = \Phi - \lambda\Psi$, with $\Psi(u_1) \ge 0$ and $\Psi(u_2) \ge 0$, then according to Lemma 2.1, u_1 and u_2 are nonnegative, and we get

$$\inf_{t\in[0,1]} \Psi(tu_1 + (1-t)u_2) \ge 0.$$

Finally, for every $\lambda \in \Lambda' \subseteq \Lambda$ (see (3.2)–(3.4)), Theorem 2.1 (with $\overline{u} = w$) and Lemma 2.1 guarantee the conclusion.

Theorem 3.1 is proved.

Now, we present the following example to illustrate Theorem 3.1.

***Example* 3.1.** Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative function and

$$\alpha(x) := \begin{cases} 1, & \text{if } x \in \left[\dfrac{3}{8}, \dfrac{5}{8}\right], \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = -1$ and $B = 1$, then we have

$$k = \left(\frac{86111}{1080}\right)^{-1}, \quad C_{A,B} = \frac{\sqrt{2}}{2}, \quad h = \left(\frac{86111}{1080}\right)^{\frac{1}{2}}, \quad E = 1, \qquad I = \frac{1080}{86111}.$$

Also let $\eta = 1$, $\theta_1 = 0.001$ and $\theta_2 = 20$. So the condition $2\theta_1 < \sqrt{2}\ \ \eta h < \theta_2$ is satisfied. Now if

$$1000000 \int_0^{0.001} f(\xi)d\xi < \frac{720}{86111}\int_0^1 f(\xi)d\xi$$

and

$$\frac{1}{400}\int_0^{20} f(\xi)d\xi < \frac{360}{86111}\int_0^1 f(\xi)d\xi,$$

then according to Theorem 3.1 for each

$$\lambda \in \left] \frac{86111}{180 \int_0^1 f(\xi) d\xi}, \min \left\{ \frac{0.000004}{\int_0^{0.001} f(\xi) d\xi}, \frac{800}{\int_0^{20} f(\xi) d\xi} \right\} \right[,$$

problem

$$\begin{gathered} u^{iv}(x) - u''(x) + u(x) = \lambda \alpha(x) f(u(x)),\ x \in \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0 \end{gathered} \tag{3.5}$$

has at least three nonnegative weak solutions u_i such that $|u_i|_\infty < 20$, $i = 1, 2, 3$.

***Remark* 3.1.** For example, in problem (3.5) we can consider

$$f(t) := \begin{cases} 18000\, t^2, & \text{if } t \le 1, \\ -1800000\, t + 1818000, & \text{if } 1 < t \le 1.01, \\ 0, & \text{if } t > 1.01. \end{cases}$$

Now, we point out the following existence result, as consequence of Theorem 3.1.

***Corollary* 3.1.** *Let $f\colon\ \mathbb{R} \to [0, +\infty[$ be a continuous and nonzero function such that*

$$\lim_{\xi \to 0^+} \frac{f(\xi)}{\xi} = \lim_{\xi \to +\infty} \frac{f(\xi)}{\xi} = 0.$$

Then, for each $\lambda > \lambda^*$, where

$$\lambda^* = \inf \left\{ \frac{3}{4|\alpha|_1 C_{A,B}^2 I} \frac{\eta^2}{\int_0^\eta f(\xi)\, d\xi} \ :\ \eta > 0, \quad \int\limits_0^\eta f(\xi)\, d\xi > 0 \right\}$$

the problem (P_λ) admits at least three distinct nonnegative weak solutions.

Proof. Suppose that $\lambda > \lambda^*$ is fixed. Let $\eta > 0$ such that $\int_0^\eta f(\xi)\, d\xi > 0$, and

$$\lambda > \frac{3}{4|\alpha|_1 C_{A,B}^2 I} \frac{\eta^2}{\int_0^\eta f(\xi)\, d\xi}.$$

Then from $\lim_{\xi \to 0^+} \dfrac{f(\xi)}{\xi} = 0$ we have $\lim_{\xi \to 0^+} \dfrac{\int_0^\xi f(t) dt}{\xi^2} = 0$ and there is $\theta_1 > 0$ such that $2\theta_1 < \sqrt{2}\eta h$, and $\dfrac{\int_0^{\theta_1} f(t) dt}{{\theta_1}^2} < \dfrac{1}{2|\alpha|_1 C_{A,B}^2 \lambda}$. Also from $\lim_{\xi \to +\infty} \dfrac{f(\xi)}{\xi} = 0$ we have $\lim_{\xi \to +\infty} \dfrac{\int_0^\xi f(t) dt}{\xi^2} = 0$ and there is $\theta_2 > 0$ such that $\sqrt{2}\ \ \eta h < \theta_2$ and $\dfrac{\int_0^{\theta_2} f(t) dt}{{\theta_2}^2} <$ $< \dfrac{1}{4|\alpha|_1 C_{A,B}^2 \lambda}$. Now we can apply Theorem 3.1 and the conclusion follows.

***Example* 3.2.** Let $\eta = 1$, $A = -1$ and $B = 1$ and so $C_{A,B} = \dfrac{\sqrt{2}}{2}$ and $I = \dfrac{1080}{86111}$. Also suppose that $f(x) = x^2 e^{-x^3}$ and $\alpha(x) = \dfrac{1}{1+x^2}$ and hence $|\alpha|_1 = \displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx = \pi$. Therefore according to Corollary 3.1 for each $\lambda > \dfrac{86111}{240\pi\left(1 - \dfrac{1}{e}\right)}$ problem

$$u^{iv}(x) - u''(x) + u(x) = \lambda \frac{u(x)^2 e^{-u(x)^3}}{1+x^2}, \qquad x \in \mathbb{R},$$

$$u(-\infty) = u(+\infty) = 0$$

admits at least three nonnegative classical solutions.

References

1. G. Bonanno, *A critical point theorem via the Ekeland variational principle*, Nonlinear Anal., **75**, 2992 – 3007 (2012).
2. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer Science+Business Media, LLC (2011), DOI: 10.1007/978-0-387-70914-7.
3. G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Different. Equat., **244**, 3031 – 3059 (2008).
4. G. Bonanno, B. Di Bella, *A boundary-value problem for fourth-order elastic beam equations*, J. Math. Anal. and Appl., **343**, 1166 – 1176 (2008).
5. G. Bonanno, B. Di Bella, *Infinitely many solutions for a fourth-order elastic beam equations*, Nonlinear Different. Equat. and Appl., **18**, 357 – 368 (2011).
6. G. Bonanno, B. Di Bella, D. O'Regan, *Non-trivial solutions for nonlinear fourth-order elastic beam equations*, Comput. Math. and Appl., **62**, 1862 – 1869 (2011).
7. V. I. Burenkov, *Sobolev spaces on domains*, Vol. 137, Teubner, Leipzig (1998).
8. G. Han, Z. Xu, *Multiple solutions of some nonlinear fourth-order beam equations*, Nonlinear Anal., **68**, 3646 – 3656 (2008).
9. X.-L. Liu, W.-T. Li, *Existence and multiplicity of solutions for fourth-order boundary-values problems with parameters*, J. Math. Anal. and Appl., **327**, 362 – 375 (2007).
10. F. Wang, Y. An, *Existence and multiplicity of solutions for a fourth-order elliptic equation*, Boundary Value Problems, № 6 (2012).

Received 26.12.17,
after revision — 27.10.19