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THE ZEROS OF THE LERCH ZETA-FUNCTION ARE UNIFORMLY DISTRIBUTED MODULO ONE

НУЛІ ДЗЕТА-ФУНКЦІЇ ЛЕРХА, РІВНОМІРНО РОЗПОДІЛЕНІ ЗА МОДУЛЕМ 1

We prove that the ordinates of the nontrivial zeros of the Lerch zeta-function are uniformly distributed modulo one.

Доведено, що ординати нетривіальних нулів дзета-функції Лерха рівномірно розподілені за модулем 1.

1. Introduction. Let $s = \sigma + it$ denote a complex variable. Denote by $\{\lambda\}$ the fractional part of a real number λ. In this paper T always tends to plus infinity and constants in big O notations may depend on parameters $\lambda,\ \alpha$ and x.

The Lerch zeta-function is defined by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}, \quad \sigma > 1,$$

where $0 < \lambda, \alpha \leq 1$. This function has an analytic continuation to the whole complex plane except for a possible simple pole at $s = 1$ (see [16, 17]). The Lerch zeta-function satisfies the functional equation (see, for example, [16], Chapter 2, or [10], formula (1))

$$L(\lambda, \alpha, 1-s) = (2\pi)^{-s}\Gamma(s)\Big(e^{\pi i \frac{s}{2} - 2\pi i \alpha\lambda} L(1-\alpha, \lambda, s) + \\ + e^{-\pi i \frac{s}{2} + 2\pi i \alpha(1-\{\lambda\})}\, L(\alpha, 1-\{\lambda\}, s)\Big). \tag{1}$$

Next we indicate zero free regions. Let l be a straight line in the complex plane $\mathbb{C}$, and denote by $\varrho(s, l)$ the distance of s from l. Define, for $\delta > 0$,

$$L_\delta(l) = \big\{ s \in \mathbb{C} \colon \varrho(s, l) < \delta \big\}.$$

In [7, 12], for $0 < \lambda < 1$ and $\lambda \neq 1/2$, it is proved that $L(\lambda, \alpha, s) \neq 0$ if $\sigma < -1$ and

$$s \notin L_{\frac{\log 4}{\pi}} \left(\sigma = \frac{\pi t}{\log \dfrac{1-\lambda}{\lambda}} + 1 \right).$$

For $\lambda = 1/2, 1$, from [7, 22] we see that $L(\lambda, \alpha, s) \neq 0$ if $\sigma < -1$ and $|t| \geq 1$. Moreover, in [7] it is shown that $L(\lambda, \alpha, s) \neq 0$ if $\sigma \geq 1 + \alpha$. We say that a zero of $L(\lambda, \alpha, s)$ is *nontrivial* if it lies in the strip $-1 \leq \sigma < 1 + \alpha$. The nontrivial zero is denoted by $\rho = \beta + i\gamma$.

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Let $\zeta(s)$ and $L(s,\chi)$ be the Riemann zeta-function and the Dirichlet L-function accordingly. We have

$$L(1,1,s)=\zeta(s) \qquad \text{and} \qquad L(1/2,1/2,s)=2^s L(s,\chi), \tag{2}$$

where χ is a Dirichlet character $\bmod 4$ with $\chi(3)=-1$. For these two cases, the Riemann hypothesis can be formulated. Similar cases are $L(1,1/2,s)=(2^s-1)\zeta(s)$ and $L(1/2,1,s)=(1-2^{1-s})\zeta(s)$. For all the other cases, it is expected that the real parts of zeros of the Lerch zeta-function form a dense subset of the interval $(1/2,1)$. This is proved for any λ and transcendental α [8] using the universality property of the Lerch zeta-function. More about the universality of the Lerch zeta-function see [6, 15, 18].

Denote by $N(\lambda,\alpha,T)$ the number of nontrivial zeros of the function $L(\lambda,\alpha,s)$ in the region $0<t\le T$. Then [7, 9]

$$N(\lambda,\alpha,T)=\frac{T}{2\pi}\log T-\frac{T}{2\pi}\log(2\pi e\alpha\lambda)+O(\log T). \tag{3}$$

A sequence $\{a_1,a_2,a_3,\ldots\}$ of real numbers is *uniformly distributed* in the interval $[a,b]$, if for any subinterval $[c,d]$ of $[a,b]$ we have

$$\lim_{n\to\infty}\frac{|\{a_1,a_2,a_3,\ldots,a_n\}\cap[c,d]|}{n}=\frac{d-c}{b-a}.$$

The notation $|\{a_1,a_2,a_3,\ldots,a_n\}\cap[c,d]|$ denotes the number of elements, out of the first n elements of the sequence, that are between c and d. A sequence $a_1,a_2,a_3,\ldots$ of real numbers is said to be *uniformly distributed modulo 1* if the sequence of the fractional parts of a_n, denoted by $\{a_n\}$, is uniformly distributed in the interval $[0,1]$.

Under the assumption of the truth of the Riemann hypothesis Rademacher [20] proved that the imaginary parts of the nonreal zeros of the Riemann zeta-function are uniformly distributed modulo one; Elliott [3] and (independently) Hlawka [14] gave unconditional proofs of this result. Further extensions and generalizations can be found in the articles [2, 4, 5, 11, 21].

The main result of this paper is the following theorem.

Theorem 1. *The imaginary parts of nontrivial zeros of the Lerch zeta-function* $L(\lambda,\alpha,s)$ *are uniformly distributed modulo one.*

The proof of Theorem 1 relies on the following proposition.

Proposition 1. *Let* x *be a fixed positive real number not equal to* 1. *Then*

$$\sum_{0<\gamma\le T}x^{\rho}=(c(x)+d(x))\frac{T}{2\pi}+O(\log T),$$

where $c(x)$ *and* $d(x)$ *are complex numbers defined by formulas* (9) *and* (16) *below.*

Proposition 1 and Theorem 1 are proved in Sections 2 and 3, respectively.

2. Proof of Proposition 1. Let $B\ge 3$ be a sufficiently large number which will be chosen later. A strip $1-B\le\sigma\le B$ contains all the nontrivial zeros and a finite number of trivial zeros. Applying the residue theorem, we get

$$\sum_{0<\gamma\le T}x^{\rho}=\frac{1}{2\pi i}\int\limits_{\square}x^s\frac{L'}{L}(\lambda,\alpha,s)\,ds+O(1), \tag{4}$$

where $\square$ denotes the counterclockwise oriented rectangular contour with vertices $B+i,\ B+iT,$ $1-B+iT,\ 1-B+i$ and

$$L'(\lambda,\alpha,s)=\frac{\partial}{\partial s}L(\lambda,\alpha,s).$$

To deal with the integral in formula (4) the following two lemmas will be useful.

Lemma 1. *If $f(s)$ is analytic and $f(s_0)\neq 0$ with*

$$\left|\frac{f(s)}{f(s_0)}\right|<e^M$$

in $\{s\colon |s-s_0|\le r\}$ with $M>1$, then

$$\left|\frac{f'}{f}(s)-\sum_{\rho'}\frac{1}{s-\rho'}\right|<C\frac{M}{r}$$

for $|s-s_0|\le \frac{r}{4}$, where C is some constant and ρ' runs through the zeros of $f(s)$ such that $|\rho'-s_0|\le\frac{r}{2}$.

For the proof, see [23] (§ 3.9).

Lemma 1 is applied in the proof of the next lemma.

Lemma 2. *Let $B,b>2$ be fixed. If T is such that $L(\lambda,\alpha,\sigma+iT)\neq 0$ for $1-b\le\sigma\le B$, then*

$$\int\limits_{1-b}^{B}\left|\frac{L'}{L}(\lambda,\alpha,\sigma+iT)\right|d\sigma\ll\log T.$$

Proof. In Lemma 1, we choose $s_0=B+iT$ and $r=4(B-(1-b))$. It is known (see, for example, [13], Lemma 3) that, for $|s-s_0|\le r$,

$$L(\lambda,\alpha,s)\ll T^c$$

with some $c>0$. Therefore we can take $M=2c\log T$. Then Lemma 1 gives

$$\frac{L'}{L}(\lambda,\alpha,s)=\sum_{|\rho-s_0|\le\frac{r}{2}}\frac{1}{s-\rho}+O(\log T) \tag{5}$$

for $|s-s_0|\le\frac{r}{4}$. Note that the points $B+iT$ and $1-b+iT$ are not very near to zeros of $L(\lambda,\alpha,s)$. Thus,

$$\int\limits_{1-b}^{B}\left|\frac{L'}{L}(\lambda,\alpha,\sigma+iT)\right|d\sigma\le\int\limits_{1-b}^{B}\sum_{|\rho-s_0|\le\frac{r}{2}}\left|\frac{1}{\sigma+iT-\rho}\right|d\sigma+O(\log T)=$$

$$=\sum_{|\rho-s_0|\le\frac{r}{2}}\int\limits_{1-b}^{B}\frac{1}{\sqrt{(\sigma-\beta)^2+(T-\gamma)^2}}\,d\sigma+O(\log T)=$$

$$= \sum_{|\rho - s_0| \le \frac{r}{2}} \left(\log \left(B - \beta + \sqrt{(T-\gamma)^2 + (B-\beta)^2} \right) - \right.$$

$$\left. - \log \left(1 - b - \beta + \sqrt{(T-\gamma)^2 + (1-b-\beta)^2)} \right) \right) + O(\log T) \ll$$

$$\ll \log T,$$

since the inequality $|\rho - s_0| \le \dfrac{r}{2}$ is satisfied with $O(\log T)$ many zeros ρ (see the asymptotic formula (3)).

Lemma 2 is proved.

***Proof of Proposition* 1.** We consider the contour integral in formula (4):

$$\int\limits_{\square} x^s \frac{L'(\lambda, \alpha, s)}{L(\lambda, \alpha, s)} \, ds =$$

$$= \left\{ \int\limits_{B+i}^{B+iT} + \int\limits_{B+iT}^{1-B+iT} + \int\limits_{1-B+iT}^{1-B+i} + \int\limits_{1-B+i}^{B+i} \right\} x^s \frac{L'(\lambda, \alpha, s)}{L(\lambda, \alpha, s)} \, ds = \sum_{j=1}^{4} I_j. \tag{6}$$

Let $x_{m+1} = x_{m+1}(\alpha) = (m+\alpha)/\alpha, \ m = 0, 1, \ldots,$ be the sequence X and define

$$S = \{ x_{k_1} x_{k_2} \ldots x_{k_m} : m \in \mathbb{N}, \ \ k_1 \in \mathbb{N}, \ldots, k_m \in \mathbb{N} \}$$

as the set of all possible products of elements of the sequence X. Let

$$1 = y_1(\alpha) < y_2(\alpha) < \ldots \tag{7}$$

be an ordered sequence of all different numbers of S. By Lemma 8 in [11] there are $\sigma_1 \ge 1$ and complex numbers $c_n, \ n = 1, 2, \ldots,$ such that the logarithmic derivative of $L(\lambda, \alpha, s)$ has an absolutely convergent Dirichlet series expansion

$$\frac{L'(\lambda, \alpha, s)}{L(\lambda, \alpha, s)} = \sum_{n=1}^{\infty} \frac{c_n}{y_n^s(\alpha)}, \quad \sigma > \sigma_1.$$

Let $B > \sigma_1$. Interchanging summation and integration, we find

$$I_1 = \sum_{n=1}^{\infty} c_n \int\limits_{B+i}^{B+iT} \left(\frac{x}{y_n(\alpha)} \right)^s ds = \sum_{n=2}^{\infty} c_n i \int\limits_{1}^{T} \exp \big((B+it) \log(x/y_n(\alpha)) \big) \, dt =$$

$$= \sum_{n=1}^{\infty} c_n i \exp \big(B \log(x/y_n(\alpha)) \big) \int\limits_{1}^{T} \exp \big(it \log(x/y_n(\alpha)) \big) \, dt.$$

In view of

$$\int\limits_{1}^{T} \exp(it \log(x/y_n(\alpha))) \, dt =$$

$$= \begin{cases} T-1 & \text{if} \quad x = y_n(\alpha), \\ \big(\exp\big(iT\log(x/y_n(\alpha))\big) - \exp\big(i\log(x/y_n(\alpha))\big)\big)/\big(i\log(x/y_n(\alpha))\big) & \text{otherwise,} \end{cases}$$

we get

$$I_1 = ic(x)T + O(1), \tag{8}$$

where

$$c(x) = \begin{cases} c_n & \text{if} \quad x = y_n(\alpha), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

To evaluate the integral

$$I_3 = -\int\limits_{1-B+i}^{1-B+iT} x^s \frac{L'(\lambda,\alpha,s)}{L(\lambda,\alpha,s)}\,ds \tag{10}$$

we will use the functional equation (1). The logarithmic derivative of the functional equation is

$$\frac{L'}{L}(\lambda,\alpha,s) = \log 2\pi\lambda - \frac{\Gamma'}{\Gamma}(1-s) - \frac{\pi i}{2} - \frac{E'}{E}(\lambda,\alpha,1-s), \tag{11}$$

where, for $\sigma < -1$,

$$E(\lambda,\alpha,1-s) := 1 + \sum_{m=1}^{\infty} \frac{e^{-2\pi i\alpha m}}{\left(\dfrac{\lambda+m}{\lambda}\right)^{1-s}} + e^{-\pi i(1-s)} e^{2\pi i\alpha(1+\lambda-\{\lambda\})} \sum_{m=0}^{\infty} \frac{e^{2\pi i\alpha m}}{\left(\dfrac{1-\{\lambda\}+m}{\lambda}\right)^{1-s}}$$

and

$$E'(\lambda,\alpha,s) = \frac{\partial}{\partial s} E(\lambda,\alpha,s).$$

We have

$$\int\limits_{1-B+i}^{1-B+iT} x^s \log(2\pi\lambda)ds = O(1). \tag{12}$$

It is known (see formula 6.3.18 in [1]) that, for $|\arg s| < \pi$,

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right), \quad s \to \infty.$$

Thus,

$$\int\limits_{1-B+i}^{1-B+iT} x^s \frac{\Gamma'}{\Gamma}(1-s)ds = ix^{1-B} \int\limits_{1}^{T} x^{it}\left(\log t - \frac{\pi}{2} + O\left(\frac{1}{t}\right)\right) dt = O(\log T). \tag{13}$$

Let

$$F(\lambda,\alpha,1-s)=1+\sum_{m=1}^{\infty}\frac{e^{-2\pi i\alpha m}}{\left(\dfrac{\lambda+m}{\lambda}\right)^{1-s}},\quad \sigma<-1,$$

and

$$F'(\lambda,\alpha,s)=\frac{\partial}{\partial s}F(\lambda,\alpha,s).$$

Then, for $\sigma<-1$,

$$\frac{E'(\lambda,\alpha,1-s)}{E(\lambda,\alpha,1-s)}=\frac{F'(\lambda,\alpha,1-s)}{F(\lambda,\alpha,1-s)}+O(e^{-t}),\quad t\to\infty. \tag{14}$$

Again, by Lemma 8 in [11] there are complex numbers $d_n,\ n=1,2,\ldots,$ such that the logarithmic derivative of $F(\lambda,\alpha,s)$ has the Dirichlet series expansion

$$\frac{F'(\lambda,\alpha,s)}{F(\lambda,\alpha,s)}=\sum_{n=1}^{\infty}\frac{d_n}{y_n^s(\lambda)},$$

which converges absolutely for $\Re s\geq B$ if B is sufficiently large. The numbers $y_n(\lambda),\ n=1,2,\ldots,$ are defined by (7). This and formula (14) give

$$\int\limits_{1-B+i}^{1-B+iT}x^s\frac{E'(\lambda,\alpha,1-s)}{E(\lambda,\alpha,1-s)}\,ds=ix^{1-B}\sum_{n=2}^{\infty}\frac{d_n}{y_n^B(\lambda)}\int\limits_1^T\left(\frac{x}{y_n(\lambda)}\right)^{it}dt+O(1)=$$

$$=id(x)T+O(1), \tag{15}$$

where

$$d(x)=\begin{cases}d_ny_n(\lambda) & \text{if}\quad x=y_n(\lambda),\\ 0 & \text{otherwise.}\end{cases} \tag{16}$$

By formulae (10)–(13) and (15) we obtain

$$I_3=id(x)T+O(\log T). \tag{17}$$

We observe that I_4 does not depend on T. Therefore,

$$I_4=O(1). \tag{18}$$

Lemma 2 gives

$$I_2=\int\limits_{1-B+iT}^{B+iT}x^s\frac{L'(\lambda,\alpha,s)}{L(\lambda,\alpha,s)}ds=\int\limits_{1-B}^{B}x^{\sigma+iT}\frac{L'(\lambda,\alpha,\sigma+iT)}{L(\lambda,\alpha,\sigma+iT)}d\sigma\ll\log T. \tag{19}$$

Formulae (4), (6), (8), (17), (18), and (19) prove Proposition 1.

3. Proof of Theorem 1. In the proof of Theorem 1 the following Weyl's criterion for the uniform distribution will be important.

Lemma 3. *A sequence of real numbers y_n is uniformly distributed modulo one if and only if, for each integer $m \neq 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} e^{2\pi i m y_j} = 0.$$

For the proof, see [24, 25].

Let $N^{(1)}(T)$ be the number of nontrivial zeros of $L(\lambda, \alpha, s)$ in

$$\beta > \frac{1}{2} + (\log\log T)^2/\log T, \quad 0 < t \leq T.$$

Let $N^{(2)}(T)$ be the number of those in

$$\beta < \frac{1}{2} - (\log\log T)^2/\log T, \quad 0 < t \leq T,$$

and let $N^{(3)}(T)$ be the number of those in

$$\frac{1}{2} - (\log\log T)^2/\log T \leq \beta \leq \frac{1}{2} + (\log\log T)^2/\log T, \quad 0 < t \leq T.$$

The following clustering of the nontrivial zeros around the critical line $\sigma = 1/2$ will be useful in the proof of Theorem 1.

Proposition 2. *We have*

$$N^{(1)}(T) \ll \frac{T\log T}{\log\log T}, \tag{20}$$

$$N^{(2)}(T) \ll \frac{T\log T}{\log\log T}, \tag{21}$$

$$N^{(3)}(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e \alpha\lambda} + O\left(\frac{T\log T}{\log\log T}\right). \tag{22}$$

To prove the proposition we will need the following two lemmas.

Lemma 4. *We have*

$$\sum_{\substack{0<\gamma\leq T\\ \beta>1/2}} \left(\beta - \frac{1}{2}\right) = O(T\log\log T).$$

Proof. From the proof of Theorem 1 in [9] we have, for $b > -3$,

$$2\pi \sum_{\substack{0<\gamma\leq T\\ \beta>-b}} (b+\beta) = \int_0^T \log\left|\alpha^{-b+it} L(\lambda, \alpha, -b+it)\right| dt + O(\log T).$$

Theorem 1 in [10] gives the bound

$$\int_0^T \left|L(\lambda, \alpha, \frac{1}{2} + it)\right|^2 dt = O(T \log T).$$

Choosing $b = -\frac{1}{2}$, by the concavity of the logarithm we get

$$2\pi \sum_{\substack{0<\gamma\leq T \\ \beta>1/2}} \left(\beta - \frac{1}{2}\right) = \int_0^T \log \left|\alpha^s L(\lambda, \alpha, \frac{1}{2} + it)\right| dt + O(\log T) =$$

$$= \int_0^T \log \left|L(\lambda, \alpha, \frac{1}{2} + it)\right| dt + O(T) \leq$$

$$\leq \frac{1}{2} T \log \left(\frac{1}{T} \int_0^T \left|L(\lambda, \alpha, \frac{1}{2} + it)\right|^2 dt \right) + O(T) \ll$$

$$\ll T \log\log T.$$

Lemma 4 is proved.

Lemma 5. *Let $b \geq 3$ be a constant. For $0 < \lambda, \alpha \leq 1$,*

$$\sum_{0<\gamma\leq T} (b + \beta) = \left(b + \frac{1}{2}\right) \frac{T}{2\pi} \log \frac{T}{4\pi e \alpha\lambda} + \frac{T}{4\pi} \log \frac{\alpha}{\lambda} + O(\log T).$$

For the proof, see [9] (Theorem 1) and [13] (Lemma 1).

***Proof of Proposition* 2.** From Lemma 4 and the definition of $N^{(1)}(T)$ we see that

$$0 \leq N^{(1)}(T)(\log\log T)^2/\log T \leq O(T \log\log T).$$

This proves the bound (20).

Next we consider the bound for $N^{(2)}(T)$. Let $\delta = (\log\log T)^2/\log T$. We have

$$\sum_{0<\gamma\leq T} (b + \beta) = \sum_{\substack{0<\gamma\leq T \\ \beta\geq 1/2-\delta}} (b + \beta) + \sum_{\substack{0<\gamma\leq T \\ \beta<1/2-\delta}} (b + \beta). \tag{23}$$

Definitions of $N^{(1)}(T)$, $N^{(3)}(T)$, and Lemma 4 give

$$\sum_{\substack{0<\gamma\leq T \\ \beta\geq 1/2-\delta}} (b + \beta) \leq \sum_{\substack{0<\gamma\leq T \\ \beta>1/2}} \left(\beta - \frac{1}{2}\right) + \left(b + \frac{1}{2}\right)(N^{(1)}(T) + N^{(3)}(T)) \leq$$

$$\leq O(T \log\log T) + \left(b + \frac{1}{2}\right)(N^{(1)}(T) + N^{(3)}(T)).$$

For the second sum in the right-hand side of formula (23), we obtain

$$\sum_{\substack{0<\gamma\leq T\\ \beta<1/2-\delta}}(b+\beta)\leq\left(b+\frac{1}{2}-(\log\log T)^2/\log T\right)N^{(2)}(T).$$

The last three formulas yield the inequality

$$\sum_{0<\gamma\leq T}(b+\beta)\leq O(T\log\log T)+\left(b+\frac{1}{2}\right)N(\lambda,\alpha,T)-(\log\log T)^2N^{(2)}(T)/\log T.$$

By this, Lemma 5, and formula (3) we see that

$$0\leq O(T\log\log T)-(\log\log T)^2N^{(2)}(T)/\log T.$$

This proves formula (21). Formulae (20) and (21) together with the asymptotic formula (3) yield the equality (22).

Proposition 2 is proved.

Proposition 2 leads to the following lemma.

Lemma 6. *We have*

$$\sum_{0<\gamma\leq T}\left|\beta-\frac{1}{2}\right|\ll\frac{T\log T}{\log\log T}.$$

Proof. As in the proof of Proposition 2 we denote $\delta=(\log\log T)^2/\log T$. Then

$$\sum_{0<\gamma\leq T}\left|\beta-\frac{1}{2}\right|=\left\{\sum_{\substack{0<\gamma\leq T\\ \beta-1/2>\delta}}+\sum_{\substack{0<\gamma\leq T\\ |\beta-1/2|\leq\delta}}+\sum_{\substack{0<\gamma\leq T\\ \beta-1/2<-\delta}}\right\}\left|\beta-\frac{1}{2}\right|. \tag{24}$$

In view of zero free regions of the Lerch zeta-function we have that $|\beta-1/2|\ll 1$. Then the expression (24) together with Proposition 2 proves Lemma 6.

***Proof of Theorem* 1.** Similarly, as in the proof of Theorem 1 in [21], we can write

$$x^{\frac{1}{2}}\sum_{0<\gamma\leq T}x^{i\gamma}=\sum_{0<\gamma\leq T}x^{\beta+i\gamma}+\sum_{0<\gamma\leq T}(x^{1/2+i\gamma}-x^{\beta+i\gamma}). \tag{25}$$

We find

$$x^{\frac{1}{2}}\left|\sum_{0<\gamma\leq T}x^{i\gamma}\right|\leq\left|\sum_{0<\gamma\leq T}x^{\beta+i\gamma}\right|+\sum_{0<\gamma\leq T}\left|x^{1/2+i\gamma}-x^{\beta+i\gamma}\right|.$$

By inequality

$$\left|\exp(y)-1\right|=\left|\int_0^y\exp(t)\,dt\right|\leq|y|\max\{1,\exp(y)\},$$

where y is a real number, we get

$$|x^{1/2+i\gamma} - x^{\beta+i\gamma}| = x^{\beta}\left|\exp\left(\left(\frac{1}{2}-\beta\right)\log x\right)-1\right| \leq$$

$$\leq \left|\beta-\frac{1}{2}\right| |\log x| \max\left\{x^{\beta}, x^{1/2}\right\}.$$

Then by Lemma 6 and formula (3) we have

$$\frac{1}{N(\lambda,\alpha,T)}\sum_{0<\gamma\leq T}|x^{1/2+i\gamma} - x^{\beta+i\gamma}| \leq$$

$$\leq \frac{|\log x| \max\left\{x^{\beta}, x^{1/2}\right\}}{N(\lambda,\alpha,T)}\sum_{0<\gamma\leq T}\left|\beta-\frac{1}{2}\right| \ll \frac{1}{\log\log T}.$$

This, formula (25), and Proposition 1 give that

$$\frac{1}{N(\lambda,\alpha,T)}\sum_{0<\gamma\leq T}x^{1/2+i\gamma} \ll \frac{1}{\log\log T}.$$

Let $x = z^m$ with some positive $z \neq 1$ and $m \in \mathbb{N}$. Then, after dividing the previous formula by $x^{\frac{1}{2}}$, we deduce

$$\lim_{T\to\infty}\frac{1}{N(\lambda,\alpha,T)}\sum_{0<\gamma\leq T}\exp(im\gamma\log z) = 0.$$

Then Weyl's criterion (Lemma 3) implies that the sequence of numbers $\gamma\log z/2\pi$ is uniformly distributed modulo 1.

Theorem 1 is proved.

References

1. M. Abramowitz, I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards, Wiley-Intersci. Publ., New York (1972).
2. A. Akbary, M. R. Murty, *Uniform distribution of zeros of Dirichlet series*, Anatomy of Integer, CRM Proc. Lecture Notes, **46**, 143 – 158 (2008).
3. P. D. T. A. Elliott, *The Riemann zeta function and coin tossing*, J. reine und angew. Math., **254**, 100 – 109 (1972).
4. K. Ford, K. Soundararajan, A. Zaharescu, *On the distribution of imaginary parts of zeros of the Riemann zeta function, II*, Math. Ann., **343**, 487 – 505 (2009).
5. A. Fujii, *On the uniformity of the distribution of zeros of the Riemann zeta function*, J. reine und angew. Math., **302**, 167 – 205 (1978).
6. R. Garunkštis, *The universality theorem with weight for the Lerch zeta-function*, New Trends in Probability and Statistics, vol. 4 (Palanga, 1996), VSP, Utrecht (1997).
7. R. Garunkštis, A. Laurinčikas, *On zeros of the Lerch zeta-function*, Number Theory and Its Applications, S. Kanemitsu, K. Gyory (eds.), Kluwer Acad. Publ., 129 – 143 (1999).
8. R. Garunkštis, A. Laurinčikas, *The Lerch zeta-function*, Integral Transforms Spec. Funct., **10**, 211 – 226 (2000).
9. R. Garunkštis, J. Steuding, *On the zero distributions of Lerch zeta-functions*, Analysis, **22**, 1 – 12 (2002).
10. R. Garunkštis, A. Laurinčikas, J. Steuding, *On the mean square of Lerch zeta-functions*, Arch. Math., **80**, 47 – 60 (2003).
11. R. Garunkštis, J. Steuding, R. Šimėnas, *The a-points of the Selberg zeta-function are uniformly distributed modulo one*, Illinois J. Math., **58**, 207 – 218 (2014).

12. R. Garunkštis, J. Steuding, *Do Lerch zeta-functions satisfy the Lindeloff hypothesis?*, Anal. and Probab. Methods Number Theory, Proc. Third Intern. Conf. Honour of J. Kubilius (Palanga, Lithuania, 24 – 28 September 2001), TEV, Vilnius (2002), p. 61 – 74.
13. R. Garunkštis, R. Tamošiūnas, *Symmetry of zeros of Lerch zeta-function for equal parameters*, Lith. Math. J., **57**, 433 – 440 (2017).
14. E. Hlawka, *Über die Gleichverteilung gewisser Folgen, welche mit den Nullstellen der Zetafunktionen zusammenhängen*, Österr. Akad. Wiss., Math.-Natur. Kl. Abt. II, **184**, 459 – 471 (1975).
15. A. Laurinčikas, *The universality of the Lerch zeta-function*, Lith. Math. J., **37**, 275 – 280 (1997).
16. A. Laurinčikas, R. Garunkštis, *The Lerch zeta-function*, Kluwer Acad. Publ., Dordrecht (2002).
17. M. Lerch, *Note sur la fonction* $\mathcal{K}(z, x, s) = \sum_{k=0}^{\infty} e^{2k\pi i x}(z + k)^{-s}$, Acta Math., **11**, 19 – 24 (1887).
18. Y. Lee, T. Nakamura, Ł. Pańkowski, *Joint universality for Lerch zeta-functions*, J. Math. Soc. Japan, **69**, 153 – 161 (2017).
19. N. Levinson, *Almost all root of* $\zeta(s) = a$ *are arbitrarily close to* $\sigma = 1/2$, Proc. Nat. Acad. Sci. USA, **72**, 1322 – 1324 (1975).
20. H. G. Rademacher, *Fourier analysis in number theory*, Symp. Harmonic Analysis and Related Integral Transforms (Cornell Univ., Ithaca, N.Y., 1956), Collected Papers of Hans Rademacher, vol. II (1974), p. 434 – 458.
21. J. Steuding, *The roots of the equation* $\zeta(s) = a$ *are uniformly distributed modulo one*, Anal. and Probab. Methods Number Theory TEV, Vilnius (2012), p. 243 – 249.
22. R. Spira, *Zeros of Hurwitz zeta-functions*, Math. Comput., **136**, 863 – 866 (1976).
23. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., rev. by D. R. Heath-Brown, Oxford Sci. Publ., Clarendon Press, Oxford (1986).
24. H. Weyl, *Sur une application de la théorie des nombres à la mécaniques statistique et la théorie des pertubations*, Enseign. Math., **16**, 455 – 467 (1914).
25. H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann., **77**, 313 – 352 (1916).

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