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DIFFERENTIAL AND INTEGRAL EQUATIONS FOR LEGENDRE – LAGUERRE BASED HYBRID POLYNOMIALS

ДИФЕРЕНЦІАЛЬНІ ТА ІНТЕГРАЛЬНІ РІВНЯННЯ ДЛЯ ГІБРИДНИХ ПОЛІНОМІВ НА БАЗІ ПОЛІНОМІВ ЛЕЖАНДРА – ЛАГЕРРА

In this article, a hybrid family of three-variable Legendre – Laguerre – Appell polynomials is explored and their properties including the series expansions, determinant forms, recurrence relations, shift operators, followed by differential, integro-differential and partial differential equations are established. The analogous results for the three-variable Hermite – Laguerre – Appell polynomials are deduced. Certain examples in terms of Legendre – Laguerre – Bernoulli, – Euler and – Genocchi polynomials are constructed to show the applications of main results. A further investigation is performed by deriving homogeneous Volterra integral equations for these polynomials and for their relatives.

Розглянуто гібридну сім'ю поліномів Лежандра – Лагерра – Аппеля та встановлено їхні властивості, які включають розклади рядів, форми детермінантів, рекурентні співвідношення, оператори зсуву, за якими йдуть диференціальні та інтегро-диференціальні рівняння, а також диференціальні рівняння з частинними похідними. Подібні результати отримано для поліномів Ерміта – Лагерра – Аппеля з трьома змінними. У термінах поліномів Лежандра – Лагерра – Бернуллі, – Ейлера та – Дженоккі побудовано деякі приклади, щоб показати застосування основних результатів. Далі, для цих та пов'язаних з ними поліномів отримано однорідне інтегральне рівняння Вольтерра.

1. Introduction and preliminaries. The study of differential equations is a wide field in pure and applied mathematics, physics and engineering. The mathematical theory of differential equations first developed together with the sciences where the equations had originated and where the results found applications. Differential equations play an important role in modeling virtually every physical, technical, or biological process, from celestial motion to bridge design, to interactions between neurons. We recall the following definitions.

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials such that $\deg(p_n(x))=n,\ n\in\mathbb{N}_0:=\{0,1,2,\ldots\}$. The differential operators Θ_n^- and Θ_n^+ satisfying the properties

$$\Theta_n^-\{p_n(x)\}=p_{n-1}(x),\qquad \Theta_n^+\{p_n(x)\}=p_{n+1}(x), \tag{1.1}$$

are called derivative and multiplicative operators, respectively. The polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ satisfying equation (1.1) is then called quasimonomial.

The derivative and multiplicative operators for a given family of polynomials give rise to some useful properties such as

$$(\Theta_{n+1}^-\Theta_n^+)\{p_n(x)\}=p_n(x),\qquad (\Theta_{n-1}^+\Theta_{n-2}^+\ldots\Theta_2^+\Theta_1^+\Theta_0^+)\{p_0(x)\}=p_n(x). \tag{1.2}$$

The technique used in obtaining differential equations via (1.2) is known as the factorization method [12, 13]. The main idea of the factorization method is to find the derivative and multiplicative operators such that equation (1.2) holds. The factorization method can be equivalently treated as monomiality principle. The monomiality principle [7] and the associated operational rules are used in [8] to explore new classes of isospectral problems leading to nontrivial generalizations of special functions.

The Appell polynomial sequences [2] arise in numerous problems of mathematics, physics, and engineering. The set of all Appell sequences is closed under the operation of umbral compositions of polynomial sequences and forms an abelian group. The Appell polynomial sequences are defined by the generating function

$$\mathcal{R}(t)e^{yt} = \sum_{n=0}^{\infty} \mathcal{R}_n(y)\frac{t^n}{n!}. \tag{1.3}$$

The power series $\mathcal{R}(t)$ is given by

$$\mathcal{R}(t) = \mathcal{R}_0 + \frac{t}{1!}\mathcal{R}_1 + \frac{t^2}{2!}\mathcal{R}_2 + \ldots + \frac{t^n}{n!}\mathcal{R}_n + \ldots = \sum_{n=0}^{\infty}\mathcal{R}_n\frac{t^n}{n!}, \quad \mathcal{R}_0, \neq 0,$$

with $\mathcal{R}_i,\ i = 0, 1, 2, \ldots,$ real coefficients. The function $\mathcal{R}(t)$ is an analytic function at $t = 0$ and for any $\mathcal{R}(t)$, the derivative of $\mathcal{R}_n(y)$ satisfies

$$\mathcal{R}_n^{'}(y) = n\mathcal{R}_{n-1}(y).$$

The Appell polynomial sequences are defined by the series expansion

$$\mathcal{R}_n(y) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{R}_k\, y^{n-k}. \tag{1.4}$$

For the suitable choices of the function $\mathcal{R}(t)$, different members belonging to the family of Appell polynomials can be obtained. These members and their related numbers are given in Table 1.1.

The Bernoulli and Euler numbers appear in the Taylor series expansions of trigonometric and hyperbolic tangent and cotangent and trigonometric and hyperbolic secant functions, respectively. The Genocchi numbers appear in counting the number of up-down ascent sequences and graph and automata theories.

We know that the generalized special polynomials provide new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. Most of the special functions of mathematical physics and their generalizations have been suggested by physical problems. Some of these special polynomials are listed below.

The two-variable Laguerre polynomials (2VLP) $L_n(x, y)$ [10] are defined by means of the generating equation

$$e^{yt}C_0(xt) = \sum_{n=0}^{\infty} L_n(x, y)\frac{t^n}{n!}, \tag{1.5}$$

where $C_0(xt)$ is the 0^{th} Tricomi function [1] defined by the operational definition

$$C_0(\alpha x) = \exp\left(-\alpha D_x^{-1}\right)\{1\}, \qquad D_x^{-n}\{1\} := \frac{x^n}{n!} \quad \text{is inverse derivative operator.} \tag{1.6}$$

The Tricomi function $C_n(x)$ is defined by the series expansion

Table 1.1. Certain members belonging to the Appell family

S.No.	Name of the polynomial and related number	$\mathcal{R}(t)$	Generating function	Series expansion
I	Bernoulli polynomials and numbers [11]	$\dfrac{t}{e^t-1}$	$\left(\dfrac{t}{e^t-1}\right)e^{yt}=\sum\limits_{n=0}^{\infty}B_n(y)\dfrac{t^n}{n!}$ $\left(\dfrac{t}{e^t-1}\right)=\sum\limits_{n=0}^{\infty}B_n(:=B_n(0)=B_n(1))\dfrac{t^n}{n!}$ $B_0=1,\ B_1=\pm\dfrac{1}{2},\ B_2=\dfrac{1}{6}$	$B_n(y)=\sum\limits_{k=0}^{n}\binom{n}{k}B_k y^{n-k}$
II	Euler polynomials and numbers [11]	$\dfrac{2}{e^t+1}$	$\left(\dfrac{2}{e^t+1}\right)e^{yt}=\sum\limits_{n=0}^{\infty}E_n(y)\dfrac{t^n}{n!}$ $\dfrac{2e^t}{e^{2t}+1}=\sum\limits_{n=0}^{\infty}E_n\left(:=2^nE_n\left(\dfrac{1}{2}\right)\right)\dfrac{t^n}{n!}$ $E_0=1,\ E_1=0,\ E_2=-1$	$E_n(y)=\sum\limits_{k=0}^{n}\binom{n}{k}\times$ $\times\dfrac{E_k}{2^k}\left(y-\dfrac{1}{2}\right)^{n-k}$
III	Genocchi polynomials and numbers [4, 14]	$\dfrac{2t}{e^t+1}$	$\left(\dfrac{2t}{e^t+1}\right)e^{yt}=\sum\limits_{n=0}^{\infty}G_n(y)\dfrac{t^n}{n!}$ $\dfrac{2t}{e^t+1}=\sum\limits_{n=0}^{\infty}G_n(:=G_n(0))\dfrac{t^n}{n!}$ $G_0=0,\ G_1=1,\ G_2=-1$	$G_n(y)=\sum\limits_{k=0}^{n}\binom{n}{k}G_k y^{n-k}$

$$C_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\,x^k}{k!\,(n+k)!}. \tag{1.7}$$

The series expansion and operational representation for the 2VLP $L_n(x,y)$ are given by [10]

$$L_n(x,y)=n!\sum_{k=0}^{n}\frac{(-1)^k\,x^k\,y^{n-k}}{(k!)^2(n-k)!}, \tag{1.8}$$

$$L_n(x,y)=\exp\left(-D_x^{-1}\frac{\partial}{\partial y}\right)\{y^n\}. \tag{1.9}$$

Next, the two-variable Legendre polynomials (2VLeP) $S_n(z,y)$ [9] are specified by means of the generating equation

$$e^{yt}\,C_0(-zt^2)=\sum_{n=0}^{\infty}S_n(z,y)\frac{t^n}{n!}. \tag{1.10}$$

The series expansion and operational representation for the 2VLeP $S_n(z,y)$ are given by [9]

$$S_n(z,y)=n!\sum_{k=0}^{[\frac{n}{2}]}\frac{z^k\,y^{n-2k}}{(k!)^2\,(n-2k)!}, \tag{1.11}$$

$$S_n(z,y)=\exp\left(D_z^{-1}\frac{\partial^2}{\partial y^2}\right)\{y^n\}. \tag{1.12}$$

***Remark* 1.1.** In fact, from equations (1.6) and (1.10), we find that $S_n(z,y) = H_n(y, D_z^{-1})$, where $H_n(y, D_z^{-1})$ are the two-variable Hermite Kampé de Feriet polynomials defined by [3]

$$e^{yt+D_z^{-1}t^2} = \sum_{n=0}^{\infty} H_n(y, D_z^{-1}) \frac{t^n}{n!}.$$

To introduce the multivariable hybrid polynomials and to characterize their properties via different generating function methods is an interesting approach. These polynomials may be useful in certain problems of number theory, combinatorics, numerical analysis, theoretical physics, approximation theory and other fields of pure and applied mathematics. This gives motivation to introduce a new hybrid family of three-variable Legendre – Laguerre – Appell polynomials (3VLeLAP). The series expansion, determinant form, recurrence relations, shift operators and differential equations for these polynomials are derived. Certain applications are framed in order to give the results for the three-variable Legendre – Laguerre – Bernoulli, – Euler and – Genocchi polynomials. The integral equations for the Legendre – Laguerre – Appell and other hybrid polynomials are also established.

2. Legendre – Laguerre based hybrid polynomials. First, we introduce a hybrid family of the three-variable Legendre – Laguerre polynomials (3VLeLP) by making use of replacement technique and slightly focus on proving some properties related to these polynomials.

Expanding the exponential function and replacing the powers of y, *that is* y^n, $n = 0, 1, 2, \ldots,$ *by the polynomials* $S_n(z, y)$, $n = 0, 1, 2, \ldots,$ *in equation* (1.5) *and then using equation* (1.10), *we get the following generating function for the 3VLeLP*:

The 3VLeVP ${}_SL_n(x, z, y)$ are defined by means of the generating function

$$e^{yt} C_0(xt) C_0(-zt^2) = \sum_{n=0}^{\infty} {}_SL_n(x, z, y) \frac{t^n}{n!}. \tag{2.1}$$

Using equations (1.5) and (1.7) or (1.10) and (1.7) appropriately in equation (2.1) and after simplification, we get the following series expansions for the 3VLeLP ${}_SL_n(x, y, z)$:

$${}_SL_n(x, z, y) = n! \sum_{k=0}^{[n/2]} \frac{L_{n-2k}(x, y) z^k}{(n-2k)!\,(k!)^2} \qquad \left(\text{or} \quad = n! \sum_{k=0}^{n} \frac{(-1)^k x^k S_{n-k}(z, y)}{(n-k)!\,(k!)^2}\right), \tag{2.2}$$

which in view of equations (1.8) or (1.11) can also be expressed as

$${}_SL_n(x, z, y) = n! \sum_{k,\ l=0}^{k+2l \leq n} \frac{z^l (-x)^k\, y^{n-k-2l}}{(n-k-2l)!\,(k!)^2 (l!)^2}.$$

Using equations (1.9) and (1.11) or (1.12) and (1.8) appropriately in equation (2.2) gives the following operational representations for the 3VLeLP ${}_SL_n(x, z, y)$:

$${}_SL_n(x, z, y) = \exp\left(-D_x^{-1} \frac{\partial}{\partial y}\right) \{S_n(z, y)\} \qquad \left(\text{or} \quad = \exp\left(D_z^{-1} \frac{\partial^2}{\partial y^2}\right) \{L_n(x, y)\}\right),$$

which on using equations (1.12) or (1.9) can also be expressed as

$${}_SL_n(x, z, y) = \exp\left(D_z^{-1} \frac{\partial^2}{\partial y^2} - D_x^{-1} \frac{\partial}{\partial y}\right) \{y^n\}.$$

Now, we introduce a hybrid family of 3VLeLAP via generating function, series expansions and determinant definition. For this, we prove the following results.

Theorem 2.1. *The 3VLeLAP are defined by the generating function*

$$\mathcal{R}(t)e^{yt}C_0(xt)C_0(-zt^2) = \sum_{n=0}^{\infty} {}_{S}{}_{L}\mathcal{R}_n(x,z,y)\frac{t^n}{n!}. \tag{2.3}$$

Proof. Expanding the exponential function e^{yt} and then replacing the powers of y, i.e., $y^0, y^1, y^2, \ldots, y^n$ by the polynomials ${}_{S}L_0(x,z,y),\ {}_{S}L_1(x,z,y), \ldots,\ {}_{S}L_n(x,z,y)$ in the left-hand side and y by the polynomial ${}_{S}L_1(x,z,y)$ in the right-hand side of equation (1.3) and after summing up the terms in the left-hand side of the resultant equation, we have

$$\mathcal{R}(t)\sum_{n=0}^{\infty} {}_{S}L_n(x,z,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{R}_n({}_{S}L_1(x,z,y))\frac{t^n}{n!},$$

which on using equation (2.1) in the left-hand side and denoting the resultant 3VLeLAP in the right-hand side by ${}_{S}{}_{L}\mathcal{R}_n(x,z,y)$ that is

$${}_{S}{}_{L}\mathcal{R}_n(x,z,y) := \mathcal{R}_n\{{}_{S}L_1(x,z,y)\}, \tag{2.4}$$

we get generating function (2.3).

Theorem 2.2. *The 3VLeLAP are defined by the series expansion*

$${}_{S}{}_{L}\mathcal{R}_n(x,z,y) = n!\sum_{k=0}^{n}\sum_{l=0}^{[n/2]} \frac{\mathcal{R}_{n-k-2l}(y)(-x)^k z^l}{(n-k-2l)!\,(k!)^2\,(l!)^2}. \tag{2.5}$$

Proof. Using equations (1.3) and (1.7) in the left-hand side of equation (2.3) and applying the Cauchy-product rule and then comparing the coefficients of like powers of $t^n/n!$ gives series expansion (2.5).

Theorem 2.3. *The 3VLeLAP of degree n are defined by*

$${}_{S}{}_{L}\mathcal{R}_0(x,z,y) = \frac{1}{\beta_0},$$

$${}_{S}{}_{L}\mathcal{R}_n(x,z,y) =$$

$$= \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & {}_{S}L_1(x,z,y) & {}_{S}L_2(x,z,y) & \ldots & {}_{S}L_{n-1}(x,z,y) & {}_{S}L_n(x,z,y) \\ \beta_0 & \beta_1 & \beta_2 & \ldots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \ldots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \ldots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{2.6}$$

where $n = 1, 2, \ldots,\ \beta_0, \beta_1, \ldots, \beta_n \in \mathbb{R},\ \beta_0 \neq 0.$

Proof. Replacing the powers $y^0, y^1, y^2, \ldots, y^n$ by the polynomials ${}_SL_0(x,z,y)$, ${}_SL_1(x,z,y), \ldots, {}_SL_n(x,z,y)$ in the right-hand side and y by the polynomial ${}_SL_1(x,z,y)$ in the left-hand side of determinant definition of the Appell polynomials ([6, p. 1533], (29), (30)) and then using equation (2.4) in the left-hand side of resultant equation, we obtain determinant definition (2.6).

Further, we focus on obtaining recurrence relations and shift operators for the 3VLeLAP ${}_{SL}\mathcal{R}_n(x,z,y)$. For this, we prove the following results.

Theorem 2.4. *The 3VLeLAP ${}_{SL}\mathcal{R}_n(x,z,y)$ satisfy the recurrence relation*

$$ {}_{SL}\mathcal{R}_{n+1}(x,z,y) = \left(y + \alpha_0 - D_x^{-1}\right){}_{SL}\mathcal{R}_n(x,z,y) + $$

$$ +2nD_z^{-1}{}_{SL}\mathcal{R}_{n-1}(x,z,y) + \sum_{k=1}^{n} \binom{n}{k} \alpha_k \, {}_{SL}\mathcal{R}_{n-k}(x,z,y), \tag{2.7} $$

where the coefficients $\{\alpha_k\}_{k\in\mathbb{N}_0}$ are given by expansions

$$ \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}. \tag{2.8} $$

Proof. We consider generating function (2.3) in the form

$$ \mathcal{R}(t) e^{(y-D_x^{-1})t + D_z^{-1}t^2} = \sum_{n=0}^{\infty} {}_{SL}\mathcal{R}_n(x,z,y) \frac{t^n}{n!}, $$

which on differentiating both sides with respect to t and using equations (2.3) and (2.8) and then applying the Cauchy-product rule in the left-hand side of the resultant equation, it follows that

$$ \sum_{n=0}^{\infty} {}_{SL}\mathcal{R}_{n+1}(x,z,y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \Bigg((y + \alpha_0 - D_x^{-1}){}_{SL}\mathcal{R}_n(x,z,y) + $$

$$ +2nD_z^{-1}{}_{SL}\mathcal{R}_{n-1}(x,z,y) + \sum_{k=1}^{n} \binom{n}{k} \alpha_k \, {}_{SL}\mathcal{R}_{n-k}(x,z,y) \Bigg) \frac{t^n}{n!}. $$

Equating the coefficients of like powers of $t^n/n!$ on both sides of the above equation yields recurrence relation (2.7).

Theorem 2.5. *The shift operators for the 3VLeLAP ${}_{SL}\mathcal{R}_n(x,z,y)$ are given by*

$$ {}_y\pounds_n^- := \frac{1}{n} D_y, \tag{2.9} $$

$$ {}_x\pounds_n^- := -\frac{1}{n} D_x, \tag{2.10} $$

$$ {}_z\pounds_n^- := \frac{1}{n} D_y^{-1} D_z, \tag{2.11} $$

$$ {}_y\pounds_n^+ := y + \alpha_0 - D_x^{-1} + 2D_z^{-1}D_y + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_y^k, \tag{2.12} $$

$$ {}_x\pounds_n^+ := y + \alpha_0 - D_x^{-1} - 2D_z^{-1}D_x + \sum_{k=1}^{n}(-1)^k\frac{\alpha_k}{k!}D_x^k, \qquad (2.13) $$

$$ {}_z\pounds_n^+ := y + \alpha_0 - D_x^{-1} + 2D_y^{-1} + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^{-k}D_z^k. \qquad (2.14) $$

Proof. Differentiating both sides of generating relation (2.3) with respect to y and then simplifying it follows that

$$ {}_y\pounds_n^-\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\} = \frac{1}{n}D_y\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\} = {}_{SL}\mathcal{R}_{n-1}(x,z,y), \qquad (2.15) $$

which proves assertion (2.9).

Again, differentiating both sides of equation (2.3) with respect to x and on simplification, we find

$$ {}_x\pounds_n^-\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\} = \frac{-1}{n}D_x\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\} = {}_{SL}\mathcal{R}_{n-1}(x,z,y), \qquad (2.16) $$

which gives assertion (2.10).

Further, differentiating both sides of generating function (2.3) with respect to z and after simplification of the resultant equation, we get

$$ {}_z\pounds_n^-\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\} = \frac{1}{n}D_y^{-1}D_z\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\} = {}_{SL}\mathcal{R}_{n-1}(x,z,y), \qquad (2.17) $$

which yields assertion (2.11).

Using equation (2.15) in the relation

$$ {}_{SL}\mathcal{R}_{n-k}(x,z,y) = \big(\pounds_{n-k+1}^-\pounds_{n-k+2}^-\ldots\pounds_{n-1}^-\pounds_n^-\big)\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\}, \qquad (2.18) $$

gives

$$ {}_{SL}\mathcal{R}_{n-k}(x,z,y) = \frac{(n-k)!}{n!}D_y^k\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\}. \qquad (2.19) $$

Making use of equation (2.19) in recurrence relation (2.7) and in view of the fact that

$$ \pounds_n^+\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\} = {}_{SL}\mathcal{R}_{n+1}(x,z,y), \qquad (2.20) $$

we obtain

$$ {}_y\pounds_n^+\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\} = \left(y + \alpha_0 - D_x^{-1} + 2D_z^{-1}D_y + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^k\right)\big\{{}_{SL}\mathcal{R}_n(x,z,y)\big\} = $$

$$ = {}_{SL}\mathcal{R}_{n+1}(x,z,y), $$

which proves assertion (2.12).

In order to derive the expression for raising operator (2.13), we use equation (2.16) in relation (2.18) and on simplification, we have

$$ {}_{S}L\mathcal{R}_{n-k}(x,z,y) = (-1)^k \frac{(n-k)!}{n!} D_x^k\big\{{}_{S}L\mathcal{R}_n(x,z,y)\big\}, $$

which on using in recurrence relation (2.7) and taking help of relation (2.20) gives

$$ {}_{x}\pounds_n^{+}\big\{{}_{S}L\mathcal{R}_n(x,z,y)\big\} = $$

$$ = \left(y+\alpha_0 - D_x^{-1} - 2D_z^{-1}D_x + \sum_{k=1}^{n}(-1)^k \frac{\alpha_k}{k!} D_x^k\right)\big\{{}_{S}L\mathcal{R}_n(x,z,y)\big\} = {}_{S}L\mathcal{R}_{n+1}(x,z,y), $$

which proves assertion (2.13).

Similarly, using equation (2.17) in relation (2.18) and after simplification it follows that

$$ {}_{S}L\mathcal{R}_{n-k}(x,z,y) = \frac{(n-k)!}{n!} D_y^{-k} D_z^k\big\{{}_{S}L\mathcal{R}_n(x,z,y)\big\}. \tag{2.21} $$

Further, in view of equations (2.21), (2.7) and (2.20), we get

$$ {}_{z}\pounds_n^{+}\big\{{}_{S}L\mathcal{R}_n(x,z,y)\big\} = $$

$$ = \left(y+\alpha_0 - D_x^{-1} + 2D_y^{-1} + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_y^{-k} D_z^k\right)\big\{{}_{S}L\mathcal{R}_n(x,z,y)\big\} = {}_{S}L\mathcal{R}_{n+1}(x,z,y), $$

which led to assertion (2.14).

Next, we establish the differential, integro-differential and partial differential equations for the 3VLeLAP ${}_{S}L\mathcal{R}_n(x,z,y)$.

Theorem 2.6. *The 3VLeLAP ${}_{S}L\mathcal{R}_n(x,z,y)$ satisfy the differential equation*

$$ \left(xyD_x^2 - (x-y)D_x - \alpha_0 D_y - 2zD_z - \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_y^{k+1} + n\right){}_{S}L\mathcal{R}_n(x,z,y) = 0. \tag{2.22} $$

Proof. Consider the factorization relation

$$ \mathcal{L}_{n+1}^{-}\mathcal{L}_n^{+}\big\{{}_{S}L\mathcal{R}_n(x,z,y)\big\} = {}_{S}L\mathcal{R}_n(x,z,y). \tag{2.23} $$

Now, making use of operators (2.9) and (2.12) in above equation and taking help of the relation

$$ (y - D_x^{-1})D_y = -xyD_x^2 + (x-y)D_x, \qquad D_y^2 = D_z z D_z, $$

we are led to differential equation (2.22).

Theorem 2.7. *The 3VLeLAP ${}_{S}L\mathcal{R}_n(x,z,y)$ satisfy the integro-differential equations*

$$ \left((y+\alpha_0)D_x - 2D_z^{-1} + \sum_{k=1}^{n}(-1)^k \frac{\alpha_k}{k!} D_x^{k+1} + n\right){}_{S}L\mathcal{R}_n(x,z,y) = 0, \tag{2.24} $$

$$ \left(\left(y+\alpha_0 - D_x^{-1} + 2D_y^{-1}\right)D_z + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_y^{-k} D_z^{k+1} - (n+1)D_y\right){}_{S}L\mathcal{R}_n(x,z,y) = 0. \tag{2.25} $$

Proof. Using expressions (2.10), (2.13) and (2.11), (2.14), respectively, in relation (2.23) give integro-differential equations (2.24) and (2.25).

Table 2.1. Result for ${}_{H L}\mathcal{R}_n(x, y, D_z^{-1})$

S.No.	Result	Expression
I	generating function	$e^{yt+D_z^{-1}t^2} C_0(xt) = {}_H L_n(x, y, D_z^{-1}) \dfrac{t^n}{n!}$
II	series definition	${}_{HL}\mathcal{R}_n(x, y, D_z^{-1}) = n! \sum\limits_{k=0}^{n} \sum\limits_{l=0}^{[n/2]} \dfrac{\mathcal{R}_{n-k-2l}(D_z^{-1})(-x)^k y^l}{(n-k-2l)!\ (k!)^2\ (l!)^2}$
III	recurrence relation	${}_{HL}\mathcal{R}_{n+1}(x, y, D_z^{-1}) = \left(D_z^{-1} + \alpha_0 - D_x^{-1}\right) {}_{HL}\mathcal{R}_n(x, y, D_z^{-1}) +$ $+ 2n D_y^{-1} {}_{HL}\mathcal{R}_{n-1}(x, y, D_z^{-1}) + \sum\limits_{k=1}^{n} \binom{n}{k} \alpha_k \ {}_{HL}\mathcal{R}_{n-k}(x, y, D_z^{-1})$
IV	shift operators	${}_{D_z^{-1}}\mathcal{L}_n^- := \dfrac{1}{n} D_{D_z^{-1}}$ ${}_x\mathcal{L}_n^- := -\dfrac{1}{n} D_x$ ${}_y\mathcal{L}_n^- := \dfrac{1}{n} D^{-1}_{D_z^{-1}} D_y,$ ${}_{D_z^{-1}}\mathcal{L}_n^+ := D_z^{-1} + \alpha_0 - D_x^{-1} + 2D_y^{-1} D_{D_z^{-1}} + \sum\limits_{k=1}^{n} \dfrac{\alpha_k}{k!} D^k_{D_z^{-1}}$ ${}_x\mathcal{L}_n^+ := y + \alpha_0 - D_x^{-1} - 2D_z^{-1} D_x + \sum\limits_{k=1}^{n} (-1)^k \dfrac{\alpha_k}{k!} D_x^k$ ${}_y\mathcal{L}_n^+ := D_z^{-1} + \alpha_0 - D_x^{-1} + 2D^{-1}_{D_z^{-1}} + \sum\limits_{k=1}^{n} \dfrac{\alpha_k}{k!} D^{-k}_{D_z^{-1}} D_y^k$
V	differential equation	$\Big(x D_z^{-1} D_x^2 - (x - D_z^{-1}) D_x - \alpha_0 D_{D_z^{-1}} - 2y D_y -$ $- \sum\limits_{k=1}^{n} \dfrac{\alpha_k}{k!} D^{k+1}_{D_z^{-1}} + n \Big) {}_{HL}\mathcal{R}_n\big(x, y, D_z^{-1}\big) = 0$
VI	integro-differential equations	$\Big(\big(D_z^{-1} + \alpha_0\big) D_x - 2D_y^{-1} + \sum\limits_{k=1}^{n} (-1)^k \dfrac{\alpha_k}{k!} D_x^{k+1} + n \Big) {}_{HL}\mathcal{R}_n(x, y, D_z^{-1}) = 0$ $\Big(\big(D_z^{-1} + \alpha_0 - D_x^{-1} + 2D^{-1}_{D_z^{-1}}\big) D_y + \sum\limits_{k=1}^{n} \dfrac{\alpha_k}{k!} D^{-k}_{D_z^{-1}} D_y^{k+1} -$ $- (n+1) D_{D_z^{-1}} \Big) {}_{HL}\mathcal{R}_n\big(x, y, D_z^{-1}\big) = 0$
VII	partial-differential equations	$\Big(\big(D_z^{-1} + \alpha_0\big) D_y^n D_x - 2D_y^{n-1} +$ $+ \sum\limits_{k=1}^{n} (-1)^k \dfrac{\alpha_k}{k!} D_y^n D_x^{k+1} + n D_y^n \Big) {}_{HL}\mathcal{R}_n(x, y, D_z^{-1}) = 0$ $\Big(\big(D_z^{-1} + \alpha_0 - D_x^{-1}\big) D^n_{D_z^{-1}} D_y + (n+2) D^{n-1}_{D_z^{-1}} D_y + 2D^{n-1}_{D_z^{-1}} D_y +$ $+ \sum\limits_{k=1}^{n} \dfrac{\alpha_k}{k!} D^{n-k}_{D_z^{-1}} D_y^{k+1} -$ $-(n+1) D^{n+1}_{D_z^{-1}} \Big) {}_{HL}\mathcal{R}_n(x, y, D_z^{-1}) = 0$

Theorem 2.8. *The 3VLeLAP* ${}_{S L}\mathcal{R}_n(x,z,y)$ *satisfy the partial-differential equations*

$$\left((y+\alpha_0)D_z^nD_x - 2D_z^{n-1} + \sum_{k=1}^{n}(-1)^k\frac{\alpha_k}{k!}D_z^nD_x^{k+1} + nD_z^n\right){}_{S L}\mathcal{R}_n(x,z,y) = 0, \tag{2.26}$$

$$\left(\left(y+\alpha_0 - D_x^{-1}\right)D_y^nD_z + (n+2)D_y^{n-1}D_z + 2D_y^{n-1}D_z + \right.$$

$$\left. + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^{n-k}D_z^{k+1} - (n+1)D_y^{n+1}\right){}_{S L}\mathcal{R}_n(x,z,y) = 0. \tag{2.27}$$

Proof. Differentiating equation (2.24) n times with respect to z and equation (2.25) n times with respect to y, respectively, yields assertions (2.26) and (2.27).

Remark **2.1.** From Remark 1.1 we conclude that the 3VLeLP ${}_S L_n(x,z,y)$ reduce to the three-variable Hermite – Laguerre polynomials (3VHLP) ${}_H L_n(x,y,D_z^{-1})$. In view of this fact, we find that the 3VLeLAP ${}_{S L}\mathcal{R}_n(x,z,y)$ reduce to the three-variable Hermite – Laguerre – Appell polynomials (3VHLAP) ${}_{H L}\mathcal{R}_n(x,y,D_z^{-1})$. We present the results for 3VHLAP in Table 2.1.

We note that by taking $\alpha_k = 0,\ k = 0,1,\ldots,n,$ in the results derived above, we can easily find the corresponding results for the 3VLeLP ${}_S L_n(x,z,y)$ and 3VHLP ${}_H L_n(x,y,D_z^{-1})$. Thus, we omit them.

In the next section, certain examples are constructed as applications of the results derived above.

3. Applications.

We study the analogous results for some members of the 3VLeLAP ${}_{S L}\mathcal{R}_n(x,z,y)$ by considering the following examples.

Example **3.1.** Taking $\mathcal{R}(t) = \left(\frac{t}{e^t-1}\right)$ and $\mathcal{R}_n(y) = B_n(y)$ in generating function (2.3) of the 3VLeLAP ${}_{S L}\mathcal{R}_n(x,z,y)$, we find the three-variable Legendre – Laguerre – Bernoulli polynomials (3VLeLBP) ${}_{S L}B_n(x,z,y)$, which are defined by the generating function

$$\left(\frac{t}{e^t-1}\right)e^{yt}C_0(xt)C_0(-zt^2) = \sum_{n=0}^{\infty}{}_{S L}B_n(x,z,y)\frac{t^n}{n!}.$$

The other results for the 3VLeLBP ${}_{S L}B_n(x,z,y)$ can be obtained by making the substitutions

$$\mathcal{R}_n(y) = B_n(y), \qquad \mathcal{R}(t) = \frac{t}{e^t-1} \quad \text{so that} \quad \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} = -\sum_{n=0}^{\infty}\frac{B_{n+1}(1)}{n+1}\frac{t^n}{n!}$$

$$\Rightarrow \quad \alpha_n = -\frac{B_{n+1}(1)}{n+1} \quad (n \geq 1), \qquad \alpha_0 = -\frac{1}{2}, \qquad \alpha_1 = -\frac{1}{12}$$

in equations (2.5), (2.7), (2.9) – (2.14), (2.22) and (2.24) – (2.27). We present these results in Table 3.1.

The determinant definition of the 3VLeLBP ${}_{S L}B_n(x,z,y)$ can be obtained by substituting $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1},\ i = 1,2,\ldots,n,$ (for which the determinant definition of the Appell polynomials reduce to the Bernoulli polynomials [5, 6]) in determinant definition (2.6) of the 3VLeLAP ${}_{S L}\mathcal{R}_n(x,z,y)$.

Table 3.1. Results for ${}_{SL}B_n(x,z,y)$

S.No.	Result	Expression
I	series definition	${}_{SL}B_n(x,z,y) = n! \sum_{k=0}^{n} \sum_{l=0}^{[n/2]} \frac{B_{n-k-2l}(y)(-x)^k z^l}{(n-k-2l)!\,(k!)^2\,(l!)^2}$
II	recurrence relation	${}_{SL}B_{n+1}(x,z,y) = \left(y - \frac{1}{2} - D_x^{-1}\right) {}_{SL}B_n(x,z,y) +$ $+ 2nD_z^{-1}\, {}_{SL}B_{n-1}(x,z,y) -$ $- \sum_{k=1}^{n} \binom{n}{k} \frac{B_{k+1}(1)}{k+1}\ {}_{SL}B_{n-k}(x,z,y)$
III	shift operators	${}_y\pounds_n^- := \frac{1}{n} D_y$ ${}_x\pounds_n^- := -\frac{1}{n} D_x$ ${}_z\pounds_n^- := \frac{1}{n} D_y^{-1} D_z$ ${}_y\pounds_n^+ := y - \frac{1}{2} - D_x^{-1} + 2D_z^{-1} D_y - \sum_{k=1}^{n} \frac{B_{k+1}(1)}{(k+1)!} D_y^k$ ${}_x\pounds_n^+ := y - \frac{1}{2} - D_x^{-1} - 2D_z^{-1} D_x - \sum_{k=1}^{n} (-1)^k \frac{B_{k+1}(1)}{(k+1)!} D_x^k$ ${}_z\pounds_n^+ := y - \frac{1}{2} - D_x^{-1} + 2D_y^{-1} - \sum_{k=1}^{n} \frac{B_{k+1}(1)}{(k+1)!} D_y^{-k} D_z^k$
IV	differential equation	$\Big(xyD_x^2 - (x-y)D_x + \frac{1}{2} D_y - 2zD_z +$ $+ \sum_{k=1}^{n} \frac{B_{k+1}(1)}{(k+1)!} D_y^{k+1} + n\Big) {}_{SL}B_n(x,z,y) = 0$
V	integro-differential equations	$\Big(\left(y - \frac{1}{2}\right) D_x - 2D_z^{-1} -$ $- \sum_{k=1}^{n} (-1)^k \frac{B_{k+1}(1)}{(k+1)!} D_x^{k+1} + n\Big) {}_{SL}B_n(x,z,y) = 0$ $\Big(\left(y - \frac{1}{2} - D_x^{-1} + 2D_y^{-1}\right) D_z -$ $- \sum_{k=1}^{n} \frac{B_{k+1}(1)}{(k+1)!} D_y^{-k} D_z^{k+1} - (n+1)D_y\Big) {}_{SL}B_n(x,y,z) = 0$
VI	partial-differential equations	$\Big(\left(y - \frac{1}{2}\right) D_z^n D_x - 2D_z^{n-1} - \sum_{k=1}^{n} (-1)^k \frac{B_{k+1}(1)}{(k+1)!} D_z^n D_x^{k+1} +$ $+ nD_z^n\Big) {}_{SL}B_n(x,z,y) = 0$ $\Big(\left(y - \frac{1}{2} - D_x^{-1}\right) D_y^n D_z + (n+2) D_y^{n-1} D_z + 2D_y^{n-1} D_z -$ $- \sum_{k=1}^{n} \frac{B_{k+1}(1)}{(k+1)!} D_y^{n-k} D_z^{k+1} - (n+1) D_y^{n+1}\Big) {}_{SL}B_n(x,z,y) = 0$

***Example* 3.2.** Taking $\mathcal{R}(t) = \left(\dfrac{2}{e^t+1}\right)$ and $\mathcal{R}_n(y) = E_n(y)$ in generating function (2.3) of the 3VLeLAP ${}_{S L}\mathcal{R}_n(x,z,y)$, we find the three-variable Legendre – Laguerre – Euler polynomials (3VLeLEP) ${}_{S L}E_n(x,z,y)$, which are defined by the generating function

$$\left(\frac{2}{e^t+1}\right)e^{yt}C_0(xt)C_0(-zt^2) = \sum_{n=0}^{\infty} {}_{S L}E_n(x,z,y)\frac{t^n}{n!}.$$

The other results for the 3VLeLEP ${}_{S L}E_n(x,z,y))$ can be obtained by making the substitutions

$$\mathcal{R}_n(y) = E_n(y), \qquad \mathcal{R}(t) = \frac{2}{e^t+1} \quad \text{so that} \quad \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} = \sum_{n=0}^{\infty} \frac{\mathcal{E}_n}{2}\frac{t^n}{n!}$$

$$\Rightarrow \quad \alpha_n = \frac{\mathcal{E}_n}{2} \quad (n \geq 1), \quad \alpha_0 = -\frac{1}{2}, \quad \alpha_1 = -\frac{1}{2} \qquad \left(\mathcal{E}_n = \frac{-1}{2^n}\sum_{k=0}^{n}\binom{n}{k}E_{n-k}\right)$$

in equations (2.5), (2.7), (2.9) – (2.14), (2.22) and (2.24) – (2.27). We present these results in Table 3.2.

The determinant definition of the 3VLeLEP ${}_{S L}E_n(x,z,y)$ can be obtained by substituting $\beta_0 = 1$ and $\beta_i = \dfrac{1}{2}$, $i = 1,2,\ldots,n$ (for which the determinant definition of the Appell polynomials reduce to the Euler polynomials [6]) in determinant definition (2.6) of the 3VLeLAP ${}_{S L}\mathcal{R}_n(x,z,y)$.

***Example* 3.3.** Taking $\mathcal{R}(t) = \left(\dfrac{2t}{e^t+1}\right)$ and $\mathcal{R}_n(y) = G_n(y)$ in generating function (2.3) of the 3VLeLAP ${}_{S L}\mathcal{R}_n(x,z,y)$, we find the three-variable Legendre – Laguerre – Genocchi polynomials (3VLeLGP) ${}_{S L}G_n(x,z,y)$, which are defined by the generating function

$$\left(\frac{2t}{e^t+1}\right)e^{yt}C_0(xt)C_0(-zt^2) = \sum_{n=0}^{\infty} {}_{S L}G_n(x,z,y)\frac{t^n}{n!}.$$

The other results for the 3VLeLGP ${}_{S L}G_n(x,z,y)$ can be obtained by making the substitutions

$$\mathcal{R}_n(y) = G_n(y), \qquad \mathcal{R}(t) = \frac{2t}{e^t+1} \quad \text{so that} \quad \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} = \sum_{n=0}^{\infty} \frac{G_n}{2}\frac{t^n}{n!}$$

$$\Rightarrow \quad \alpha_n = \frac{G_n}{2} \quad (n \geq 2), \quad \alpha_0 = 1, \quad \alpha_1 = -1$$

in equations (2.5), (2.7), (2.9) – (2.14), (2.22) and (2.24) – (2.27). We present these results in Table 3.3.

The determinant definition of the 3VLeLGP ${}_{S L}G_n(x,z,y)$ can be obtained by substituting $\beta_0 = 1$ and $\beta_i = \dfrac{1}{2(i+1)}$, $i = 1,2,\ldots,n$ (for which the determinant definition of the Appell polynomials reduce to the Genocchi polynomials) in determinant definition (2.6) of the 3VLeLAP ${}_{S L}\mathcal{R}_n(x,z,y)$.

In view of Remark 2.1, we note that the corresponding results for the three-variable Hermite – Laguerre – Bernoulli, – Euler and – Genocchi polynomials ${}_{H L}B_n(x,z,y)$, ${}_{H L}E_n(x,z,y)$ and ${}_{H L}G_n(x,z,y)$, respectively, can be obtained easily. Thus, we omit them.

In the next section, we derive the Volterra integral equations for the 3VLeLAP and for their relatives.

Table 3.2. Results for ${}_{S L}E_n(x,z,y)$

S.No.	Result	Expression
I	series definition	${}_{S L}E_n(x,z,y) = n! \sum_{k=0}^{n} \sum_{l=0}^{[n/2]} \frac{E_{n-k-2l}(y)(-x)^k z^l}{(n-k-2l)!\ (k!)^2\ (l!)^2}$
II	recurrence relation	${}_{S L}E_{n+1}(x,z,y) = \left(y - \frac{1}{2} - D_x^{-1}\right) {}_{S L}E_n(x,z,y) +$ $+ 2nD_z^{-1}\, {}_{S L}E_{n-1}(x,z,y) + \sum_{k=1}^{n} \binom{n}{k} \frac{\mathcal{E}_k}{2}\, {}_{S L}E_{n-k}(x,z,y)$
III	shift operators	${}_y\pounds_n^- := \frac{1}{n} D_y$ ${}_x\pounds_n^- := -\frac{1}{n} D_x$ ${}_z\pounds_n^- := \frac{1}{n} D_y^{-1} D_z$ ${}_y\pounds_n^+ := y - \frac{1}{2} - D_x^{-1} + 2D_z^{-1} D_y + \sum_{k=1}^{n} \frac{\mathcal{E}_k}{2\ k!} D_y^k$ ${}_x\pounds_n^+ := y - \frac{1}{2} - D_x^{-1} - 2D_z^{-1} D_x + \sum_{k=1}^{n} (-1)^k \frac{\mathcal{E}_k}{2\ k!} D_x^k$ ${}_z\pounds_n^+ := y - \frac{1}{2} - D_x^{-1} + 2D_y^{-1} + \sum_{k=1}^{n} \frac{\mathcal{E}_k}{2\ k!} D_y^{-k} D_z^k$
IV	differential equation	$\Big(xyD_x^2 - (x-y)D_x + \frac{1}{2} D_y - 2zD_z - \sum_{k=1}^{n} \frac{\mathcal{E}_k}{2\ k!} D_y^{k+1} +$ $+ n\Big) {}_{S L}E_n(x,z,y) = 0$
V	integro-differential equations	$\Big(\left(y - \frac{1}{2}\right) D_x - 2D_z^{-1} + \sum_{k=1}^{n} (-1)^k \frac{\mathcal{E}_k}{2\ k!} D_x^{k+1} +$ $+ n\Big) {}_{S L}E_n(x,z,y) = 0$ $\Big(\left(y - \frac{1}{2} - D_x^{-1} + 2D_y^{-1}\right) D_z + \sum_{k=1}^{n} \frac{\mathcal{E}_k}{2\ k!} D_y^{-k} D_z^{k+1} -$ $- (n+1) D_y\Big) {}_{S L}E_n(x,z,y) = 0$
VI	partial-differential equations	$\Big(\left(y - \frac{1}{2}\right) D_z^n D_x - 2D_z^{n-1} + \sum_{k=1}^{n} (-1)^k \frac{\mathcal{E}_k}{2\ k!} D_z^n D_x^{k+1} +$ $+ nD_z^n\Big) {}_{S L}E_n(x,z,y) = 0$ $\Big(\left(y - \frac{1}{2} - D_x^{-1}\right) D_y^n D_z + (n+2) D_y^{n-1} D_z + 2D_y^{n-1} D_z +$ $+ \sum_{k=1}^{n} \frac{\mathcal{E}_k}{2\ k!} D_y^{n-k} D_z^{k+1} - (n+1) D_y^{n+1}\Big) {}_{S L}E_n(x,z,y) = 0$

Table 3.3. Results for ${}_{S L}G_n(x,z,y)$

S.No.	Result	Expression
I	series definition	${}_{S L}G_n(x,z,y)=n!\sum_{k=0}^{n}\sum_{l=0}^{[n/2]}\frac{G_{n-k-2l}(y)(-x)^k z^l}{(n-k-2l)!\,(k!)^2\,(l!)^2}$
II	recurrence relation	${}_{S L}G_{n+1}(x,z,y)=\big(y+1-D_x^{-1}\big){}_{S L}G_n(x,z,y)+$ $+\,2nD_z^{-1}\,{}_{S L}\mathcal{G}_{n-1}(x,z,y)+$ $+\sum_{k=1}^{n}\binom{n}{k}\frac{G_k}{2}\,{}_{S L}G_{n-k}(x,z,y)$
III	shift operators	${}_y\pounds_n^-:=\frac{1}{n}D_y$ ${}_x\pounds_n^-:=-\frac{1}{n}D_x$ ${}_z\pounds_n^-:=\frac{1}{n}D_y^{-1}D_z$ ${}_y\pounds_n^+:=y+1-D_x^{-1}+2D_z^{-1}D_y+\sum_{k=1}^{n}\frac{G_k}{2\,k!}D_y^k$ ${}_x\pounds_n^+:=y+1-D_x^{-1}-2D_z^{-1}D_x+\sum_{k=1}^{n}(-1)^k\frac{G_k}{2\,k!}D_x^k$ ${}_z\pounds_n^+:=y+1-D_x^{-1}+2D_y^{-1}+\sum_{k=1}^{n}\frac{G_k}{2\,k!}D_y^{-k}D_z^k$
IV	differential equation	$\Big(xyD_x^2-(x-y)D_x-D_y-2zD_z-\sum_{k=1}^{n}\frac{G_k}{2\,k!}D_y^{k+1}+$ $+n\Big){}_{S L}G_n(x,z,y)=0$
V	integro-differential equations	$\Big((y+1)D_x-2D_z^{-1}+\sum_{k=1}^{n}(-1)^k\frac{G_k}{2\,k!}D_x^{k+1}+n\Big){}_{S L}G_n(x,z,y)=0$ $\Big(\big(y+1-D_x^{-1}+2D_y^{-1}\big)D_z+\sum_{k=1}^{n}\frac{G_k}{2\,k!}D_y^{-k}D_z^{k+1}-$ $-(n+1)D_y\Big){}_{S L}G_n(x,z,y)=0$
VI	partial-differential equations	$\Big((y+1)D_z^nD_x-2D_z^{n-1}+\sum_{k=1}^{n}(-1)^k\frac{G_k}{2\,k!}D_z^nD_x^{k+1}+$ $+nD_z^n\Big){}_{S L}G_n(x,z,y)=0$ $\Big(\big(y+1-D_x^{-1}\big)D_y^nD_z+(n+2)D_y^{n-1}D_z+2D_y^{n-1}D_z+$ $+\sum_{k=1}^{n}\frac{G_k}{2\,k!}D_y^{n-k}D_z^{k+1}-(n+1)D_y^{n+1}\Big){}_{S L}G_n(x,z,y)=0$

4. Volterra integral equations. Integral equations arise in many scientific and engineering problems, such as diffraction problems scattering in quantum mechanics, conformal mapping and water waves etc. In order to further stress the importance of integral equations, we derive the integral equations for the 3VLeLAP by proving the following result.

Theorem 4.1. *The 3VLeLAP satisfy the homogeneous Volterra integral equation*

$$\phi(y)=\frac{1}{\alpha_1}\big(xyD_x^2-(x-y)D_x-2zD_z+n\big)\times$$

$$\times\left(n\sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{n-r-1}{k}(-1)^r\mathcal{R}_k\frac{x^r}{r!}\times\right.$$

$$\left.\times y^{n-k-r-1}+\sum_{r=0}^{n}\sum_{k=0}^{n-r}\binom{n}{r}\binom{n-r}{k}(-1)^r\mathcal{R}_k\frac{x^r}{r!}y^{n-k-r}\right)-$$

$$-\frac{\alpha_0}{\alpha_1}n\sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{n-r-1}{k}(-1)^r\mathcal{R}_k\frac{x^r}{r!}y^{n-k-r-1}+$$

$$+\int\limits_0^y\left(\frac{1}{\alpha_1}\big(xyD_x^2-(x-y)D_x-2zD_z+n\big)(y-\xi)+\frac{\alpha_0}{\alpha_1}\right)\phi(\xi)d\xi. \tag{4.1}$$

Proof. We consider the following second order differential equation of the 3VLeLAP:

$$\left(D_y^2+\frac{\alpha_0}{\alpha_1}D_y-\frac{1}{\alpha_1}\big(-xyD_x^2+(x-y)D_x+2zD_z-n\big)\right){}_{S}{}_{L}\mathcal{R}_n(x,z,y)=0. \tag{4.2}$$

By taking help of equations (1.3), (1.4) and (2.3), we deduce the initial conditions

$${}_{S}{}_{L}\mathcal{R}_n(x,0,y)={}_{L}\mathcal{R}_n(x,y)=\sum_{r=0}^{n}\sum_{k=0}^{n-r}\binom{n}{r}\binom{n-r}{k}(-1)^r\mathcal{R}_k\frac{x^r}{r!}y^{n-k-r}, \tag{4.3}$$

$$\frac{d}{dy}\big\{{}_{S}{}_{L}\mathcal{R}_n(x,0,y)\big\}=$$

$$=n\,{}_{S}{}_{L}\mathcal{R}_{n-1}(x,0,y)=n\sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{n-r-1}{k}(-1)^r\mathcal{R}_k\frac{x^r}{r!}y^{n-k-r-1}. \tag{4.4}$$

Now, consider

$$D_y^2\big\{{}_{S}{}_{L}\mathcal{R}_n(x,z,y)\big\}=\phi(y),$$

which on integrating using initial conditions (4.3) and (4.4) gives

$$D_y\big\{{}_{S}{}_{L}\mathcal{R}_n(x,z,y)\big\}=$$

$$=\int\limits_0^y\phi(\xi)d\xi+n\sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{n-r-1}{k}(-1)^r\mathcal{R}_k\frac{x^r}{r!}y^{n-k-r-1}, \tag{4.5}$$

$${}_{S}{}_{L}\mathcal{R}_n(x,z,y)=\int\limits_0^y\phi(\xi)d\xi^2+\sum_{r=0}^{n}\sum_{k=0}^{n-r}\binom{n}{r}\binom{n-r}{k}(-1)^r\mathcal{R}_k\frac{x^r}{r!}y^{n-k-r}. \tag{4.6}$$

Use of expressions (4.5) and (4.6) in equation (4.2) led to integral equation (4.1).

***Remark* 4.1.** By substituting the values of coefficients $\alpha_0 = -\frac{1}{2}$, $\alpha_1 = -\frac{B_2(1)}{2} = -\frac{1}{12}$ and $\mathcal{R}_k = B_k$ in integral equation (4.1), we find that, for the 3VLeLBP ${}_{S L}B_n(x,z,y)$, the following homogeneous Volterra integral equation holds true:

$$\phi(y) = -12\big(xyD_x^2 - (x-y)D_x - 2zD_z + n\big)\times$$
$$\times\left(n\sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{n-r-1}{k}(-1)^r B_k\frac{x^r}{r!}y^{n-k-r-1}+\right.$$
$$\left.+\sum_{r=0}^{n}\sum_{k=0}^{n-r}\binom{n}{r}\binom{n-r}{k}(-1)^r B_k\frac{x^r}{r!}y^{n-k-r}\right)-$$
$$-6n\sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{n-r-1}{k}(-1)^r B_k\frac{x^r}{r!}y^{n-k-r-1}+$$
$$+\int_0^y\Big(-12\Big(xyD_x^2 - (x-y)D_x - 2zD_z + n\Big)(y-\xi)+6\Big)\,\phi(\xi)d\xi.$$

***Remark* 4.2.** By substituting the values of coefficients $\alpha_0 = -\frac{1}{2}$, $\alpha_1 = \frac{\mathcal{E}_1}{2} = -\frac{1}{2}$ and $\mathcal{R}_k = E_k$ in integral equation (4.1), we find that, for the 3VLeLEP ${}_{S L}E_n(x,z,y)$, the following homogeneous Volterra integral equation holds true:

$$\phi(y) = -2\big(xyD_x^2 - (x-y)D_x - 2zD_z + n\big)\times$$
$$\times\left(n\sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{n-r-1}{k}(-1)^r E_k\frac{x^r}{r!}y^{n-k-r-1}+\right.$$
$$\left.+\sum_{r=0}^{n}\sum_{k=0}^{n-r}\binom{n}{r}\binom{n-r}{k}(-1)^r E_k\frac{x^r}{r!}y^{n-k-r}\right)-$$
$$-n\sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{n-r-1}{k}(-1)^r E_k\frac{x^r}{r!}y^{n-k-r-1}+$$
$$+\int_0^y\Big(-2\big(xyD_x^2 - (x-y)D_x - 2zD_z + n\big)(y-\xi)+1\Big)\phi(\xi)d\xi.$$

***Remark* 4.3.** By substituting $\alpha_0 = 1$, $\alpha_1 = -1$, and $\mathcal{R}_k = G_k$ in integral equation (4.1), we find that, for the 3VLeLGP ${}_{S L}G_n(x,z,y)$, the following homogeneous Volterra integral equation holds true:

$$\phi(y) = -\big(xyD_x^2 - (x-y)D_x - 2zD_z + n\big)\times$$
$$\times\left(n\sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{n-r-1}{k}(-1)^r G_k\frac{x^r}{r!}^{n-k-r-1}y^{n-k-r-1}+\right.$$

$$+\sum_{r=0}^{n}\sum_{k=0}^{n-r}\binom{n}{r}\binom{n-r}{k}(-1)^r G_k \frac{x^r}{r!} y^{n-k-r}\Bigg)+$$

$$+n\sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{n-r-1}{k}(-1)^r G_k \frac{x^r}{r!} y^{n-k-r-1}+$$

$$+\int_0^y \Big(-\big(xyD_x^2-(x-y)D_x-2zD_z+n\big)(y-\xi)-1\Big)\phi(\xi)d\xi.$$

To study the combination of operational representations with the integral transforms and their applications to the theory of fractional calculus for the 3VLeLAP ${}_{SL}\mathcal{R}_n(x,z,y)$ and for their relatives will be taken in further investigation.

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