DOI: 10.37863/umzh.v73i5.895

UDC 514.7

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LEGENDRE CURVES AND THE SINGULARITIES OF RULED SURFACES OBTAINED BY USING ROTATION MINIMIZING FRAME

КРИВІ ЛЕЖАНДРА ТА СИНГУЛЯРНОСТІ ЛІНІЙЧАТИХ ПОВЕРХОНЬ, ЯКІ ОТРИМАНО ЗА ДОПОМОГОЮ РЕПЕРА З МІНІМАЛЬНИМ ОБЕРТАННЯМ

In this paper, Legendre curves in unit tangent bundle are given using rotation minimizing vector fields. Ruled surfaces corresponding to these curves are represented. Singularities of these ruled surfaces are also analyzed and classified.

У цій роботі криві Лежандра в одиничному дотичному жмутку наведено за допомогою векторних полів з мінімальним обертанням. Описано лінійчаті поверхні, що відповідають цим кривим. Також проаналізовано та класифіковано сингулярності таких поверхонь.

1. Introduction. One of the most known orthonormal frame on a space curve is the Frenet–Serret frame, comprising the tangent vector field T, the principal normal vector field N and the binormal vector field $B = T \times N$. When this frame is used to orient a body along a path, its angular velocity vector (known also as the Darboux vector) W satisfies $\langle W, N\rangle = 0$, i.e., it has no component in the principal normal vector direction. This means that the body exhibits no instantaneous rotation about the unit normal vector N from point to point along the path.

Bishop introduced rotation minimizing frame (RMF) which is an alternative to the Frenet–Serret frame (see [5]). This alternative frame does not have an instantaneous rotation about the unit tangent vector field T. Nowadays, RMF is widely used in mathematical researches and computer aided geometric desing (see, e.g., [1, 8, 13]).

More precisely, in n-dimensional Riemannian manifold $(M, g = \langle , \rangle)$, a RMF along a curve γ is an orthonormal frame defined by the tangent vector field T (of the curve γ in M) and by $n-1$ normal vector fields N_i, which do not rotate with respect to the tangent vector field (i.e., $\nabla_T N_i$ is proportional to $T = \gamma'(s)$, where ∇ is the Levi–Civita connection of g). This type of a normal vector field along a curve is said to be a rotation minimizing vector field (RM vector field). Any orthonormal basis $\left\{T(s_0), N_1(s_0), \ldots, N_{n-1}(s_0)\right\}$ at a point $\gamma(s_0)$ defines a unique RMF along the curve γ. Thus, such a RMF is uniquely designated modula of a rotation in $(n-1)$-dimensional real vector space $\mathbb{R}^{n-1}$. The notion of RMF particularizes to that of Bishop frame in Euclidean case (see [7]). The Frenet type equations of the RMF are given by

$$\nabla_T T(s) = \sum_{i=1}^{n-1} \kappa_i(s) N_i(s) \qquad \text{and} \qquad \nabla_T N_i(s) = \kappa_i(s) T(s),$$

where $\kappa_i(s)$ are called the natural curvatures along the curve γ.

On the other hand, Legendre curves (especially in the tangent bundle of 2-sphere, $T\mathbb{S}^2$) are studied by many authors (see, e.g., [10, 11]). We call the pair $\Gamma = (\gamma, v) \subset T\mathbb{S}^2$ satisfying $\langle \gamma', v \rangle = 0$ as Legendre curve. We prove that any two RM vector fields correspond to a Legendre curve in (the unit tangent bundle of 2-sphere) $UT\mathbb{S}^2$, see Theorems 1 and 2.

In [11], we have shown that to any Legendre curve in $T\mathbb{S}^2$ corresponds a developable ruled surface. Using RMF along a curve in 3-dimensional manifold, one can define six ruled surfaces. In this study, we want to describe how the local shape of a curve in $T\mathbb{S}^2$ is affected by the offsetting process. In particular, we want to classify the singularities of these six ruled surfaces. We have observed that these six ruled surfaces can be one of the following depending on their singularities: *Cuspidal edge* $C \times \mathbb{R}$, *Swallowtail* SW, *Cuspidal crosscap* CCR or a *cone surface*.

It is important to emphasize that in [9], Haiming and Donghe studied Legendrian dualities between spherical indicatrixes of curves in 3-dimensional Euclidean space $\mathbb{E}^3$ by using the theory of Legendrian duality. Moreover, they classified the singularities of two ruled surfaces, which are the first and second type ruled surfaces obtained by using Bishop frame. However, in this paper we classify four extra ones. As stated in Corollary 1, Theorems 3.1 and 3.2 given in [9] are obtained as particular cases of our study. Another advantage of this paper is the use of the theorems in [12] to accelerate the singularity calculations.

This paper is divided into two parts: In Section 2, we give some definitions and notions about the Legendre curves in $UT\mathbb{S}^2$ and about the RM vector fields. By Theorems 1 and 2 and by Example 1, we give some relationships between these curves and vector fields. In Section 3, we show that the ruled surfaces obtained from RMF are developable and we analyze the singularities of these ruled surfaces.

All curves and manifolds considered in this paper are of class C^∞ unless otherwise stated.

2. Legendre curves and RM vectors fields. Let $\gamma\colon I \subset \mathbb{R} \to M$ be a regular curve with arc-length parameter s in 3-dimensional Riemannian manifold $(M, g = \langle , \rangle)$. Then there exists an accompanying 3-frame $\{T, N, B\}$ known as the *Frenet–Serret frame* of $\gamma = \gamma(s)$. In this case, the moving Frenet–Serret formulas in M are given by

$$\begin{pmatrix} \nabla_T T(s) \\ \nabla_T N(s) \\ \nabla_T B(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{1}$$

where $\kappa(s) \neq 0$ and $\tau(s)$ are called the *curvature* and the *torsion* of the curve γ at s, respectively. The set $\{T, N, B, \kappa, \tau\}$ is also called the *Frenet-frame apparatus*.

Definition 1. *Let γ be a curve in (M, g). A normal vector field N over γ is said to be a RM vector field if it is parallel with respect to the normal connection of γ. This means that $\nabla_{\gamma'} N$ and γ' are proportional.*

A RMF along a curve $\gamma = \gamma(s)$ in (M^3, g) is an orthonormal frame defined by the tangent vector T and by two normal vector fields N_1 and N_2, whose derivatives are proportional to T. Any orthonormal basis $\{T, N_1, N_2\}$ at a point $\gamma(s_0)$ defines a unique RMF along the curve γ. Let ∇ be the Levi–Civita connection of the metric g. Then Frenet type equations read as

$$\begin{pmatrix} \nabla_T T(s) \\ \nabla_T N_1(s) \\ \nabla_T N_2(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ -\kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}. \tag{2}$$

Here, the functions $\kappa_1(s)$ and $\kappa_2(s)$ are called the *natural curvatures* of RMF given by

$$\kappa(s) = \sqrt{\kappa_1^2(s) + \kappa_2^2(s)} \qquad \text{and} \qquad \tau(s) = \theta'(s) = \frac{\kappa_1(s)\kappa_2'(s) - \kappa_1'(s)\kappa_2(s)}{\kappa_1^2(s) + \kappa_2^2(s)},$$

where $\theta(s) = \arg(\kappa_1(s), \kappa_2(s)) = \arctan \dfrac{\kappa_2(s)}{\kappa_1(s)}$ and $\theta'(s)$ is the derivative of $\theta(s)$ with respect to the arc-length.

If (M, g) is the Euclidean 3-space $\big(\mathbb{R}^3, \langle , \rangle\big)$, then the notion of RMF particularizes to that of Bishop frame.

Let $\mathbb{S}^2$ be the unit 2-sphere in $\mathbb{R}^3$. Then the tangent bundle of $\mathbb{S}^2$ is given by

$$T\mathbb{S}^2 = \big\{(\gamma, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |\gamma| = 1 \ \text{ and } \ \langle \gamma, v\rangle = 0\big\}$$

and the unit tangent bundle of $\mathbb{S}^2$ is given by

$$UT\mathbb{S}^2 = \big\{(\gamma, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |\gamma| = |v| = 1 \ \text{ and } \ \langle \gamma, v\rangle = 0\big\} =$$
$$= \big\{(\gamma, v) \in \mathbb{S}^2 \times \mathbb{S}^2 : \langle \gamma, v\rangle = 0\big\}, \tag{3}$$

which is a 3-dimensional contact manifold and its canonical contact 1-form is θ. Here, where $\langle , \rangle$ and $| , |$ denote the usual *inner product* and the *norm* in $\mathbb{R}^3$, respectively. For further information see [10, 15].

In general, in any Riemannian manifold, a curve γ is said to be *Legendre* if it is an integral curve of the contact distribution $D = \ker\theta$, i.e., $\theta(\gamma') = 0$ (see [2]). In particular, Legendre curves in 3-dimensional contact manifold $UT\mathbb{S}^2$ on $\mathbb{S}^2$ can be given by the following definition.

Definition 2. *The smooth curve*

$$\Gamma(s) = (\gamma(s), v(s)) : I \subset \mathbb{R} \to UT\mathbb{S}^2 \subset \mathbb{S}^2 \times \mathbb{S}^2$$

is called a Legendre curve in $UT\mathbb{S}^2$ *if*

$$\big\langle \gamma'(s), v(s)\big\rangle = 0. \tag{4}$$

The Legendre curve condition in $UT\mathbb{S}^2$ can be seen in [9] as a definition of Δ-dual to each other in $\mathbb{S}^2$. By the following theorem we give the relationship between RM vector fields and the Legendre curve conditions in $UT\mathbb{S}^2$.

Theorem 1. *If* $\{U, V, W\}$ *is an orthonormal frame* (*along a curve*) *such that* U *and* V *have derivatives parallel to* W, *then* (U, V) *is Legendre in* $UT\mathbb{S}^2$.

Example **1.** Let $\gamma : I \subset \mathbb{R} \to \mathbb{S}^2$ be a regular unit speed curve with the frame apparatus $\{T, N, B, \kappa, \tau\}$. Then the following three cases can be given:

1. If $N_1(s)$ and $N_2(s)$ are RM vector fields along γ, then the curve $\big(N_1(s), N_2(s)\big)$ is Legendre in $UT\mathbb{S}^2$.

2. If $\overline{N}_1(s)$ and $\overline{N}_2(s)$ are RM vectors along B-direction curve $\overline{\beta}(s) = \displaystyle\int B(s)ds$, then the curve $\big(\overline{N}_1(s), \overline{N}_2(s)\big)$ is Legendre in $UT\mathbb{S}^2$.

3. If $B(s)$ and $T(s)$ are RM vector fields along N-direction curve $\beta(s) = \displaystyle\int N(s)ds$, then the curve $\big(B(s), T(s)\big)$ is Legendre in $UT\mathbb{S}^2$.

Let us verify these three cases: assume that $\gamma\colon I \subset \mathbb{R} \to \mathbb{S}^2$ is a regular unit speed curve with the frame apparatus $\{T, N, B, \kappa, \tau\}$. Then

1. Consider the curve $\Gamma(s) = \big(N_1(s), N_2(s)\big) \in UT\mathbb{S}^2$. Since $N_1(s)$ and $N_2(s)$ are RM vector fields along $\gamma(s)$, from equation (2) we get

$$\big\langle N_1'(s), N_2(s)\big\rangle = -\kappa_1(s)\big\langle T(s), N_2(s)\big\rangle = 0.$$

Thus, from equation (4) we can say that Γ is a Legendre curve in $UT\mathbb{S}^2$.

2. Consider the curve $\Gamma(s) = \big(\overline{N}_1(s), \overline{N}_2(s)\big) \in UT\mathbb{S}^2$ along the B-direction curve $\overline{\beta}(s)$. The Frenet type equations can be given as

$$\begin{pmatrix} B'(s) \\ \overline{N}_1'(s) \\ N_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & \bar{\kappa}_1(s) & \bar{\kappa}_2(s) \\ -\bar{\kappa}_1(s) & 0 & 0 \\ -\bar{\kappa}_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} B(s) \\ \overline{N}_1(s) \\ \overline{N}_2(s) \end{pmatrix} \tag{5}$$

with the natural curvatures

$$\bar{\kappa}(s) = \sqrt{\bar{\kappa}_1^2(s) + \bar{\kappa}_2^2(s)} \qquad \text{and} \qquad \bar{\tau}(s) = \theta'(s) = \frac{\bar{\kappa}_1'(s)\bar{\kappa}_2(s) - \bar{\kappa}_1'(s)\bar{\kappa}_2(s)}{\bar{\kappa}_1^2(s) + \bar{\kappa}_2^2(s)}.$$

From equation (5), we have

$$\big\langle \overline{N}_1'(s), \overline{N}_2(s)\big\rangle = -\bar{\kappa}_1(s)\big\langle B(s), \overline{N}_2(s)\big\rangle = 0.$$

Thus, from equation (4), we can say that Γ is a Legendre curve in $UT\mathbb{S}^2$. The proof of Case 3 can be given by the similar way as Cases 1 and 2.

From the definition of the set $UT\mathbb{S}^2$, we know that for a smooth curve $\Gamma(s) = \big(\gamma(s), v(s)\big)$ in $T\mathbb{S}^2$ we have $\langle \gamma(s), v(s)\rangle = 0$. Thus, we can define a new frame using the unit vector $\eta(s) = \gamma(s) \wedge v(s)$, where $\wedge$ denotes the usual *vector product* in $\mathbb{R}^3$. It is obvious that $\langle \gamma(s), \eta(s)\rangle = \langle v(s), \eta(s)\rangle = 0$. Hence, we get the following Frenet frame $\big\{\gamma(s), v(s), \eta(s)\big\}$ along $\gamma(s)$:

$$\begin{pmatrix} \gamma'(s) \\ v'(s) \\ \eta'(s) \end{pmatrix} = \begin{pmatrix} 0 & l(s) & m(s) \\ -l(s) & 0 & n(s) \\ -m(s) & -n(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ v(s) \\ \eta(s) \end{pmatrix}, \tag{6}$$

where $l(s) = \langle \gamma'(s), v(s)\rangle$, $m(s) = \big\langle \gamma'(s), \mu(s)\rangle$, $n(s) = \langle v'(s), \mu(s)\big\rangle$. The triple $\{l, m, n\}$ is called the *curvature functions* of Γ.

We know that if $l(s) = 0$, then the curve $\Gamma(s) = (\gamma(s), v(s))$ is Legendre in $UT\mathbb{S}^2$ with the curvature functions (m, n).

Theorem 2. *Let* $\Gamma(s) = (\gamma(s), v(s))$ *be a smooth curve in* $UT\mathbb{S}^2$. *If* $\Gamma(s)$ *is Legendre, then the vectors* $\gamma(s)$ *and* $v(s)$ *are RM vector fields along the* η*-direction curve* β, *i.e.,* $\beta(s) = \int \eta(s)ds$, *and the triple vector field set* $\{\gamma, v, \eta\}$ *is a RMF.*

Proof. Let $\Gamma(s) = (\gamma(s), v(s))$ be a smooth Legendre curve in $UT\mathbb{S}^2$. Then the Frenet frame given by equation (6) can be given for the Legendre condition (that is, $l(s) = 0$) as

$$\begin{pmatrix} \eta'(s) \\ \gamma'(s) \\ v'(s) \end{pmatrix} = \begin{pmatrix} 0 & -m(s) & -n(s) \\ m(s) & 0 & 0 \\ n(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta(s) \\ \gamma(s) \\ v(s) \end{pmatrix}. \tag{7}$$

From equation (2), we can say that $\{\eta, \gamma, v\}$ is a RMF along the η-direction curve $\beta(s) = \int \eta(s)ds$.

Theorem 2 is proved.

3. Singularities of ruled surfaces obtained by using RMF. A *ruled surface* in $\mathbb{R}^3$ is locally the map

$$\Phi_{(\beta,\alpha)} : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$$

defined by

$$\Phi_{(\beta,\alpha)}(s, u) = \beta(s) + u\alpha(s),$$

where β and α are smooth mappings defined from an open interval I (or a unit circle $\mathbb{S}^1$) to $\mathbb{R}^3$. β is the *base curve* (or *directrix* and the non-null curve α is the *director curve.* The straight lines $u \longrightarrow \beta(s) + u\alpha(s)$ are the *rulings.*

The *striction curve* of the ruled surface $\Phi_{(\beta,\alpha)}(s, u) = \beta(s) + u\alpha(s)$ is defined by

$$\bar{\beta}(s) = \beta(s) - \frac{\big\langle \beta'(s), \alpha'(s) \big\rangle}{\langle \alpha'(s), \alpha'(s) \rangle} \alpha(s). \tag{8}$$

If $\big\langle \beta'(s), \alpha'(s) \big\rangle = 0$, then the striction curve $\bar{\beta}(s)$ coincides with the base curve $\beta(s)$.

A ruled surface $\Phi_{(\beta,\alpha)}(s, u) = \beta(s) + u\alpha(s)$ is said to be *developable* if

$$\det \big(\beta'(s), \alpha(s), \alpha'(s)\big) = 0.$$

From Theorem 2, we can say that if Γ is a Legendre curve, then the vector set $\{\eta, \gamma, v\}$ is a RMF along the η-direction curve $\beta(s) = \int \eta(s)ds$. One can define by this frame the following six ruled surfaces:

$$\Phi_{(a_{1i}, a_{2i})}(s, u) = a_{1i}(s) + u_i a_{2i}(s) \quad \text{for} \quad i = 1, \ldots, 6, \tag{9}$$

where $a_{1i}(s)$ and $a_{2i}(s)$ are different unit curves from the set $\big\{\beta(s), \gamma(s), v(s)\big\}$.

Proposition 1. *Ruled surfaces* $\Phi_{(a_{1i}, a_{2i})}(s, u)$ *for* $i = 1, \ldots, 6$ *given by equation* (9) *are developable.*

Proof. Let $\Phi_{(a_{11}, a_{21})}(s, u) = \beta(s) + u\gamma(s)$ be a ruled surface defined by equation (9). By using equation (7), we get

$$\det \big(\beta'(s), \gamma(s), \gamma'(s)\big) = \det \big(\eta(s), \gamma(s), m(s)\eta(s)\big) = 0,$$

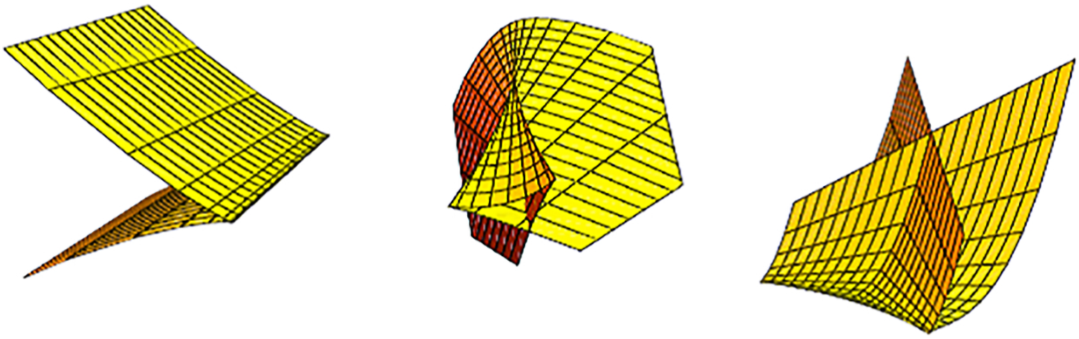

Fig. 1. Left surface is the *Cuspidal edge* $C \times \mathbb{R}$, middle surface is the *Swallowtail* SW and right surface is the *Cuspidal crosscap* CCR.

which is the developability condition of the ruled surface $\Phi_{(a_{11},a_{21})}$. Proof of the other ruled surfaces $\Phi_{(a_{1i},a_{2i})}$ for $i = 2, \ldots, 6$ can be given by the similar way.

Now, recall the parametric equations of the surfaces *Cuspidal edge*, *Swallowtail* and *Cuspidal crosscap* in $\mathbb{R}^3$ given by Fig. 1 (see [12]):

(i) *Cuspidal edge*: $C \times \mathbb{R} = \{(x_1, x_2);\ x_1^2 = x_2^3\} \times \mathbb{R}$,

(ii) *Swallowtail*: SW$= \{(x_1, x_2, x_3);\ x_1 = 3u^4 + u^2 v,\ x_2 = 4u^3 + 2uv,\ x_3 = v\}$,

(iii) *Cuspidal crosscap*: CCR$= \{(x_1, x_2, x_3);\ x_1 = u^3,\ x_2 = u^3 v^3,\ x_3 = v^2\}$.

By the following theorem, we give the local classification of singularities of the ruled surfaces defined by equation (9).

Theorem 3. *Let $\Gamma(s) = (\gamma(s), v(s))$ be a smooth Legendre curve in $UT\mathbb{S}^2$ and $\{\eta, \gamma, v\}$ be the RMF given in Theorem 2. Then we have the following*:

1) $\Phi_{(\beta,\gamma)}(s,u) = \beta(s) + u\gamma(s)$ *which is locally diffeomorphic to*:

(a) $C \times R$ *at* $\Phi_{(\beta,\gamma)}(s_0,u_0)$ *if and only if* $u_0 = -m(s_0)^{-1} \neq 0$ *and* $m'(s_0) \neq 0$;

(b) SW *at* $\Phi_{(\beta,\gamma)}(s_0,u_0)$ *if and only if* $u_0 = -m(s_0)^{-1} \neq 0$, $m'(s_0) = 0$ *and* $m(s_0)^{-1})''(s_0) \neq \neq 0$,

2) $\Phi_{(\beta,v)}(s,u) = \beta(s) + uv(s)$ *which is locally diffeomorphic to*:

(a) $C \times R$ *at* $\Phi_{(\beta,v)}(s_0,u_0)$ *if and only if* $u_0 = -n(s_0)^{-1} \neq 0$ *and* $u'(s_0) \neq 0$;

(b) SW *at* $\Phi_{(\beta,v)}(s_0,u_0)$ *if and only if* $u_0 = -n(s_0)^{-1} \neq 0$, $n'(s_0) = 0$ *and* $(n(s_0)^{-1})''(s_0) \neq \neq 0$,

3) $\Phi_{(\beta,\gamma)}(s,u) = \beta(s) + u\gamma(s)$ (*resp*., $\Phi_{(\beta,v)}(s,u) = \beta(s) + uv(s)$) *which is a cone surface if and only if* $m(s)$ (*resp*., $n(s)$) *is constant.*

Proof. Assume that $\Gamma(s) = (\gamma(s), v(s))$ is a smooth Legendre curve in $UT\mathbb{S}^2$ depending on the RMF $\{\eta, \gamma, v\}$ along the η-direction curve $\beta(s)$. By using equation (9) and $\Phi_{(\beta,\gamma)}(s,u) = = \beta(s) + u\gamma(s)$, we get

$$\frac{\partial \Phi_{(\beta,\gamma)}}{\partial s}(s,u) = (1 + um(s))\eta,$$

$$\frac{\partial \Phi_{(\beta,\gamma)}}{\partial u}(s,u) = \gamma,$$

$$\frac{\partial \Phi_{(\beta,\gamma)}}{\partial s}(s,u) \wedge \frac{\partial \Phi_{(\beta,\gamma)}}{\partial u}(s,u) = (1 + um(s))v.$$

Singularities of the normal vector field of $\Phi_{(\beta,\gamma)} = \Phi_{(\beta,\gamma)}(s,u)$ are

$$u = \frac{-1}{m(s)}.$$

From Theorem 3.3 of the paper [12], we know that if there exists a parameter s_0 such that $u_0 = \frac{-1}{m(s_0)} \neq 0$ and $u_0' = \frac{m'(s_0)}{m^2(s_0)} \neq 0$ (i.e., $m'(s_0) \neq 0$), then $\Phi(s,u)$ is locally diffeomorphic to $C \times \mathbb{R}$ at $\Phi_{(\beta,\gamma)}(s_0,u_0)$. This completes the proof of Assertion 1 (a). Again from the Theorem 3.3 of [12], we know that if there exists a parameter s_0 such that $u_0 = \frac{-1}{m(s_0)} \neq 0$, $u_0' = \frac{m'(s_0)}{m^2(s_0)} = 0$ and $(m(s_0)^{-1})''(s_0) \neq 0$, then $\Phi_{(\beta,\gamma)}$ is locally diffeomorphic to SW at $\Phi_{(\beta,\gamma)}(s_0,u_0)$, and this completes the proof of Assertion 1 (b).

Proof of Assertion 2 can be given similar to the proof of Assertion 1. To prove Assertion 3, note that the singularity points are equal to the striction curve of Φ and can be given by

$$\varphi_{(\beta,\gamma)}(s) = \Phi_{(\beta,\gamma)}\left(s, \frac{-1}{m(s)}\right) = \beta(s) - \frac{1}{m(s)}\gamma(s)$$

$$\left(\text{resp., } \varphi_{(\beta,v)}(s) = \Phi_{(\beta,v)}\left(s, \frac{-1}{m(s)}\right) = \beta(s) - \frac{1}{m(s)}v(s)\right).$$

Thus, we have

$$\varphi'_{(\beta,\gamma)}(s) = -\left(\frac{1}{m(s)}\right)'\gamma(s) \quad \left(\text{resp., } \varphi'_{(\beta,v)}(s) = -\left(\frac{1}{m(s)}\right)'v(s)\right),$$

which means that if $m(s)$ is a constant function, then

$$\varphi'_{(\beta,\gamma)}(s) = \varphi'_{(\beta,v)}(s) = 0.$$

Thus, $\Phi_{(\beta,\gamma)}$ (resp., $\Phi_{(\beta,v)}$) has only one singularity point. This means that $\Phi_{(\beta,\gamma)}$ and $\Phi_{(\beta,v)}$ are cone surfaces.

Theorem 3 is proved.

***Corollary* 1.** *Let $\alpha\colon I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth curve with frame apparatus $\{N_1, N_2, \kappa_1, \kappa_2\}$ given by equation* (5). *If we choose $\Gamma(s) = (\gamma(s), v(s)) = \Gamma\big(N_1(s), N_2(s)\big)$, then we obtain the Theorem* 3.1 *given in* [9]. *And if we choose $\Gamma(s) = (\gamma, v) = \Gamma(N_2(s), N_1(s))$, then we obtain Theorem* 3.2 *given in* [9].

Proof. Let $\alpha\colon I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth curve with frame apparatus $\{N_1, N_2, \kappa_1, \kappa_2\}$ given by equation (2). The vector fields $\{T, N_1, N_2\}$ is a RMF along the T-direction curve $\beta(s) = \alpha(s) = \int T(s)ds$. This means that $\Gamma\big(N_1(s), N_2(s)\big)$ is a Legendre curve in $T\mathbb{S}^2$. By using Theorem 3, we complete the proof, where $m(s) = \kappa_1(s)$ and $n(s) = \kappa_2(s)$.

Theorem 4. *Let $\Gamma(s) = (\gamma(s), v(s))$ be a smooth Legendre curve in $UT\mathbb{S}^2$ and $\{\eta, \gamma, v\}$ be the RMF given in Theorem* 2. *Then we have the following*:

1) $\Phi_{(\gamma,\beta)}(s,u) = \gamma(s) + u\beta(s)$ *which is locally diffeomorphic to*:

(a) $C \times R$ *at* $\Phi_{(\gamma,\beta)}(s_0,u_0)$ *if and only if* $u_0 = -m(s_0) \neq 0$ *and* $m'(s_0) \neq 0$;

(b) SW *at* $\Phi_{(\gamma,\beta)}(s_0,u_0)$ *if and only if* $u_0 = -m(s_0) \neq 0$, $m'(s_0) = 0$ *and* $m''(s_0) \neq 0$;

(c) CCR *at* $\Phi_{(\gamma,\beta)}(s_0,u_0)$ *if and only if* $u_0 = -m(s_0) = 0$ *and* $m'(s_0) \neq 0$,

2) $\Phi_{(v,\beta)}(s,u) = v(s) + u\beta(s)$ *which is locally diffeomorphic to*:

(a) $C \times R$ *at* $\Phi_{(v,\beta)}(s_0, u_0)$ *if and only if* $u_0 = -n(s_0) \neq 0$ *and* $n'(s_0) \neq 0$;

(b) SW *at* $\Phi_{(v,\beta)}(s_0, u_0)$ *if and only if* $u_0 = -n(s_0)$, $n'(s_0) = 0$ *and* $n''(s_0) \neq 0$;

(c) CCR *at* $\Phi_{(v,\beta)}(s_0, u_0)$ *if and only if* $u_0 = -n(s_0) = 0$ *and* $n'(s_0) \neq 0$,

3) $\Phi_{(\gamma,\beta)}(s,u) = \gamma(s) + u\beta(s)$ (*resp*., $\Phi_{(v,\beta)}(s,u) = v(s) + u\beta(s)$) *which is a cone surface if and only if* $m(s)$ (*resp*., $n(s)$) *is constant.*

Theorem 5. *Let* $\Gamma(s) = (\gamma(s), v(s))$ *be a smooth Legendre curve in* $UT\mathbb{S}^2$ *with curvature functions* $\{m, n\}$. *Then we have the following*:

1) *ruled surface* $\Phi_{(\gamma,v)}(s,u) = \gamma(s) + uv(s)$ *is locally diffeomorphic to*:

(a) $C \times R$ *at* $\Phi_{(\gamma,v)}(s_0, u_0)$ *if and only if* $u_0 = -\dfrac{m}{n}(s_0) \neq 0$ *and* $\left(\dfrac{m}{n}\right)'(s_0) \neq 0$;

(b) SW *at* $\Phi_{(\gamma,v)}(s_0, u_0)$ *if and only if* $u_0 = -\dfrac{m}{n}(s_0) \neq 0$, $\left(\dfrac{m}{n}\right)'(s_0) = 0$ *and* $\left(\dfrac{m}{n}\right)''(s_0) \neq$ $\neq 0$;

(c) CCR *at* $\Phi_{(\gamma,v)}(s_0, u_0)$ *if and only if* $u_0 = -\dfrac{m}{n}(s_0) = 0$ (*i.e.*, $m(s_0) = 0$ *and* $n(s_0) \neq 0$) *and* $\left(\dfrac{m}{n}\right)'(s_0) \neq 0$;

2) *ruled surface* $\Phi_{(v,\gamma)}(s,u) = v(s) + u\gamma(s)$ *is locally diffeomorphic to*:

(a) $C \times R$ *at* $\Phi_{(v,\gamma)}(s_0, u_0)$ *if and only if* $u_0 = -\dfrac{n}{m}(s_0) \neq 0$ *and* $\left(\dfrac{n}{m}\right)'(s_0) \neq 0$;

(b) SW *at* $\Phi_{(v,\gamma)}(s_0, u_0)$ *if and only if* $u_0 = -\dfrac{n}{m}(s_0) \neq 0$, $\left(\dfrac{n}{m}\right)'(s_0) = 0$ *and* $\left(\dfrac{n}{m}\right)''(s_0) \neq$ $\neq 0$;

(c) CCR *at* $\Phi_{(v,\gamma)}(s_0, u_0)$ *if and only if* $u_0 = -\dfrac{n}{m}(s_0) = 0$ (*i.e.*, $n(s_0) = 0$ *and* $m(s_0) \neq 0$) *and* $\left(\dfrac{n}{m}\right)'(s_0) \neq 0$,

3) *ruled surface* $\Phi_{(\gamma,v)}(s,u) = \gamma(s) + uv(s)$ (*resp*., $\Phi_{(v,\gamma)}(s,u) = v(s) + u\gamma(s)$) *is a cone surface if and only if* $\dfrac{n}{m}(s)$ $\Big($*resp*., $\dfrac{m}{n}(s)\Big)$ *is constant.*

Proofs of Theorems 4 and 5 can be given similar to the proof of Theorem 3.

***Corollary* 2.** *Let* $\alpha\colon I \subset \mathbb{R} \to \mathbb{R}^3$ *be a smooth curve with frame apparatus* $\{T, N, B, \kappa, \tau\}$. *If we choose* $\Gamma(s) = (\gamma(s), v(s)) = \Gamma\big(B(s), T(s)\big)$, *we obtain Theorem* 3.2 *given in* [12].

Proof. Since T and B are RM vector fields along the T-direction curve $\beta(s) = \alpha(s) =$ $= \int T(s)ds$, the curve $\Gamma\big(B(s), T(s)\big)$ is a Legendre in $T\mathbb{S}^2$. By using Theorem 4 and taking $m(s) = \kappa_1(s)$, $n(s) = \kappa_2(s)$, we get the proof.

***Corollary* 3.** *Let* $\alpha\colon I \subset \mathbb{R} \to \mathbb{R}^3$ *be a smooth curve with frame apparatus* $\{N, C, W = \overline{D}, f, g\}$ (*see* [3, 4]). *If we choose* $\Gamma(s) = (\gamma(s), v(s)) = \Gamma(W(s), N(s))$, *then we obtain the Theorem* 3.3 *given in* [12], *where*

$$W(s) = \overline{D}(s) = \frac{\tau(s)T(s) + \kappa(s)B(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}$$

is the unit Darboux vector field.

Proof. Let $\alpha\colon I\subset\mathbb{R}\to\mathbb{R}^3$ be a smooth curve with frame apparatus $\{N, C, W=\overline{D}, f, g\}$. Then the curve $\Gamma(s)=(\gamma(s),v(s))=\Gamma(W(s),N(s))$ is Legendre in $T\mathbb{S}^2$. By using Theorem 5, we get the slant helix condition

$$\frac{m}{n}(s)=\left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'\right)(s)=\sigma(s),$$

which completes the proof.

We close this section by giving some examples to illustrate the main results. The first example is an application of Theorem 5.

***Example* 2.** Let us take a smooth curve $\gamma\colon I\subset\mathbb{R}\to\mathbb{R}^3$ given by

$$\gamma(s)=\frac{1}{\sqrt{2}}\big(-\cos(s),-\sin(s),1\big)$$

and a unit vector given by

$$v(s)=\frac{1}{\sqrt{2}}\big(\cos(s),\sin(s),0\big).$$

Then we have

$$\big\langle\gamma'(s),v(s)\big\rangle=0.$$

Thus, $\Gamma(s)=(\gamma,v)$ is a Legendre curve in $UT\mathbb{S}^2$. The RMF $\{\eta,\gamma,v\}$ along the η-direction curve $\beta(s)=\displaystyle\int\eta(s)ds$ can be given as

$$\begin{pmatrix}\eta'(s)\\ \gamma'(s)\\ v'(s)\end{pmatrix}=\begin{pmatrix}0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & 0 & 0\\ \frac{1}{\sqrt{2}} & 0 & 0\end{pmatrix}\begin{pmatrix}\eta(s)\\ \gamma(s)\\ v(s)\end{pmatrix}.$$

The ruled surface

$$\Phi_{(v,\gamma)}(s,u)=v(s)+u\gamma(s)=\frac{1}{\sqrt{2}}\big(\cos(s)-u\cos(s),\sin(s)-u\sin(s),u\big)$$

represents a cone surface (see Fig. 2).

The second example is an application of Theorem 4.

***Example* 3.** Let $\alpha\colon I\subset\mathbb{R}\to\mathbb{R}^3$ be a smooth curve defined by

$$\gamma(s)=\left(\cos\left(\frac{s}{\sqrt{2}}\right),\sin\left(\frac{s}{\sqrt{2}}\right),\frac{s}{\sqrt{2}}\right).$$

Then the tangent and binormal vector fields of α are, respectively,

$$T(s)=\frac{1}{\sqrt{2}}\left(-\sin\left(\frac{s}{\sqrt{2}}\right),\cos\left(\frac{s}{\sqrt{2}}\right),1\right),$$

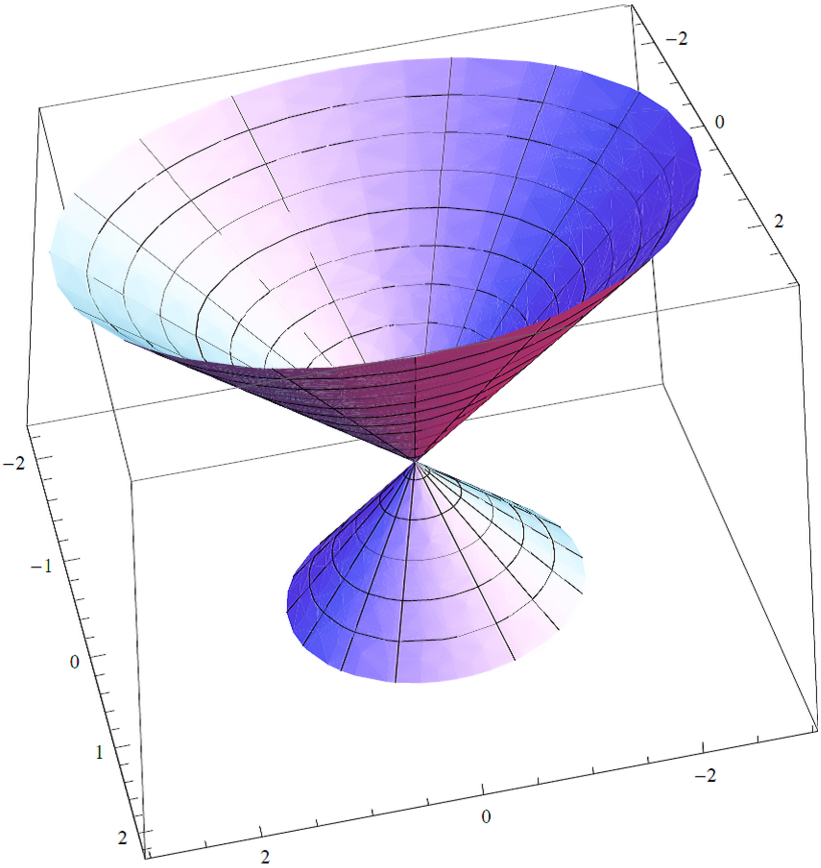

Fig. 2. Cone surface with one singularity point.

$$B(s) = \frac{1}{\sqrt{2}}\left(\sin\left(\frac{s}{\sqrt{2}}\right), \cos\left(\frac{s}{\sqrt{2}}\right), 1\right)$$

with the curvature $\kappa = \dfrac{1}{2}$ and the torsion $\tau = \dfrac{1}{2}$. So, γ is a helix. The curve $\Gamma(s) = (B, T)$ is Legendre in $UT\mathbb{S}^2$ and the ruled surface

$$\Phi_{(B,T)}(s,u) = B(s) + uT(s) =$$
$$= \frac{1}{\sqrt{2}}\left((1-u)\sin\left(\frac{s}{\sqrt{2}}\right), (u+1)\cos\left(\frac{s}{\sqrt{2}}\right), 1+u\right)$$

is a cone. We get the singularity point for $u = 1$ on the point $\Phi_{(B,T)}(s,1) = (0,0,\sqrt{2})$ (see Fig. 3).

The last example is an application of Theorem 3.

***Example* 4.** Let $\alpha\colon I = [0, A] \to \mathbb{R}^3$ be a smooth curve (for $0 < A \leq 2\pi$) defined by

$$\gamma(s) = \frac{1}{4}\Big(3\cos(s) - \cos(3s), 3\sin(s) - \sin(3s), 2\sqrt{3}\cos(s)\Big),$$

$$v(s) = \frac{1}{4}\Big(3\sin(s) - \sin(3s), -3\cos(s) - \cos(3s), -2\sqrt{3}\sin(s)\Big),$$

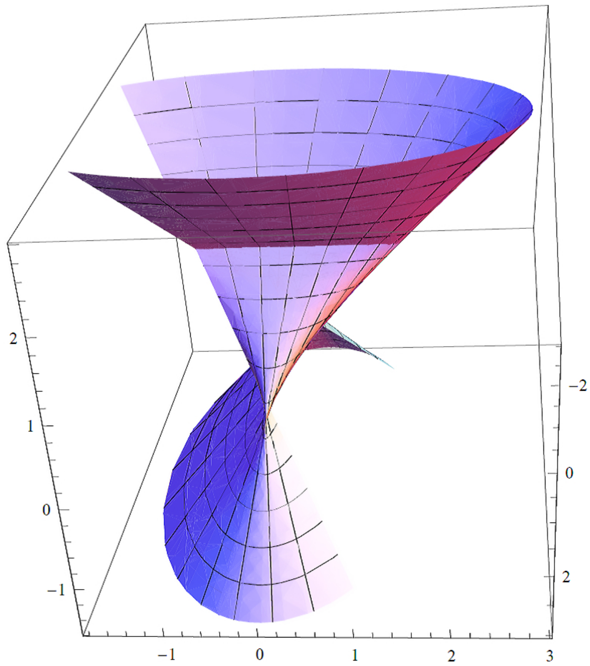

Fig. 3. Cone surface with helix singularity curve.

$$\eta(s) = \frac{1}{2}\left(\sqrt{3}\cos(2s), \sqrt{3}\sin(2s), -1\right).$$

Then $\Gamma(s) = (\gamma(s), v(s))$ is a Legendre curve with Legendre curvature function

$$m(s) = \sqrt{3}\sin(s)$$

and we have the following:

1. If $A = \pi$, then $m\left(\frac{\pi}{2}\right) = \sqrt{3} \neq 0$, $m'\left(\frac{\pi}{2}\right) = 0$ and $m''\left(\frac{\pi}{2}\right) = -\sqrt{3} \neq 0$. Then the ruled surface

$$\Phi_{(\beta,\gamma)}(s,u) = \beta(s) + u\gamma(s) =$$

$$= \left(\frac{\sqrt{3}}{2}\sin(2s) + \frac{3}{4}u\cos(s) - \frac{1}{4}u\cos(3s),\right.$$

$$\left.-\frac{\sqrt{3}}{2}\cos(2s) - \frac{3}{4}u\sin(s) - \frac{1}{4}u\sin(3s), -\frac{s}{2} + \frac{\sqrt{3}}{2}u\cos(s)\right)$$

is locally diffeomorphic to $C \times \mathbb{R}$ at $\Phi_{(\beta,\gamma)}\left(\frac{\pi}{2}, \frac{-1}{\sqrt{3}}\right)$ (see Fig. 4).

2. If $A = \frac{\pi}{2}$, then $u_0 = m^{-1}(s_0) \neq 0$, $(m^{-1})'(s_0) \neq 0$. Then the ruled surface $\Phi_{(\beta,\gamma)}(s,u)$ is locally diffeomorphic to SW at $\Phi_{(\beta,\gamma)}\left(\frac{\pi}{2}, u_0\right)$ (see Fig. 5).

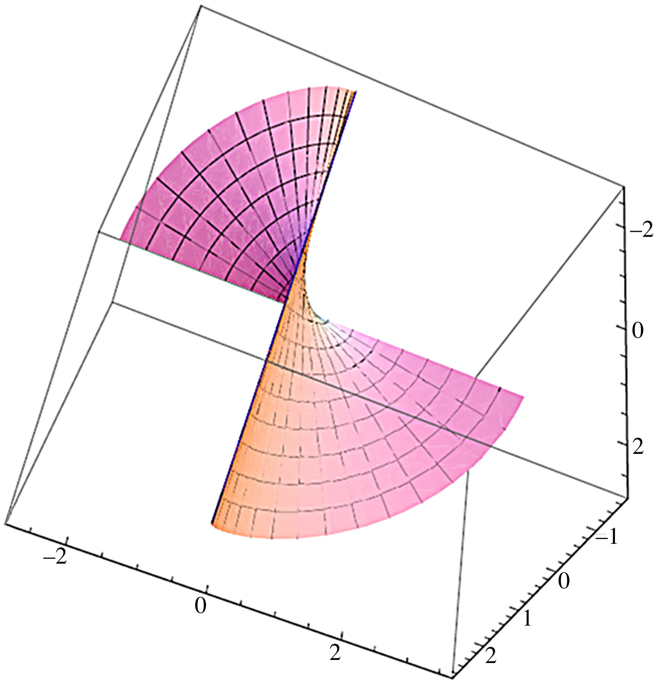

Fig. 4. Cuspidal edge $C \times \mathbb{R}$.

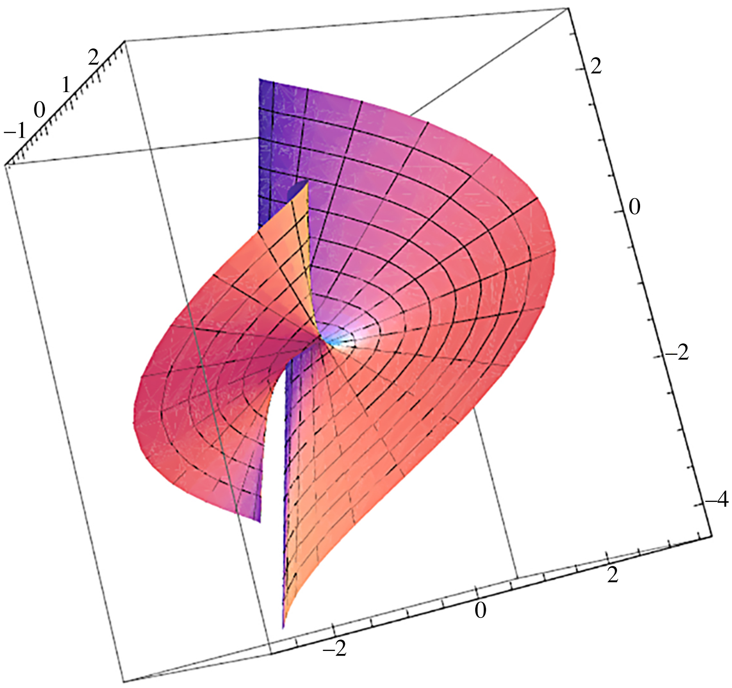

Fig. 5. Swallowtail SW.

4. Conclusions. In this paper, we give the Legendre curves on the unit tangent bundle by using the RM vector fields. We represent the ruled surfaces corresponding to these Legendre curves and discuss their singularities. For some special cases, given by Corollaries 1, 2, and 3, we get the main ideas of the studies [9, 12].

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Received 22.02.18