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ESTIMATES FOR ANALYTIC FUNCTIONS CONCERNED WITH HANKEL DETERMINANT

ОЦІНКИ ДЛЯ АНАЛІТИЧНИХ ФУНКЦІЙ, ПОВ'ЯЗАНІ З ДЕТЕРМІНАНТОМ ГАНКЕЛЯ

We give an upper bound of Hankel determinant of the first order $(H_2(1))$ for the classes of an analytic function. In addition, an evaluation with the Hankel determinant from below will be given for the second angular derivative of $f(z)$ analytic function. For new inequalities, the results of Jack's lemma and Hankel determinant were used. Moreover, in a class of analytic functions on the unit disc, assuming the existence of an angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained.

Отримано верхню границю детермінанта Ганкеля першого порядку $(H_2(1))$ для класів аналітичної функції. Також встановлено оцінку знизу з детермінантом Ганкеля для другої кутової похідної аналітичної функції $f(z)$. Для отримання нових нерівностей використано лему Джека та детермінант Ганкеля. Крім того, для класу аналітичних функцій на одиничному диску за умови існування кутової границі для межової точки отримано оцінки знизу для модуля кутової похідної.

1. Introduction. The most classical version of the Schwarz lemma examines the behavior of a bounded, analytic function mapping the origin to the origin in the unit disc $U = \{z : |z| < 1\}$. It is possible to see its effectiveness in the proofs of many important theorems. The Schwarz lemma, which has broad applications and is the direct application of the maximum modulus principle, is given in the most basic form as follows:

Let U be the unit disc in the complex plane $\mathbb{C}$. Let $f : U \to U$ be an analytic function with $f(z) = c_p z^p + \ldots$. Under these conditions, $|f(z)| \leq |z|^p$ for all $z \in U$ and $|c_p| \leq 1$. In addition, if the equality $|f(z)| = |z|^p$ holds for any $z \neq 0$ or $|c_p| = 1$, then f is a rotation, that is, $f(z) = z^p e^{i\theta}$, θ real [5, p. 329]. Schwarz lemma has several applications in the field of electrical and electronics engineering. Usage of positive real function and boundary analysis of these functions for circuit synthesis can be given as an exemplary application of the Schwarz lemma in electrical engineering. Furthermore, it is also used for analysis of transfer functions in control engineering and multinotch filter design in signal processing [12, 13].

In order to derive our main results, we have to recall here the following lemma [6].

Lemma 1 (Jack's lemma). *Let $f(z)$ be a nonconstant analytic function in U with $f(0) = 0$. If*

$$|f(z_0)| = \max\{|f(z)| : |z| \leq |z_0|\},$$

then there exists a real number $k \geq 1$ such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions $f(z) = z + c_2 z^2 + c_3 z^3 + \ldots$ that are analytic in U. Also, let $\mathcal{M}$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying

$$\Re\left[\left(\frac{\left(\frac{z}{f(z)}\right)^2 f'(z)}{\left(\frac{z}{f(z)}\right)^2 f'(z)-1}\right)\left(2+\frac{zf''(z)}{f'(z)}-2\frac{zf'(z)}{f(z)}\right)\right]<1. \tag{1.1}$$

The certain analytic functions which is in the class of $\mathcal{M}$ on the unit disc U are considered in this paper. The subject of the present paper is to discuss some properties of the function $f(z)$ which belongs to the class of $\mathcal{M}$ by applying Jack's lemma.

In this paper, we will give the estimates for the Hankel determinant of the first order for the class of analytic function $f \in \mathcal{A}$ will satisfy the condition (1.1). In particular, upper bounds on $H_2(1)$ will be obtained for the class $\mathcal{M}$. In addition, the relationship between the coefficients of the Hankel determinant and the angular derivative of the function f, which provides the class $\mathcal{M}$, will be examined. In this examine, the coefficients c_2, c_3 and c_4 will be used. Let $f \in \mathcal{A}$. The qth Hankel determinant of f for $n \geq 0$ and $q \geq 1$ is stated by Thomas and Noonan [19] as

$$H_q(n)=\begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+q-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+q-1} & c_{n+q} & \cdots & c_{n+2q-2} \end{vmatrix}, \quad c_1=1.$$

From the Hankel determinant for $n=1$ and $q=2$, we have

$$H_2(1)=\begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix}=c_3-c_2^2.$$

Here, the Hankel determinant $H_2(1)=c_3-c_2^2$ is well-known as Fekete–Szegö functional [18]. In [19], the authors have obtained the upper bounds of the Hankel determinant $\left|c_2c_4-c_3^2\right|$. Also, in [16], the author have obtained the upper bounds the Hankel determinant $A_n^{(k)}$. Moreover, in [17], the authors have given bounds for the second Hankel determinant for class $\mathcal{M}_\alpha$.

Let $f \in \mathcal{M}$ and consider the function

$$\Psi(z)=2\left[\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right].$$

It is an analytic function in U and $\Psi(0)=0$. Now, let us show that $|\Psi(z)|<1$ in U. If the logarithm differentiation of both sides is taken in the last equation, we obtain

$$\ln\left(1+\frac{1}{2}\Psi(z)\right)=\ln\left(\left(\frac{z}{f(z)}\right)^2 f'(z)\right),$$

$$\frac{\frac{1}{2}\Psi'(z)}{1+\frac{1}{2}\Psi(z)}=\frac{2}{z}-2\frac{f'(z)}{f(z)}+\frac{f''(z)}{f'(z)}$$

and

$$\frac{\frac{1}{2}z\Psi'(z)}{1+\frac{1}{2}\Psi(z)} = 2 - 2\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}.$$

Therefore, we have

$$\Re\left[\left(\frac{\left(\frac{z}{f(z)}\right)^2 f'(z)}{\left(\frac{z}{f(z)}\right)^2 f'(z)-1}\right)\left(2+\frac{zf''(z)}{f'(z)}-2\frac{zf'(z)}{f(z)}\right)\right] = \Re\left(\frac{z\Psi'(z)}{\Psi(z)}\right).$$

We suppose that there exists a $z_0 \in U$ such that

$$\max_{|z|\le|z_0|}|\Psi(z)| = \left|\Psi(z_0)\right| = 1.$$

From Jack's lemma, we get

$$\Psi(z_0) = e^{i\theta} \qquad \text{and} \qquad \frac{z_0\Psi'(z_0)}{\Psi(z_0)} = k.$$

Therefore, we have

$$\Re\left[\left(\frac{\left(\frac{z_0}{f(z_0)}\right)^2 f'(z_0)}{\left(\frac{z_0}{f(z_0)}\right)^2 f'(z_0)-1}\right)\left(2+\frac{z_0f''(z_0)}{f'(z_0)}-2\frac{z_0f'(z_0)}{f(z_0)}\right)\right] =$$

$$= \Re\left(\frac{z_0\Psi'(z_0)}{\Psi(z_0)}\right) = \Re\left(\frac{k\Psi(z_0)}{\Psi(z_0)}\right) = \Re(k) = k \ge 1.$$

This contradicts the $f \in \mathcal{M}$. This means that there is no point $z_0 \in U$ such that

$$\max_{|z|\le|z_0|}|\Psi(z)| = \left|\Psi(z_0)\right| = 1.$$

Hence, we take $|\Psi(z)| < 1$ in U. From the Schwarz lemma, we obtain

$$\Psi(z) = 2\left[\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right] =$$

$$= 2\left[\left(\frac{z}{z+c_2z^2+c_3z^3+\ldots}\right)^2\left(1+2c_2z+3c_3z^2+\ldots\right)-1\right] =$$

$$= 2\left[(c_3-c_2^2)z^2 + (2c_4-4c_2c_3+2c_2^3)z^3+\ldots\right],$$

$$\frac{\Psi(z)}{z^2} = 2\left[(c_3-c_2^2) + (2c_4-4c_2c_3+2c_2^3)z+\ldots\right],$$

$$2|c_3-c_2^2| = 2\left|H_2(1)\right| \le 1$$

and

$$\left|H_2(1)\right| \le \frac{1}{2}.$$

We thus obtain the following lemma.

Lemma 2. *If* $f \in \mathcal{M}$, *then we have the inequality*

$$\left|H_2(1)\right| \le \frac{1}{2}. \tag{1.2}$$

Consider the product

$$B(z) = \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i} z}.$$

The function $B(z)$ is called a finite Blaschke product, where $z_1, z_2, \ldots, z_n \in U$. Let the function $\Psi(z)$ satisfy the condition of the Schwarz lemma and also have zeros $z_1, z_2, \ldots, z_n$. Thus, one can see that the inequality (1.2) can be strengthened by standard methods as follows:

$$\left|H_2(1)\right| \le \frac{1}{2} \prod_{i=1}^{n} |z_i|.$$

Since the area of applicability of Schwarz lemma is quite wide, there exist many studies about it. Some of these studies, which are called the boundary version of Schwarz lemma, are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz lemma is given as follows:

If f extends continuously to some boundary point b with $|b| = 1$ and if $\left|f(b)\right| = 1$ and $f'(b)$ exists, then $\left|f'(b)\right| \ge 1$, which is known as the Schwarz lemma on the boundary. In addition to conditions of the boundary Schwarz lemma, if f fixes the point zero, that is, $f(z) = c_p z^p + c_{p+1} z^{p+1} + \ldots$, then the inequality

$$\left|f'(b)\right| \ge p + \frac{1 - |c_p|}{1 + |c_p|} \tag{1.3}$$

and

$$\left|f'(b)\right| \ge p \tag{1.4}$$

are obtained [11]. Inequality (1.3) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1 – 4, 7, 9 – 14]. Mercer [8] proves a version of the Schwarz lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [9]. In addition, he obtains a new boundary Schwarz lemma, for analytic functions mapping the unit disk to itself [10].

The following lemma, known as the Julia – Wolff lemma, is needed in the sequel (see [15]).

Lemma 3 (Julia – Wolff lemma). *Let* f *be an analytic function in* U, $f(0) = 0$ *and* $f(U) \subset U$. *If, in addition, the function* f *has an angular limit* $f(b)$ *at* $b \in \partial U$, $\left|f(b)\right| = 1$, *then the angular derivative* $f'(b)$ *exists and* $1 \le \left|f'(b)\right| \le \infty$.

***Corollary* 1.** *The analytic function* f *has a finite angular derivative* $f'(b)$ *if and only if* f' *has the finite angular limit* $f'(b)$ *at* $b \in \partial U$.

2. Main results. In this section, we discuss different versions of the boundary Schwarz lemma and the Hankel determinant for $\mathcal{M}$ class. Assuming the existence of angular limit on a boundary point, we obtain some estimations from below for the moduli of derivatives of analytic functions from a certain class. In the inequalities obtained, the relationship between the Hankel determinant and the second angular derivative of the $f(z)$ function was established.

Theorem 1. *Let $f \in \mathcal{M}$. Assume that, for some $b \in \partial U$, f has an angular limit $f(b)$ at b, $f(b) = \frac{2b}{3}$ and $f'(b) = \frac{2}{3}$. Then we have the inequality*

$$\left|f''(b)\right| \geq \frac{4}{9}. \tag{2.1}$$

Proof. Let us consider the function

$$\Psi(z) = 2\left[\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right].$$

$\Psi(z)$ is an analytic function in U, $\Psi(0) = 0$ and $|\Psi(z)| < 1$ for $z \in U$. Also, since $f(b) = \frac{2b}{3}$ and $f'(b) = \frac{2}{3}$, we take

$$\Psi(b) = 2\left[\left(\frac{b}{f(b)}\right)^2 f'(b) - 1\right] =$$

$$= 2\left[\left(\frac{b}{\frac{2b}{3}}\right)^2 \frac{2}{3} - 1\right] = 2\left[\frac{9}{4}\frac{2}{3} - 1\right] = 1$$

and

$$\left|\Psi(b)\right| = 1.$$

Therefore, from (1.4) for $p = 2$, we get

$$2 \leq |\Psi'(b)| = 2\left|\frac{\left(2bf'(b) + f''(b)b^2\right)\left(f(b)\right)^2 - 2f(b)f'(b)b^2 f'(b)}{\left(f(b)\right)^4}\right| =$$

$$= 2\left|\frac{2bf'(b)}{\left(f(b)\right)^2} + \frac{f''(b)b^2}{\left(f(b)\right)^2} - \frac{2b^2\left(f'(b)\right)^2}{\left(f(b)\right)^3}\right| =$$

$$= 2\left|\frac{2b\frac{2}{3}}{\left(\frac{2b}{3}\right)^2} + \frac{f''(b)b^2}{\left(\frac{2b}{3}\right)^2} - \frac{2b^2\left(\frac{2}{3}\right)^2}{\left(\frac{2b}{3}\right)^3}\right| =$$

$$= 2\left|\frac{3}{b} + f''(b)\frac{9}{4} - \frac{3}{b}\right| =$$

$$= 2\left|f''(b)\right|\frac{9}{4} = \frac{9}{2}\left|f''(b)\right|$$

and

$$\left|f''(b)\right| \geq \frac{4}{9}.$$

Theorem 1 is proved.

Inequality (2.1) can be strengthened as below by taking into account c_2 and c_3 which is second and third coefficients in the expansion of the function $f(z) = z + c_2z^2 + c_3z^3 + \ldots$.

Theorem 2. *Under the same assumptions as in Theorem* 1, *we have*

$$\left|f''(b)\right| \geq \frac{2}{9}\left(1 + \frac{2}{1 + \left|H_2(1)\right|}\right). \tag{2.2}$$

Proof. Let $\Phi(z)$ be the same as in the proof of Theorem 1. Therefore, from (1.3) for $p = 2$, we obtain

$$2 + \frac{1 - |a_2|}{1 + |a_2|} \leq |\Psi'(b)| = \frac{9}{2}\left|f''(b)\right|,$$

where $|a_2| = \dfrac{\left|\Psi''(0)\right|}{2!} = 2|c_3 - c_2^2| = 2\left|H_2(1)\right|$.

Therefore, we take

$$2 + \frac{1 - 2\left|H_2(1)\right|}{1 + 2\left|H_2(1)\right|} \leq \frac{9}{2}\left|f''(b)\right|,$$

$$1 + \frac{2}{1 + 2\left|H_2(1)\right|} \leq \frac{9}{2}\left|f''(b)\right|,$$

and

$$\left|f''(b)\right| \geq \frac{2}{9}\left(1 + \frac{2}{1 + \left|H_2(1)\right|}\right).$$

Theorem 2 is proved.

In the following theorem, inequality (2.2) has been strenghened by adding the consecutive term c_4 of $f(z)$ function.

Theorem 3. *Let $f \in \mathcal{M}$. Assume that, for some $b \in \partial U$, f has an angular limit $f(b)$ at b, $f(b) = \dfrac{2b}{3}$ and $f'(b) = \dfrac{2}{3}$. Then we have the inequality*

$$\left|f''(b)\right| \geq \frac{2}{9}\left(2 + \frac{\left(1 - \left|H_2(1)\right|\right)^2}{1 - \left(2\left|H_2(1)\right|\right)^2 + 4\left|c_4 - c_2(c_2^2 + 2H_2(1))\right|}\right). \tag{2.3}$$

Proof. Let $\Psi(z)$ be the same as in the proof of Theorem 1 and $B_0(z) = z^2$. By the maximum principle, for each $z \in U$, we have the inequality $|\Psi(z)| \leq |B_0(z)|$. Therefore,

$$m(z) = \frac{\Psi(z)}{B_0(z)} = \frac{2\left[\left(\dfrac{z}{f(z)}\right)^2 f'(z) - 1\right]}{z^2} =$$

$$= 2\Big[\left(c_3 - c_2^2\right) + (2c_4 - 4c_2c_3 + 2c_2^3)z + \ldots\Big]$$

is analytic function in U and $\left|m(z)\right| \leq 1$ for $|z| < 1$. In particular, we have

$$|m(0)| = 2|c_3 - c_2^2| = 2\left|H_2(1)\right| \tag{2.4}$$

and

$$|m'(0)| = 2\left|2c_4 - 4c_2c_3 + 2c_2^3\right| = 4\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|.$$

Furthermore, the geometric meaning of the derivative and the inequality $|\Psi(z)| \leq |B_0(z)|$ imply the inequality

$$\frac{b\Psi'(b)}{\Psi(b)} = |\Psi'(b)| \geq |B_0'(b)| = \frac{bB_0'(b)}{B_0(b)}.$$

The composite function

$$H(z) = \frac{m(z) - m(0)}{1 - \overline{m(0)}m(z)}$$

is analytic in U, $H(0) = 0$, $|H(z)| < 1$ for $|z| < 1$ and $|H(b)| = 1$ for $b \in \partial U$. For $p = 1$, from (1.3), we obtain

$$\frac{2}{1 + \left|H'(0)\right|} \leq |H'(b)| = \frac{1 - |m(0)|^2}{\left|1 - \overline{m(0)}m(b)\right|^2}|m'(b)| \leq$$

$$\leq \frac{1 + |m(0)|}{1 - |m(0)|}\left\{|\Psi'(b)| - |B_0'(b)|\right\} =$$

$$= \frac{1 + 2\left|H_2(1)\right|}{1 - 2\left|H_2(1)\right|}\left(\frac{9}{2}\left|f''(b)\right| - 2\right).$$

Since

$$H'(z) = \frac{1 - |m(0)|^2}{\left(1 - \overline{m(0)}m(z)^2\right)}m'(z)$$

and

$$\left|H'(0)\right| = \frac{|m'(0)|}{1 - |m(0)|^2} = \frac{4\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{1 - \left(2\left|H_2(1)\right|\right)^2},$$

we get

$$\frac{2}{1 + \dfrac{4\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{1 - \left(2\left|H_2(1)\right|\right)^2}} \leq \frac{1 + 2\left|H_2(1)\right|}{1 - 2\left|H_2(1)\right|}\left(\frac{9}{2}\left|f''(b)\right| - 2\right)$$

and

$$\left|f''(b)\right| \geq \frac{2}{9}\left(2 + \frac{\left(1 - |H_2(1)|\right)^2}{1 - \left(2|H_2(1)|\right)^2 + 4\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}\right).$$

Theorem 3 is proved.

If $f(z) - z$ a have zeros different from $z = 0$, taking into account these zeros, inequality (2.3) can be strengthened in another way. This is given by the following theorem.

Theorem 4. *Let $f \in \mathcal{M}$. Assume that, for some $b \in \partial U$, f has an angular limit $f(b)$ at b, $f(b) = \dfrac{2b}{3}$ and $f'(b) = \dfrac{2}{3}$. Let $z_1, z_2, \ldots, z_n$ be zeros of the function $f(z) - z$ in U that are different from zero. Then we have the inequality*

$$\left|f''(b)\right| \geq \frac{2}{9}\left(2 + \sum_{i=1}^{n} \frac{1-|z_i|^2}{|b-z_i|^2} + \right.$$

$$\left. + \frac{2\left(\prod_{i=1}^{n}|z_i| - 2\left|H_2(1)\right|\right)^2}{\left(\prod_{i=1}^{n}|z_i|\right)^2 - 4\left|H_2(1)\right|^2 + 2\prod_{i=1}^{n}|z_i| \left|2\left(c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right) + H_2(1)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}\right). \tag{2.5}$$

Proof. Let $\Psi(z)$ be as in the proof of Theorem 1 and $z_1, z_2, \ldots, z_n$ be zeros of the function $f(z) - z$ in U that are different from zero. Let

$$B_1(z) = z^2 \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i} z}.$$

$B(z)$ is an analytic function in U and $|B(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in U$, we have $|\Psi(z)| \leq |B_1(z)|$. Consider the function

$$R(z) = \frac{\Psi(z)}{B_1(z)} = 2\left[\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right] \frac{1}{z^2 \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i} z}} =$$

$$= 2\frac{\left(c_3 - c_2^2\right)z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right)z^3 + \ldots}{z^2 \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i} z}} =$$

$$= 2\frac{\left(c_3 - c_2^2\right) + \left(2c_4 - 4c_2c_3 + 2c_2^3\right)z + \ldots}{\prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i} z}}.$$

$R(z)$ is analytic in U and $|R(z)| < 1$ for $z \in U$. In particular, we have

$$|R(0)| = 2\frac{|c_3 - c_2^2|}{\prod_{i=1}^{n}|z_i|} = \frac{2\left|H_2(1)\right|}{\prod_{i=1}^{n}|z_i|}$$

and

$$|R'(0)| = 2\frac{\left|2c_4 - 4c_2c_3 + 2c_2^3 + \left(c_3 - c_2^2\right)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\prod_{i=1}^{n}|z_i|}.$$

Moreover, with the simple calculations, we get

$$\frac{b\Psi'(b)}{\Psi(b)} = |\Psi'(b)| \geq |B_1'(b)| = \frac{bB_1'(b)}{B_1(b)}$$

and

$$|B_1'(b)| = 2 + \sum_{i=1}^{n} \frac{1-|z_i|^2}{|b-z_i|^2}.$$

The auxiliary function

$$\Phi(z) = \frac{R(z)-R(0)}{1-\overline{R(0)}R(z)}$$

is analytic in the unit disc U, $\Phi(0)=0$, $|\Phi(z)|<1$ for $z\in U$ and $|\Phi(b)|=1$ for $b\in\partial U$. From (1.3) for $p=1$, we obtain

$$\frac{2}{1+|\Phi'(0)|} \le \left|\Phi'(b)\right| = \frac{1+|R(0)|^2}{\left|1-\overline{R(0)}R(b)\right|^2}|R'(b)| \le$$

$$\le \frac{1+|R(0)|}{1-|R(0)|}\left\{|\Psi'(b)| - |B_1'(b)|\right\}.$$

Since

$$|\Phi'(0)| = \frac{|R'(0)|}{1-|R(0)|^2} = \frac{\dfrac{2\left|2c_4-4c_2c_3+2c_2^3+\left(c_3-c_2^2\right)\sum_{i=1}^{n}\dfrac{1-|z_i|^2}{z_i}\right|}{\prod_{i=1}^{n}|z_i|}}{1-\left(\dfrac{2\left|H_2(1)\right|}{\prod_{i=1}^{n}|z_i|}\right)^2} =$$

$$= 2\prod_{i=1}^{n}|z_i|\frac{\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)+H_2(1)\sum_{i=1}^{n}\dfrac{1-|z_i|^2}{z_i}\right|}{\left(\prod_{i=1}^{n}|z_i|\right)^2-4\left|H_2(1)\right|^2},$$

we get

$$\frac{2}{1+2\prod_{i=1}^{n}|z_i|\dfrac{\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)+H_2(1)\sum_{i=1}^{n}\dfrac{1-|z_i|^2}{z_i}\right|}{\left(\prod_{i=1}^{n}|z_i|\right)^2-4\left|H_2(1)\right|^2}} \le$$

$$\le \frac{1+\dfrac{2\left|H_2(1)\right|}{\prod_{i=1}^{n}|z_i|}}{1-\dfrac{2\left|H_2(1)\right|}{\prod_{i=1}^{n}|z_i|}}\left\{\frac{9}{2}\left|f''(b)\right|-2-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2}\right\},$$

$$\frac{2\left(\left(\prod_{i=1}^{n}|z_i|\right)^2-4\left|H_2(1)\right|^2\right)}{\left(\prod_{i=1}^{n}|z_i|\right)^2-4\left|H_2(1)\right|^2+2\prod_{i=1}^{n}|z_i|\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)+H_2(1)\sum_{i=1}^{n}\dfrac{1-|z_i|^2}{z_i}\right|} \le$$

$$\leq \frac{\prod_{i=1}^{n}|z_i| + 2\left|H_2(1)\right|}{\prod_{i=1}^{n}|z_i| - 2\left|H_2(1)\right|}\left\{\frac{9}{2}\left|f''(b)\right| - 2 - \sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2}\right\},$$

$$\frac{2\left(\prod_{i=1}^{n}|z_i| - 2\left|H_2(1)\right|\right)^2}{\left(\prod_{i=1}^{n}|z_i|\right)^2 - 4\left|H_2(1)\right|^2 + 2\prod_{i=1}^{n}|z_i|\left|2\left(c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right) + H_2(1)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|} \leq$$

$$\leq \frac{9}{2}\left|f''(b)\right| - 2 - \sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2}$$

and

$$\left|f''(b)\right| \geq \frac{2}{9}\left(2 + \sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2} + \right.$$

$$\left. + \frac{2\left(\prod_{i=1}^{n}|z_i| - 2\left|H_2(1)\right|\right)^2}{\left(\prod_{i=1}^{n}|z_i|\right)^2 - 4\left|H_2(1)\right|^2 + 2\prod_{i=1}^{n}|z_i|\left|2\left(c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right) + H_2(1)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}\right).$$

Theorem 4 is proved.

If $f(z) - z$ has no zeros different from $z = 0$ in Theorem 3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 5. *Let $f \in \mathcal{M}$ and $c_3 > c_2^2$ $(c_2 > 0,\ c_3 > 0)$. Also, $f(z) - z$ has no zeros in U except $z = 0$. Further assume that, for some $b \in \partial U$, f has an angular limit $f(b)$ at b, $f(b) = \frac{2b}{3}$ and $f'(b) = \frac{2}{3}$. Then we have the inequalities*

$$\left|f''(b)\right| \geq \frac{4}{9}\left(1 - \frac{1}{2}\frac{H_2(1)\ln\left(2H_2(1)\right)}{H_2(1)\ln\left(2H_2(1)\right) - \left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}\right)$$

and

$$\left|c_4 - c_2(c_2^2 + 2H_2(1))\right| \leq \left|H_2(1)\ln(2H_2(1))\right|.$$

Proof. Let $c_3 > c_2^2$ and $\Psi(z)$, $m(z)$ be as in the proof of Theorem 3. Having in mind inequality (2.4), we denote by $\ln m(z)$ the analytic branch of the logarithm normed by the condition

$$\ln m(0) = \ln\left(2(c_3 - c_2^2)\right) = \ln 2H_2(1) < 0.$$

The function

$$l(z) = \frac{\ln m(z) - \ln m(0)}{\ln m(z) + \ln m(0)}$$

is analytic in the unit disc U, $|l(z)| < 1$ for $z \in U$, $l(0) = 0$ and $|l(b)| = 1$ for $b \in \partial U$. From (1.3) for $p = 1$, we obtain

$$\frac{2}{1+|l'(0)|} \leq \left|l'(b)\right| = \frac{|2\ln m(0)|}{|\ln m(b)+\ln m(0)|^2}\left|\frac{m'(c)}{m(c)}\right| =$$

$$= \frac{-2\ln m(0)}{\ln^2 m(0)+\arg^2 m(b)}\left\{|\Psi'(b)|-|B_0'(b)|\right\}.$$

Since

$$\left|l'(0)\right| = \frac{1}{|2\ln m(0)|}\left|\frac{m'(0)}{m(0)}\right| = \frac{-1}{2\ln\left(2H_2(1)\right)}\frac{4\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{2\left|H_2(1)\right|} =$$

$$= \frac{-1}{2\ln\left(2H_2(1)\right)}\frac{2\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{H_2(1)} =$$

$$= \frac{-1}{\ln\left(2H_2(1)\right)}\frac{\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{H_2(1)},$$

we have

$$\frac{1}{1-\dfrac{\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{H_2(1)\ln\left(2H_2(1)\right)}} \leq \frac{-\ln m(0)}{\ln^2 m(0)+\arg^2 m(b)}\left(\frac{9}{2}\left|f''(b)\right|-2\right).$$

Replacing $\arg^2 m(b)$ by zero, we take

$$\frac{1}{1-\dfrac{\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{H_2(1)\ln\left(2H_2(1)\right)}} \leq \frac{-1}{\ln m(0)}\left(\frac{9}{2}\left|f''(b)\right|-2\right) = \frac{-1}{\ln\left(2H_2(1)\right)}\left(\frac{9}{2}\left|f''(b)\right|-2\right),$$

$$2-\frac{H_2(1)\ln\left(2H_2(1)\right)}{H_2(1)\ln\left(2H_2(1)\right)-\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|} \leq \frac{9}{2}\left|f''(b)\right|$$

and

$$\left|f''(b)\right| \geq \frac{4}{9}\left(1-\frac{1}{2}\frac{H_2(1)\ln\left(2H_2(1)\right)}{H_2(1)\ln\left(2H_2(1)\right)-\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}\right).$$

Similarly, the function $l(z)$ satisfies the assumptions of the Schwarz lemma, we obtain

$$1 \geq \left|l'(0)\right| = \frac{|2\ln m(0)|}{|\ln m(0)+\ln m(0)|^2}\left|\frac{m'(0)}{m(0)}\right| = \frac{-1}{2\ln m(0)}\left|\frac{m'(0)}{m(0)}\right| =$$

$$= \frac{-1}{2\ln\left(2H_2(1)\right)}\frac{4\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{2\left|H_2(1)\right|} =$$

$$= \frac{-1}{\ln\left(2H_2(1)\right)}\frac{\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{\left|H_2(1)\right|}$$

and

$$\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right| \leq \left|H_2(1)\ln\left(2H_2(1)\right)\right|.$$

Theorem 5 is proved.

Theorem 6. *Under hypotheses of Theorem* 5, *we have*

$$\left|f''(b)\right| \geq \frac{4}{9}\left(1-\frac{1}{4}\ln\left(2H_2(1)\right)\right).$$

Proof. From the proof of Theorem 5, using inequality (1.3) for the function $l(z)$, for $p=1$, we obtain

$$1 \leq \left|l'(b)\right| = \frac{|2\ln m(0)|}{|\ln m(b)+\ln m(0)|^2}\left|\frac{m'(b)}{m(b)}\right| = \frac{-2}{\ln\left(2H_2(1)\right)}\left(\frac{9}{2}\left|f''(b)\right|-2\right)$$

and

$$\left|f''(b)\right| \geq \frac{4}{9}\left(1-\frac{1}{4}\ln\left(2H_2(1)\right)\right).$$

Theorem 6 is proved.

References

1. T. Akyel, B. N. Örnek, *Sharpened forms of the generalized Schwarz inequality on the boundary*, Proc. Indian Acad. Sci. Math. Sci., **126**, № 1, 69 – 78 (2016).
2. T. A. Azeroğlu, B. N. Örnek, *A refined Schwarz inequality on the boundary*, Complex Var. and Elliptic Equat., **58**, 571 – 577 (2013).
3. H. P. Boas, *Julius and Julia: mastering the art of the Schwarz lemma*, Amer. Math. Monthly, **117**, № 9, 770 – 785 (2010).
4. V. N. Dubinin, *The Schwarz inequality on the boundary for functions regular in the disc*, J. Math. Sci., **122**, 3623 – 3629 (2004).
5. G. M. Golusin, *Geometric theory of functions of complex variable* (in Russian), 2nd ed., Moscow (1966).
6. I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc., **3**, 469 – 474 (1971).
7. M. Mateljević, *Rigidity of holomorphic mappings & Schwarz and Jack lemma*; DOI:10.13140/RG.2.2.34140.90249.
8. P. R. Mercer, *Sharpened versions of the Schwarz lemma*, J. Math. Anal. and Appl., **205**, 508 – 511 (1997).
9. P. R. Mercer, *Boundary Schwarz inequalities arising from Rogosinski's lemma*, J. Class. Anal., **12**, 93 – 97 (2018).
10. P. R. Mercer, *An improved Schwarz lemma at the boundary*, Open Math., **16**, 1140 – 1144 (2018).
11. R. Osserman, *A sharp Schwarz inequality on the boundary*, Proc. Amer. Math. Soc., **128**, 3513 – 3517 (2000).
12. B. N. Örnek, T. Düzenli, *Bound estimates for the derivative of driving point impedance functions*, Filomat, **32**, № 18, 6211 – 6218 (2018).
13. B. N. Örnek, T. Düzenli, *Boundary analysis for the derivative of driving point impedance functions*, IEEE Trans. Circuits and Syst. Pt. II: Express Briefs, **65**, № 9, 1149 – 1153 (2018).
14. B. N. Örnek, *Sharpened forms of the Schwarz lemma on the boundary*, Bull. Korean Math. Soc., **50**, № 6, 2053 – 2059 (2013).
15. Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, Berlin (1992).
16. Ch. Pommerenke, *On the Hankel determinants of univalent functions*, Matematika, **14**, 108 – 112 (1967).
17. J. Sokól, D. K. Thomas, *The second Hankel determinant for alpha-convex functions*, Lith. Math. J., **58**, № 2, 212 – 218 (2018); DOI 10.1007/s10986-018-9397-0.
18. G. Szegö, M. Fekete, *Eine Bemerkung über ungerade schlichte Funktionen*, J. London Math. Soc., **2**, 85 – 89 (1933).
19. D. K. Thomas, J. W. Noonan, *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc., **223**, 337 – 346 (1976).

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