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S. Rashid (COMSATS Univ., Islamabad, Pakistan),
M. A. Noor (Government College Univ., Faisalabad, Pakistan),
K. I. Noor (COMSATS Univ., Islamabad, Pakistan)

INEQUALITIES INVOLVING NEW FRACTIONAL INTEGRALS TECHNIQUE VIA EXPONENTIALLY CONVEX FUNCTIONS

НЕРІВНОСТІ, ОТРИМАНІ ЗА ДОПОМОГОЮ ДРОБОВИХ ІНТЕГРАЛІВ ІЗ ВИКОРИСТАННЯМ ЕКСПОНЕНЦІАЛЬНО ОПУКЛИХ ФУНКЦІЙ

We establish some new Hermite – Hadamard type inequalities involving fractional integral operators with the exponential kernel. Meanwhile, we present many useful estimates on these types of new Hermite – Hadamard type inequalities via exponentially convex functions.

За допомогою дробових інтегральних операторів із експоненціальним ядром отримано кілька нерівностей типу Ерміта – Адамара. Серед іншого запропоновано багато корисних оцінок для цих нових нерівностей типу Ерміта – Адамара з використанням експоненціально опуклих функцій.

1. Introduction. Fractional calculus can be seen as a generalization of the ordinary differentiation and integration to an arbitrary non-integer order which has been recognized as one of the most powerful tools to describe long-memory processes in the last decades. Many phenomena from physics, chemistry, mechanics and electricity can be modeled by ordinary differential equations involving fractional derivatives (see [5, 9 – 12, 20, 21] and the references therein). There were several studies in the literature that include further properties such as expansion formulas, variational calculus applications, control theoretical applications, convexity and integral inequalities and Hermite – Hadamard type inequalities of this new operator and similar operators.

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. One of the most famous inequalities for convex functions is Hermite – Hadamard inequality, stated as [8]:

Let $\varphi\colon I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $\varsigma, \eta \in I$ with $\varsigma < \eta$. Then

$$\varphi\Big(\frac{\varsigma+\eta}{2}\Big) \le \frac{1}{\eta-\varsigma}\int\limits_{\varsigma}^{\eta}\varphi(x)dx \le \frac{\varphi(\varsigma)+\varphi(\eta)}{2}. \qquad (1.1)$$

Both inequalities hold in the reversed direction for φ to be concave.

In recent years, numerous generalizations, extensions and variants of Hermite – Hadamard inequality (1.1) were studied extensively by many researchers and appeared in a number of papers (see [1, 4, 6, 13 – 16, 18, 22 – 27]).

In [7], Fejér obtained the weighted generalizations of Hermite – Hadamard inequality (1.1) as follows:

Let $\varphi\colon [\varsigma, \eta] \to \mathbb{R}$ be a convex function. Then the inequality

$$\varphi\Big(\frac{\varsigma+\eta}{2}\Big)\int\limits_{\varsigma}^{\eta}\chi(x)dx \leq \frac{1}{\eta-\varsigma}\int\limits_{\varsigma}^{\eta}\varphi(x)\chi(x)dx \leq \frac{\varphi(\varsigma)+\varphi(\eta)}{2}\int\limits_{\varsigma}^{\eta}\chi(x)dx$$

holds for a nonnegative, integrable function $\chi\colon[\varsigma,\eta]\to\mathbb{R}$, which is symmetric to $\dfrac{\varsigma+\eta}{2}$.

Exponentially convex functions have emerged an a significant new class of convex functions, which have important applications in technology, data science and statistics. The main motivation of this paper depends on a new identity that has been proved via new fractional integrals operators with exponential kernel and applied for exponentially convex functions. This identity offers new upper bounds and estimations of Hadamard type integral inequalities. Some special cases such for $\alpha\to 1$ have been discussed, which can be deduced from these results.

To the best of our knowledge, a comprehensive investigation of exponentially convex functions as fractional integral with exponential kernel in the present paper is new one. The class of exponentially convex functions was introduced by Antczak [3] and Dragomir [6]. Motivated by these facts, Awan et al. [4] introduced and investigated another class of convex functions, which is called exponentially convex function and is significantly different from the class introduced by [3, 6, 19]. The growth of research on big data analysis and deep learning has recently increased the interest in information theory involving exponentially convex functions. The smoothness of exponentially convex function is exploited for statistical learning, sequential prediction and stochastic optimization (see [2, 3, 17] and the references therein).

It is known [6] that a function φ is exponentially convex if and only if φ satisfies the inequality

$$e^{\varphi(\frac{\varsigma+\eta}{2})} \leq \frac{1}{\eta-\varsigma}\int\limits_{\varsigma}^{\eta}e^{\varphi(x)}dx \leq \frac{e^{\varphi(\varsigma)}+e^{\varphi(\eta)}}{2}. \tag{1.2}$$

The inequality (1.2) is called the Hermite–Hadamard inequality and provides the upper and lower estimates for the exponential integral (see [22–24] and the references therein).

In this paper, we will establish here some new Hermite–Hadamard type inequalities involving fractional integral with an exponential kernel via exponentially convex functions. Meanwhile, we present many useful estimates on these types of new Hermite–Hadamard type inequalities for fractional integrals with exponential kernels.

2. Essential preliminaries. We now recall some well-known concepts and basic results, which are needed in the derivation of our results.

Definition 2.1. *A set $K\subset\mathbb{R}$ is said to be convex, if*

$$\tau x+(1-\tau)y\in K \quad \forall x,y\in K, \quad \tau\in[0,1].$$

Definition 2.2. *A function $\varphi\colon K\to\mathbb{R}$ is said to be a convex function if and only if*

$$\varphi(\tau x+(1-\tau)y)\leq\tau\varphi(x)+(1-\tau)\varphi(y) \quad \forall x,y\in K, \quad \tau\in[0,1].$$

We now consider class of exponentially convex function, which are mainly due to [3, 6].

Definition 2.3 [3, 6]. *A positive real-valued function $\varphi\colon K\subseteq\mathbb{R}\longrightarrow(0,\infty)$ is said to be exponentially convex on K, if the inequality*

$$e^{\varphi(\tau x+(1-\tau)y)}\leq\tau e^{\varphi(x)}+(1-\tau)e^{\varphi(y)}$$

holds for $x,y\in K$ and $t\in[0,1]$.

For $t = \frac{1}{2}$, we have Jensen type exponentially convex functions for Definition 2.3:

$$e^{\varphi(\frac{x+y}{2})} \leq \frac{e^{\varphi(x)} + e^{\varphi(y)}}{2}.$$

Exponentially convex functions are used to manipulate for statistical learning, sequential prediction and stochastic optimization (see [2, 3, 17] and the references therein).

It is known that $x \in K$ is the minimum of the differentiable exponentially convex functions φ if and only if $x \in K$ satisfies the inequality

$$\langle (e^{\varphi(x)})', y - x \rangle = \langle \varphi'(x) e^{\varphi(x)}, y - x \rangle \geq 0 \quad \forall y \in K. \tag{2.1}$$

The inequality of the type (2.1) is known as the exponentially variational inequality which appears to be new one.

For formulation, applications and other aspects of variational inequalities, see Noor [13 – 15].

Let us give some basic examples of exponentially convex functions (for details, see [19]).

(i) For every $\alpha > 0$, the function $\varphi(x) = e^{-\alpha\sqrt{x}}$ is exponentially convex on $(0, \infty)$, where

$$e^{-\alpha\sqrt{x}} = \int\limits_0^\infty \frac{\alpha}{2\sqrt{\pi\tau^3}} e^{-\tau(x + (\frac{\alpha}{2\tau})^2)} d\tau, \quad x > 0.$$

(ii) $\varphi(x) = x^{-\alpha}$ is exponentially convex on $(0, \infty)$ for any $\alpha > 0$.

(iii) Let ς, η be positive real numbers, $I = (0, \infty)$ and family $F = \{\varphi_\tau : \tau \in I\}$ of function defined on $C[\varsigma, \eta]$ with $\varphi_t(x) = \dfrac{e^{-x\sqrt{\tau}}}{(-\sqrt{\tau})^n}$ is an exponentially convex on $(0, \infty)$.

Now, some necessary definitions and mathematical preliminaries of fractional calculus theory are presented, which are used further in this paper.

Definition 2.4 [10]. *Let $\varphi \in L^1[\varsigma, \eta]$. The fractional integrals $\mathcal{I}_\varsigma^\alpha$ and $\mathcal{I}_\eta^\alpha$ of order $\alpha \in (0, 1)$ are defined as*

$$\mathcal{I}_\varsigma^\alpha \varphi(x) = \frac{1}{\alpha} \int\limits_\varsigma^x e^{\frac{\alpha-1}{\alpha}(x-u)} \varphi(u) du, \quad x > \varsigma, \tag{2.2}$$

and

$$\mathcal{I}_\eta^\alpha \varphi(x) = \frac{1}{\alpha} \int\limits_x^\eta e^{\frac{\alpha-1}{\alpha}(u-x)} \varphi(u) du, \quad x < \eta,$$

respectively.

If $\alpha = 1$, then

$$\lim_{\alpha \to 1} \mathcal{I}_\varsigma^\alpha \varphi(x) = \int\limits_\varsigma^x \varphi(u) du, \qquad \lim_{\alpha \to 1} \mathcal{I}_\eta^\alpha \varphi(x) = \int\limits_x^\eta \varphi(u) du.$$

Moreover,

$$\lim_{\alpha\to 0}\frac{1}{\alpha}e^{\frac{\alpha-1}{\alpha}(x-u)}=\vartheta(x-u),$$

we conclude that

$$\lim_{\alpha\to 0}\mathcal{I}_{\varsigma}^{\alpha}\varphi(x)=\varphi(x),\qquad \lim_{\alpha\to 1}\mathcal{I}_{\eta}^{\alpha}\varphi(x)=\varphi(x).$$

Definition 2.5. *The left and right Riemann – Liouville fractional integrals $J_{\varsigma+}^{\alpha}$ and $J_{\eta-}^{\alpha}$ of order $\alpha\in\mathbb{R}$ $(\alpha>0)$ are given by*

$$J_{\varsigma+}^{\alpha}\varphi(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{\varsigma}^{x}(x-u)^{\alpha-1}\varphi(u)du,\quad x>\varsigma,$$

and

$$J_{\eta-}^{\alpha}\varphi(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{x}^{\eta}(u-x)^{\alpha-1}\varphi(u)du,\quad x<\eta,$$

respectively. Here $\Gamma(\alpha)$ is the Euler gamma-function.

In the sequel of the paper, let $I\subset\mathbb{R}$ be a convex set in the finite dimensional Euclidean space $\mathbb{R}^n$. From now onwards we take $I=[\varsigma,\eta]$, unless otherwise specified. We henceforth set $\sigma=\dfrac{1-\alpha}{\alpha}(\eta-\varsigma)$.

3. Main results. An important Hermite – Hadamard inequality involving fractional integral with exponential kernel (with $\alpha\in\mathbb{R},\ \alpha\geq 0$) can be represented as follows.

Theorem 3.1. *Let $\alpha\in(0,1)$ and $\varphi\colon[\varsigma,\eta]\to\mathbb{R}$ be a positive function with $0\leq\varsigma<\eta$ and $e^{\varphi}\in L^1[\varsigma,\eta]$. If φ is a convex function on $[\varsigma,\eta]$, then the following inequalities for fractional integrals hold*:

$$e^{\varphi(\frac{\varsigma+\eta}{2})}\leq\frac{(1-\alpha)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\left[\mathcal{I}_{\varsigma}^{\alpha}e^{\varphi(\eta)}+\mathcal{I}_{\eta}^{\alpha}e^{\varphi(\varsigma)}\right]\leq\frac{e^{\varphi(\varsigma)}+e^{\varphi}(\eta)}{2}. \tag{3.1}$$

Proof. Since φ is an exponentially convex function on $[\varsigma,\eta]$, we have, for $x,y\in[\varsigma,\eta]$ with $\tau=\dfrac{1}{2}$,

$$e^{\varphi(\frac{x+y}{2})}\leq\frac{e^{\varphi(x)}+e^{\varphi(y)}}{2},$$

i.e., with $x=\tau\varsigma+(1-\tau)\eta,\ y=(1-\tau)\varsigma+\tau\eta$,

$$e^{\varphi(\frac{\varsigma+\eta}{2})}\leq\frac{e^{\varphi(\tau\varsigma+(1-\tau)\eta)}+e^{\varphi((1-\tau)\varsigma+\tau\eta)}}{2}. \tag{3.2}$$

Multiplying both sides of the above inequality by $e^{-\sigma\tau}$ and then integrating the resulting inequality with respect to τ over $[0,1]$, we obtain

$$\frac{4\sinh\left(\dfrac{\sigma}{2}\right)}{\sigma e^{\frac{\sigma}{2}}}e^{\varphi(\frac{\varsigma+\eta}{2})}\leq\int\limits_{0}^{1}e^{-\sigma\tau}e^{\varphi(\tau\varsigma+(1-\tau)\eta)}d\tau+\int\limits_{0}^{1}e^{-\sigma\tau}e^{\varphi((1-\tau)\varsigma+\tau\eta)}d\tau=$$

$$= \frac{1}{\eta - \varsigma} \int\limits_{\varsigma}^{\eta} e^{-\frac{1-\alpha}{\alpha}(\eta - u)} e^{f(u)} du + \frac{1}{\eta - \varsigma} \int\limits_{\varsigma}^{\eta} e^{-\frac{1-\alpha}{\alpha}(u - \varsigma)} e^{f(u)} du =$$

$$= \frac{\alpha}{\eta - \varsigma} \left[\mathcal{I}_{\varsigma}^{\alpha} e^{\varphi(\eta)} + \mathcal{I}_{\eta}^{\alpha} e^{\varphi(\varsigma)} \right]$$

and the first inequality in (3.1) is proved.

Because e^{φ} is convex function, we have

$$e^{\varphi(\tau\varsigma + (1-\tau)\eta)} + e^{\varphi((1-\tau)\varsigma + \tau\eta)} \leq \left[e^{\varphi(\varsigma)} + e^{\varphi(\eta)} \right].$$

Then multiplying both sides of the above inequality by $e^{-\sigma\tau}$ and integrating the resulting inequality with respect to τ over $[0, 1]$, we obtain the right-sided inequality in (3.1).

Theorem 3.1 is proved.

***Remark* 3.1.** For $\alpha \to 1$, observe that

$$\lim_{\alpha \to 1} \frac{(1-\alpha) e^{\frac{\sigma}{2}}}{4 \sinh\left(\dfrac{\sigma}{2}\right)} = \frac{1}{\eta - \varsigma}$$

and (3.1) reduces to Theorem 1 in [6].

***Remark* 3.2.** One can follow the same ideas to construct fractional version $\mathcal{F}_n^{(1)}$, $\mathcal{F}_n^{(2)}$, $\mathcal{F}_n^{(3)}$ (see [27]) try to extend to study using exponential kernel for Hermite–Hadamard inequalities in n variables based on these fundamental results. We shall study such interesting problems in the forthcoming works.

We now prove the Hermite–Hadamard–Fejér type inequality for exponentially convex function for new fractional integral operator technique.

Theorem 3.2. *Let $\varphi \colon [\varsigma, \eta] \to \mathbb{R}$ be convex and integrable function with $\varsigma < \eta$. If $\chi \colon [\varsigma, \eta] \to \mathbb{R}$ is nonnegative, integrable and symmetric with respect to $\dfrac{\varsigma + \eta}{2}$, that is, $\chi(\varsigma + \eta - x) = \chi(x)$, then the following fractional integral inequalities hold*:

$$e^{\varphi(\frac{\varsigma+\eta}{2})} \left[\mathcal{I}_{\varsigma}^{\alpha} e^{\chi(\eta)} + \mathcal{I}_{\eta}^{\alpha} e^{\chi(\varsigma)} \right] \leq \left[\mathcal{I}_{\varsigma}^{\alpha} e^{\varphi(\eta)} e^{\chi(\eta)} + \mathcal{I}_{\eta}^{\alpha} e^{\varphi(\varsigma)} e^{\chi(\varsigma)} \right] \leq$$

$$\leq \frac{\left[e^{\varphi(\varsigma)} + e^{\varphi(\eta)} \right]}{2} \left[\mathcal{I}_{\varsigma}^{\alpha} e^{\chi(\eta)} + \mathcal{I}_{\eta}^{\alpha} e^{\chi(\varsigma)} \right]. \tag{3.3}$$

Proof. Multiplying (3.2) with $e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma + \tau\eta)}$, we get

$$e^{\varphi(\frac{\varsigma+\eta}{2})} e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma + \tau\eta)} \leq$$

$$\leq e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma + \tau\eta)} e^{\varphi(\tau\varsigma + (1-\tau)\eta)} +$$

$$+ e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma + \tau\eta)} e^{\varphi((1-\tau)\varsigma + \tau\eta)}.$$

Integrating with respect to τ over $[0, 1]$, we have

$$e^{\varphi(\frac{\varsigma+\eta}{2})} \int\limits_0^1 e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma + \tau\eta)} d\tau \leq$$

$$\leq \int_0^1 e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma+\tau\eta)} e^{\varphi(\tau\varsigma+(1-\tau)\eta)} d\tau +$$

$$+ \int_0^1 e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma+\tau\eta)} e^{\varphi((1-\tau)\varsigma+\tau\eta)} d\tau. \tag{3.4}$$

If we put $u = \varsigma\tau + (1-\tau)\eta$, then $\tau = \dfrac{\eta - u}{\eta - \varsigma}$. So one has

$$e^{\varphi(\frac{\varsigma+\eta}{2})} \int_0^1 e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma+\tau\eta)} d\tau \leq$$

$$\leq \frac{1}{\eta-\varsigma} \int_\varsigma^\eta e^{-\frac{1-\alpha}{\alpha}(\eta-u)} e^{\varphi(u)} e^{\chi(u)} du + \frac{1}{\eta-\varsigma} \int_\varsigma^\eta e^{-\frac{1-\alpha}{\alpha}(u-\varsigma)} e^{\varphi(\varsigma+\eta-u)} e^{\chi(u)} du =$$

$$= \frac{1}{\eta-\varsigma} \int_\varsigma^\eta e^{-\frac{1-\alpha}{\alpha}(\eta-u)} e^{\varphi(u)} e^{\chi(\varsigma+\eta-u)} du + \frac{1}{\eta-\varsigma} \int_\varsigma^\eta e^{-\frac{1-\alpha}{\alpha}(u-\varsigma)} e^{\varphi(u)} e^{\chi(u)} du =$$

$$= \frac{\alpha}{\eta-\varsigma} \left[\mathcal{I}_\varsigma^\alpha e^{\varphi(\eta)} e^{\chi(\eta)} + \mathcal{I}_\eta^\alpha e^{\varphi(\varsigma)} e^{\chi(\varsigma)} \right].$$

By the symmetry of the function χ about $\dfrac{\varsigma+\eta}{2}$ one can see $\chi(\varsigma+\eta-x) = \chi(x),\ x \in [\varsigma, \eta]$, therefore, using this fact and Definition 2.4, we have

$$e^{\varphi(\frac{\varsigma+\eta}{2})} \left[\mathcal{I}_\varsigma^\alpha e^{\chi(\eta)} + \mathcal{I}_\eta^\alpha e^{\chi(\varsigma)} \right] \leq \frac{\alpha}{\eta-\varsigma} \left[\mathcal{I}_\varsigma^\alpha e^{\varphi(\eta)} e^{\chi(\eta)} + \mathcal{I}_\eta^\alpha e^{\varphi(\varsigma)} e^{\chi(\varsigma)} \right].$$

Now multiplying (3.4) with $e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma+\tau\eta)}$ and integrating with respect to τ over $[0,1]$, we get

$$\int_0^1 e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma+\tau\eta)} e^{\varphi(\tau\varsigma+(1-\tau)\eta)} d\tau +$$

$$+ \int_0^1 e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma+\tau\eta)} e^{\varphi((1-\tau)\varsigma+\tau\eta)} d\tau \leq$$

$$\leq \left[e^{\varphi(\varsigma)} + e^{\varphi(\eta)} \right] \int_0^1 e^{-\sigma\tau} e^{\chi((1-\tau)\varsigma+\tau\eta)} d\tau.$$

From this by setting $x = \varsigma(1-\tau) + \tau\eta$ and using $e^{\varphi(\varsigma+\eta-x)} = e^{\varphi(x)}$ it can be seen

$$\left[\mathcal{I}_\varsigma^\alpha e^{\varphi(\eta)} e^{\chi(\eta)} + \mathcal{I}_\eta^\alpha e^{\varphi(\varsigma)} e^{\chi(\varsigma)} \right] \leq \frac{\left[e^{\varphi(\varsigma)} + e^{\varphi(\eta)} \right]}{2} \left[\mathcal{I}_\varsigma^\alpha e^{\chi(\eta)} + \mathcal{I}_\eta^\alpha e^{\chi(\varsigma)} \right].$$

Theorem 3.2 is proved.

***Corollary* 3.1.** *If we choose* $\alpha = 1$, *then the inequality* (3.3) *becomes*

$$e^{\varphi(\frac{\varsigma+\eta}{2})}\int\limits_{\varsigma}^{\eta} e^{\chi(x)}dx \le \int\limits_{\varsigma}^{\eta} e^{\varphi(x)}e^{\chi(x)}dx \le \frac{e^{\varphi(\varsigma)}+e^{\varphi(\eta)}}{2}\int\limits_{\varsigma}^{\eta} e^{\chi(x)}dx.$$

***Remark* 3.3.** If we take $e^{\chi(x)} = 1$ and $\alpha = 1$, then the inequality (3.3) reduces to Theorem 1 (see [6]).

For next result we need the following lemma.

Lemma 3.1. *Let* $\varphi\colon [\varsigma,\eta] \to \mathbb{R}$ *be a differentiable mapping on* (ς,η) *with* $\varsigma < \eta$. *If* $(e^{\varphi})' \in$ $\in L^1[\varsigma,\eta]$, *then the following equality for fractional integrals holds*:

$$\frac{e^{\varphi(\varsigma)}+e^{\varphi(\eta)}}{2} - \frac{(1-\alpha)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\left[\mathcal{I}_{\varsigma}^{\alpha}e^{\varphi(\eta)} + \mathcal{I}_{\eta}^{\alpha}e^{\varphi(\varsigma)}\right] =$$

$$= \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\int\limits_0^1 \left[e^{-\sigma\tau} - e^{-\sigma(1-\tau)}\right]e^{\varphi(\varsigma\tau+(1-\tau)\eta)}\varphi'(\varsigma\tau+(1-\tau)\eta)d\tau. \tag{3.5}$$

Proof. Consider the right-hand side of (3.5):

$$\frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\int\limits_0^1 \left[e^{-\sigma\tau} - e^{-\sigma(1-\tau)}\right]e^{\varphi(\varsigma\tau+(1-\tau)\eta)}\varphi'(\varsigma\tau+(1-\tau)\eta)d\tau =$$

$$= \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\left[\int\limits_0^1 e^{-\sigma\tau}e^{\varphi(\varsigma\tau+(1-\tau)\eta)}\varphi'(a\tau+(1-\tau)\eta)d\tau - \right.$$

$$\left. - \int\limits_0^1 e^{-\sigma(1-\tau)}e^{\varphi(\varsigma(1-\tau)+\tau\eta)}\varphi'(\varsigma\tau+(1-\tau)\eta)d\tau\right].$$

Now we compute the first and the second terms of last expression as follows:

$$\frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\int\limits_0^1 e^{-\sigma\tau}e^{\varphi(\varsigma\tau+(1-\tau)\eta)}\varphi'(\varsigma\tau+(1-\tau)\eta)d\tau =$$

$$= \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\left[\frac{e^{-\sigma\tau}e^{\varphi(\tau\varsigma+(1-\tau)\eta)}}{\varsigma-\eta} + \frac{\sigma}{\varsigma-\eta}\int\limits_0^1 e^{-\sigma\tau}e^{\varphi(\tau\varsigma+(1-\tau)\eta)}d\tau\right] =$$

$$= \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\left[\frac{e^{-\sigma}e^{\varphi(\varsigma)-e^{\varphi(\eta)}}}{\varsigma-\eta} + \frac{\sigma}{\varsigma-\eta}\int\limits_0^1 e^{-\sigma\tau}e^{\varphi(\tau\varsigma+(1-\tau)\eta)}d\tau\right] =$$

$$= \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\left[\frac{e^{\varphi(\eta)}-e^{\sigma}e^{\varphi(\varsigma)}}{\eta-\varsigma} - \frac{1-\alpha}{\alpha}\int\limits_{\varsigma}^{\eta} e^{-\frac{1-\alpha}{\alpha}(\eta-u)}e^{\varphi(u)du}\right] =$$

$$= \frac{e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\Big[e^{\varphi(\eta)} - e^{-\sigma}e^{\varphi(\varsigma)} - (1-\alpha)\mathcal{I}_{\varsigma}^{\alpha}e^{\varphi(\eta)}\Big].$$

Analogously we have

$$\frac{\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\int\limits_0^1 e^{-\sigma(1-\tau)}e^{\varphi(\varsigma(1-\tau)+\tau\eta)}\varphi'(\varsigma\tau+(1-\tau)\eta)d\tau =$$

$$= \frac{e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\Big[e^{\varphi(\varsigma)} - e^{-\sigma}e^{\varphi(\eta)} - (1-\alpha)\mathcal{I}_{\eta}^{\alpha}e^{\varphi(\varsigma)}\Big].$$

Hence, the required inequality can be established.

Lemma 3.1 is proved.

Using the above lemma we establish the bounds of a difference of (3.1).

Theorem 3.3. *Let* $\varphi\colon [\varsigma,\eta] \to \mathbb{R}$ *be a differentiable mapping on* (ς,η) *with* $\varsigma < \eta$ *such that* $e^{\varphi} \in L^1[\varsigma,\eta]$. *If* $(e^{\varphi})'$ *is convex on* $[\varsigma,\eta]$, *then the following inequality for fractional integral holds*:

$$\left|\frac{e^{\varphi(\varsigma)}+e^{\varphi(\eta)}}{2} - \frac{(1-\alpha)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\big[\mathcal{I}_{\varsigma}^{\alpha}e^{\varphi(\eta)} + \mathcal{I}_{\eta}^{\alpha}e^{\varphi(\varsigma)}\big]\right| \le$$

$$\le \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\left\{\left[\frac{(\sigma^2+8)e^{\frac{\sigma}{2}} + e^{-\sigma}(\sigma^2+2\sigma+4) - (\sigma^2-2\sigma+4)}{\sigma^3}\right]\times\right.$$

$$\times\big[|e^{\varphi(\varsigma)}\varphi'(\varsigma)| + |e^{\varphi(\eta)}\varphi'(\eta)|\big] +$$

$$\left. + \frac{(\sigma^2-8)e^{-\frac{\sigma}{2}} + 2(\sigma+2)e^{-\sigma} - 2(\sigma-2)}{\sigma^3}\Theta(\varsigma,\eta)\right\},$$

where

$$\Theta(\varsigma,\eta) = \big|e^{\varphi(\varsigma)}\varphi'(\eta)\big| + \big|e^{\varphi(\eta)}\varphi'(\varsigma)\big|.$$

Proof. Using Lemma 3.1 and the exponentially convexity of φ, we have

$$\left|\frac{e^{\varphi(\varsigma)}+e^{\varphi(\eta)}}{2} - \frac{(1-\alpha)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\big[\mathcal{I}_{\varsigma}^{\alpha}e^{\varphi(\eta)} + \mathcal{I}_{\eta}^{\alpha}e^{\varphi(\varsigma)}\big]\right| \le$$

$$\le \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\int\limits_0^1 |e^{-\sigma\tau} - e^{-\sigma(1-\tau)}|\big|e^{\varphi(\varsigma\tau+(1-\tau)\eta)}\varphi'(\varsigma\tau+(1-\tau)\eta)\big|d\tau \le$$

$$\le \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\int\limits_0^1 |e^{-\sigma\tau} - e^{-\sigma(1-\tau)}|\times$$

$$\times\big\{\tau|e^{\varphi}(\varsigma)| + (1-\tau)|e^{\varphi}(\eta)|\big\}\big\{\tau|\varphi'(\varsigma)| + (1-\tau)|\varphi'(\eta)|\big\}d\tau =$$

$$= \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)} \int\limits_0^1 |e^{-\sigma\tau} - e^{-\sigma(1-\tau)}| \Big\{\tau^2 |e^{\varphi(\varsigma)}\varphi'(\varsigma)| + (1-\tau^2)|e^{\varphi(\eta)}\varphi'(\eta)| +$$

$$+\tau(1-\tau)\big\{|e^{\varphi(\eta)}\varphi'(\varsigma)| + |e^{\varphi(\varsigma)}\varphi'(\eta)|\big\}\Big\} d\tau =$$

$$= \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)} \int\limits_0^1 |e^{-\sigma\tau} - e^{-\sigma(1-\tau)}| \Big\{\tau^2 |e^{\varphi(\varsigma)}\varphi'(\varsigma)| +$$

$$+(1-\tau^2)|e^{\varphi(\eta)}\varphi'(\eta)| + \tau(1-\tau)\Theta(\varsigma,\eta)\Big\} d\tau =$$

$$= \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)} \left[\int\limits_0^{\frac{1}{2}} [e^{-\sigma(1-\tau)} - e^{-\sigma\tau}] \Big\{\tau^2 |e^{\varphi(\varsigma)}\varphi'(\varsigma)| + \right.$$

$$+(1-\tau^2)|e^{\varphi(\eta)}\varphi'(\eta)| + \tau(1-\tau)\Theta(\varsigma,\eta)\Big\} d\tau +$$

$$\left. + \int\limits_{\frac{1}{2}}^1 \big[e^{-\sigma\tau} - e^{-\sigma(1-\tau)}\big] \Big\{\tau^2 |e^{\varphi(\varsigma)}\varphi'(\varsigma)| + (1-\tau^2)|e^{\varphi(\eta)}\varphi'(\eta)| + \tau(1-\tau)\Theta(\varsigma,\eta)\Big\} d\tau \right] :=$$

$$:= \frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)} \big[K_1 + K_2\big]. \tag{3.6}$$

Calculating K_1 and K_2, we get

$$K_1 \leq \big|e^{\varphi(\varsigma)}\varphi'(\varsigma)\big| \left(\int\limits_0^{\frac{1}{2}} [e^{-\sigma(1-\tau)} - e^{-\sigma\tau}]\tau^2\right) d\tau +$$

$$+ \big|\,e^{\varphi(\eta)}\varphi'(\eta)\big| \left(\int\limits_0^{\frac{1}{2}} [e^{-\sigma(1-\tau)} - e^{-\sigma\tau}](1-\tau)^2\right) d\tau +$$

$$+ \Theta\ (\varsigma,\eta) \left(\int\limits_0^{\frac{1}{2}} [e^{-\sigma(1-\tau)} - e^{-\sigma\tau}]\tau(1-\tau)\right) d\tau =$$

$$= \big|e^{\varphi(\varsigma)}\varphi'(\varsigma)\big| \left\{\frac{(\sigma^2+8)e^{-\frac{\sigma}{2}} - 4(e^{-\sigma}+1)}{2\sigma^3}\right\} +$$

$$+\mid e^{\varphi(\eta)}\varphi'(\eta)|\left\{\frac{(\sigma^2+8)e^{-\frac{\sigma}{2}}-2e^{-\sigma}(\sigma^2+2\sigma+2)-2(\sigma^2-2\sigma+2)}{2\sigma^3}\right\}+$$

$$+\,\Theta\,(\varsigma,\eta)\left\{\frac{(\sigma^2-8)e^{-\frac{\sigma}{2}}+2(\sigma+2)e^{-\sigma}-2(\sigma-2)}{2\sigma^3}\right\} \tag{3.7}$$

and

$$K_2\le\int\limits_{\frac{1}{2}}^{1}\left[e^{-\sigma\tau}-e^{-\sigma(1-\tau)}\right]\left\{\tau^2|e^{\varphi(\varsigma)}\varphi'(\varsigma)|+(1-\tau^2)|e^{\varphi(\eta)}\varphi'(\eta)|+\tau(1-\tau)\Theta(\varsigma,\eta)\right\}d\tau=$$

$$=\left|e^{\varphi(\varsigma)}\varphi'(\varsigma)\right|\left\{\frac{(\sigma^2+8)e^{-\frac{\sigma}{2}}-2e^{-\sigma}(\sigma^2+2\sigma+2)-2(\sigma^2-2\sigma+2)}{2\sigma^3}\right\}+$$

$$+\mid e^{\varphi(\eta)}\varphi'(\eta)|\left\{\frac{(\sigma^2+8)e^{-\frac{\sigma}{2}}-4(e^{-\sigma}+1)}{2\sigma^3}\right\}+$$

$$+\,\Theta\,(\varsigma,\eta)\left\{\frac{(\sigma^2-8)e^{-\frac{\sigma}{2}}+2(\sigma+2)e^{-\sigma}-2(\sigma-2)}{2\sigma^3}\right\}. \tag{3.8}$$

Thus, if we use (3.7) and (3.8) in (3.6), we obtain the result.

Theorem 3.3 is proved.

***Remark* 3.4.** For $\alpha\to 1$, we find that

$$\lim_{\alpha\to 1}\frac{(1-\alpha)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}=\frac{1}{2(\eta-\varsigma)},$$

$$\lim_{\alpha\to 1}\frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\left\{\frac{(\sigma^2+8)e^{\frac{\sigma}{2}}+e^{-\sigma}(\sigma^2+2\sigma+4)-(\sigma^2-2\sigma+4)}{\sigma^3}\right\}=\frac{\varsigma-\eta}{6}$$

and

$$\lim_{\alpha\to 1}\frac{(\eta-\varsigma)e^{\frac{\sigma}{2}}}{4\sinh\left(\dfrac{\sigma}{2}\right)}\left\{\frac{(\sigma^2-8)e^{-\frac{\sigma}{2}}+2(\sigma+2)e^{-\sigma}-2(\sigma-2)}{\sigma^3}\right\}=\frac{\varsigma-\eta}{6}.$$

Thus, Theorem 3.3 reduces to

$$\left|\frac{e^{\varphi(\varsigma)}+e^{\varphi(\eta)}}{2}-\frac{1}{\eta-\varsigma}\int\limits_{\varsigma}^{\eta}e^{\varphi(x)}dx\right|\le\frac{\eta-\varsigma}{6}\left[\left|e^{\varphi(\varsigma)}\varphi'(\varsigma)\right|+\left|e^{\varphi(\eta)}\varphi'(\eta)\right|+\Theta(\varsigma,\eta)\right].$$

Our next result about the Pachpatte-type inequality for exponentially convex function for new fractional integral operator technique.

Theorem 3.4. *Let φ and χ be real-valued, nonnegative and exponentially convex functions on $[\varsigma,\eta]\subset\mathbb{R}$. Then the following inequality for fractional integral holds*:

$$e^{\varphi(\frac{\varsigma+\eta}{2})}e^{\chi(\frac{\varsigma+\eta}{2})}\le$$

$$\leq \frac{(1-\alpha)e^{\frac{\sigma}{2}}}{8\sinh\left(\dfrac{\sigma}{2}\right)}\left[\mathcal{I}_{\varsigma}^{\alpha}e^{\varphi(\eta)}e^{\chi(\eta)}+\mathcal{I}_{\eta}^{\alpha}e^{\varphi(\varsigma)}e^{\chi(\varsigma)}\right]+$$

$$+\left[\,e^{\varphi(\varsigma)}e^{\chi(\varsigma)}+e^{\varphi(\eta)}e^{\chi(\eta)}\right]\frac{[\sigma-2+e^{-\sigma}(\sigma+2)]e^{\frac{\sigma}{2}}}{4\sigma^2\sinh\left(\dfrac{\sigma}{2}\right)}+$$

$$+\frac{\left[\sigma^2-2\sigma+4-(\sigma^2+2\sigma+4)e^{-\sigma}\right]e^{\frac{\sigma}{2}}}{8\sigma^2\sinh\left(\dfrac{\sigma}{2}\right)}\left[e^{\varphi(\varsigma)}e^{\chi(\eta)}+e^{\varphi(\eta)}e^{\chi(\varsigma)}\right] \tag{3.9}$$

and

$$\frac{\alpha}{\eta-\varsigma}\left[\mathcal{I}_{\varsigma}^{\alpha}e^{\varphi(\eta)}e^{\chi(\eta)}+\mathcal{I}_{\eta}^{\alpha}e^{\varphi(\varsigma)}e^{\chi(\varsigma)}\right]\leq$$

$$\leq\left[e^{\varphi(\varsigma)}e^{\chi(\varsigma)}+e^{\varphi(\eta)}e^{\chi(\eta)}\right]\frac{\sigma^2-2\sigma+4-e^{\sigma}(\sigma^2+2\sigma+4)}{\sigma^3}+$$

$$+\frac{e^{-\sigma}(\sigma+2)+\sigma-2}{\sigma^3}\left[e^{\varphi(\varsigma)}e^{\chi(\eta)}+e^{\varphi(\eta)}e^{\chi(\varsigma)}\right]. \tag{3.10}$$

Proof. Using the exponentially convexity of φ and χ on $[\varsigma,\eta]$, we have

$$e^{\varphi(\frac{\varsigma+\eta}{2})}e^{\chi(\frac{\varsigma+\eta}{2})}=$$

$$=e^{\varphi(\frac{\tau\varsigma+(1-\tau)\eta}{2}+\frac{(1-\tau)\varsigma+\tau\eta}{2})}e^{\chi(\frac{\tau\varsigma+(1-\tau)\eta}{2}+\frac{(1-\tau)\varsigma+\tau\eta}{2})}\leq$$

$$\leq\left(\frac{e^{\varphi(\tau\varsigma+(1-\tau)\eta)}+e^{\varphi((1-\tau)\varsigma+\tau\eta)}}{2}\right)\left(\frac{e^{\chi(\tau\varsigma+(1-\tau)\eta)}+e^{\chi((1-\tau)\varsigma+\tau\eta)}}{2}\right)\leq$$

$$\leq\frac{e^{\varphi(\tau\varsigma+(1-\tau)\eta)}e^{\chi(\tau\varsigma+(1-\tau)\eta)}}{4}+\frac{e^{\varphi(\tau\varsigma+(1-\tau)\eta)}e^{\chi(\tau\varsigma+(1-\tau)\eta)}}{4}+$$

$$+\frac{\tau(1-\tau)}{2}\left[e^{\varphi(\varsigma)}e^{\chi(\varsigma)}+e^{\varphi(\eta)}e^{\chi(\eta)}\right]+\frac{2\tau^2-2\tau+1}{4}\left[e^{\varphi(\varsigma)}e^{\chi(\eta)}+e^{\varphi(\varsigma)}e^{\chi(\eta)}\right]. \tag{3.11}$$

Multiplying both sides of (3.11) by $e^{\sigma\tau}$ and then integrating the resulting inequality with respect to $\tau\in[0,1]$, we obtain

$$\frac{2\sinh e^{\frac{\sigma}{2}}}{\sigma e^{\frac{\sigma}{2}}}e^{\varphi(\frac{\varsigma+\eta}{2})}e^{\chi(\frac{\varsigma+\eta}{2})}\leq$$

$$\leq\int\limits_0^1 e^{-\sigma\tau}\frac{e^{\varphi(\tau\varsigma+(1-\tau)\eta)}e^{\chi(\tau\varsigma+(1-\tau)\eta)}}{4}d\tau+$$

$$+\int\limits_0^1 e^{-\sigma\tau}\frac{e^{\varphi(\tau\varsigma+(1-\tau)\eta)}e^{\chi(\tau\varsigma+(1-\tau)\eta)}}{4}d\tau+$$

$$+\int\limits_0^1 e^{-\sigma\tau}\frac{\tau(1-\tau)}{2}\left[e^{\varphi(\varsigma)}e^{\chi(\varsigma)}+e^{\varphi(\eta)}e^{\chi(\eta)}\right]d\tau+$$

$$+\int\limits_0^1 e^{-\sigma\tau}\frac{2\tau^2-2\tau+1}{4}\big[e^{\varphi(\varsigma)}e^{\chi(\eta)}+e^{\varphi(\varsigma)}e^{\chi(\eta)}\big]d\tau=$$

$$=\frac{\alpha}{\eta-\varsigma}\big[\mathcal{I}_{\varsigma}^{\alpha}e^{\varphi(\eta)}e^{\chi(\eta)}+\mathcal{I}_{\eta}^{\alpha}e^{\varphi(\varsigma)}e^{\chi(\varsigma)}\big]+$$

$$+\big[\,e^{\varphi(\varsigma)}e^{\chi(\varsigma)}+e^{\varphi(\eta)}e^{\chi(\eta)}\big]\frac{\sigma-2+e^{-\sigma}(\sigma+2)}{2\sigma^3}+$$

$$+\frac{\sigma^2-2\sigma+4-(\sigma^2+2\sigma+4)e^{-\sigma}}{4\sigma^3}\,\big[e^{\varphi(\varsigma)}e^{\chi(\eta)}+e^{\varphi(\eta)}e^{\chi(\varsigma)}\big].$$

Therefore, we get the inequality

$$e^{\varphi(\frac{\varsigma+\eta}{2})}e^{\chi(\frac{\varsigma+\eta}{2})}\leq\frac{(1-\alpha)e^{\frac{\sigma}{2}}}{8\sinh\left(\frac{\sigma}{2}\right)}\big[\mathcal{I}_{\varsigma}^{\alpha}e^{\varphi(\eta)}e^{\chi(\eta)}+\mathcal{I}_{\eta}^{\alpha}e^{\varphi(\varsigma)}e^{\chi(\varsigma)}\big]+$$

$$+\big[e^{\varphi(\varsigma)}e^{\chi(\varsigma)}+e^{\varphi(\eta)}e^{\chi(\eta)}\big]\frac{[\sigma-2+e^{-\sigma}(\sigma+2)]e^{\frac{\sigma}{2}}}{4\sigma^2\sinh\left(\frac{\sigma}{2}\right)}+$$

$$+\frac{\big[\sigma^2-2\sigma+4-(\sigma^2+2\sigma+4)e^{-\sigma}\big]e^{\frac{\sigma}{2}}}{8\sigma^2\sinh\left(\frac{\sigma}{2}\right)}\,\big[e^{\varphi(\varsigma)}e^{\chi(\eta)}+e^{\varphi(\eta)}e^{\chi(\varsigma)}\big],$$

which completes the proof of (3.9).

Since φ and χ are exponentially convex on $[\varsigma,\eta]$, then, for $\tau\in[0,1]$, follows from Definition 2.4 that

$$e^{\varphi(\tau\varsigma+(1-\tau)\eta)}e^{\chi(\tau\varsigma+(1-\tau)\eta)}\leq$$

$$\leq\tau^2e^{\varphi(\varsigma)}e^{\chi(\varsigma)}+(1-\tau)^2e^{\varphi(\eta)}e^{\chi(\eta)}+\tau(1-\tau)\big[e^{\varphi(\varsigma)}e^{\chi(\eta)}+e^{\varphi(\eta)}e^{\chi(\varsigma)}\big]$$

and

$$e^{\varphi((1-\tau)\varsigma+\tau\eta)}e^{\chi((1-\tau)\varsigma+\tau b)}\leq$$

$$\leq(1-\tau)^2e^{\varphi(\varsigma)}e^{\chi(\varsigma)}+\tau^2e^{\varphi(\eta)}e^{\chi(\eta)}+\tau(1-\tau)\big[e^{\varphi(\varsigma)}e^{\chi(\eta)}+e^{\varphi(\eta)}e^{\chi(\varsigma)}\big].$$

Consequently, we have

$$e^{\varphi(\tau\varsigma+(1-\tau)b)}e^{\chi(\tau\varsigma+(1-\tau)\eta)}+e^{\varphi((1-\tau)\varsigma+\tau\eta)}e^{\chi((1-\tau)\varsigma+\tau\eta)}\leq$$

$$\leq(2\tau^2-2\tau+1)\big[e^{\varphi(\varsigma)}e^{\chi(\varsigma)}+e^{\varphi(\eta)}e^{\chi(b)}\big]+$$

$$+2\tau(1-\tau)\big[e^{\varphi(\varsigma)}e^{\chi(\eta)}+e^{\varphi(\eta)}e^{\chi(\varsigma)}\big]. \tag{3.12}$$

Multiplying both sides of inequality (3.12) by $e^{-\sigma\tau}$ and integrating the resulting inequality with respect to $\tau\in[0,1]$, we obtain

$$\int\limits_0^1 e^{-\sigma\tau}e^{\varphi(\tau\varsigma+(1-\tau)\eta)}e^{\chi(\tau\varsigma+(1-\tau)\eta)}d\tau+$$

$$+\int\limits_0^1 e^{-\sigma\tau} e^{\varphi((1-\tau)\varsigma+\tau\eta)} e^{\chi((1-\tau)\varsigma+\tau\eta)} d\tau \leq$$

$$\leq \left[e^{\varphi(\varsigma)}e^{\chi(\varsigma)} + e^{\varphi(\eta)}e^{\chi(\eta)}\right] \int\limits_0^1 e^{-\sigma\tau}(2\tau^2 - 2\tau + 1)d\tau +$$

$$+ \left[e^{\varphi(\varsigma)}e^{\chi(\eta)} + e^{\varphi(\eta)}e^{\chi(\varsigma)}\right] \int\limits_0^1 e^{-\sigma\tau} 2\tau(1-\tau)d\tau =$$

$$= \left[e^{\varphi(\varsigma)}e^{\chi(\varsigma)} + e^{\varphi(\eta)}e^{\chi(\eta)}\right] \frac{\sigma^2 - 2\sigma + 4 - e^{\sigma}(\sigma^2 + 2\sigma + 4)}{\sigma^3} +$$

$$+ \frac{e^{-\sigma}(\sigma+2) + \sigma - 2}{\sigma^3} \left[e^{\varphi(\varsigma)}e^{\chi(\eta)} + e^{\varphi(\eta)}e^{\chi(\varsigma)}\right].$$

Thus, we have the inequality

$$\frac{\alpha}{\eta - \varsigma}[\mathcal{I}_{\varsigma}^{\alpha} e^{\varphi(\eta)}e^{\chi(\eta)} + \mathcal{I}_{\eta}^{\alpha} e^{\varphi(\varsigma)}e^{\chi(\varsigma)}] \leq$$

$$\leq \left[e^{\varphi(\varsigma)}e^{\chi(\varsigma)} + e^{\varphi(\eta)}e^{\chi(\eta)}\right] \frac{\sigma^2 - 2\sigma + 4 - e^{\sigma}(\sigma^2 + 2\sigma + 4)}{\sigma^3} +$$

$$+ \frac{e^{-\sigma}(\sigma+2) + \sigma - 2}{\sigma^3} \left[e^{\varphi(\varsigma)}e^{\chi(\eta)} + e^{\varphi(\eta)}e^{\chi(\varsigma)}\right],$$

which proves the inequality (3.10).

Theorem 3.4 is proved.

***Corollary* 3.2.** *In limiting case, under the assumption of Theorem* 3.4, *we obtain*

$$2e^{\varphi\left(\frac{\varsigma+\eta}{2}\right)} e^{\chi\left(\frac{\varsigma+\eta}{2}\right)} \leq \frac{1}{\eta - \varsigma} \int\limits_{\varsigma}^{\eta} e^{\varphi(x)}e^{\chi(x)}dx \leq$$

$$\leq \frac{e^{\varphi(\varsigma)}e^{\chi(\varsigma)} + e^{\varphi(\eta)}e^{\chi(\eta)}}{6} + \frac{e^{\varphi(\varsigma)}e^{\chi(\eta)} + e^{\varphi(\eta)}e^{\chi(\varsigma)}}{3}$$

and

$$\frac{1}{\eta - \varsigma} \int\limits_{\varsigma}^{\eta} e^{\varphi(x)}e^{\chi(x)}dx \leq \frac{e^{\varphi(\varsigma)}e^{\chi(\varsigma)} + e^{\varphi(\eta)}e^{\chi(\eta)}}{3} + \frac{e^{\varphi(\varsigma)}e^{\chi(\eta)} + e^{\varphi(\eta)}e^{\chi(\varsigma)}}{6}.$$

Conclusion. In this paper, we established the several inequalities within the scope of fractional integral with exponential kernel. This new fractional integral operator helped in proving the Hermite – Hadamard type and Hermite – Hadamard – Fejér type inequalities for exponentially convex functions and in finding bounds of these inequalities. In addition, an immediate consequences of the results derived in this paper.

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