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T. Miezaki (Waseda Univ., Tokyo, Japan)

A CONSTRUCTION OF SPHERICAL 3-DESIGNS*
ПОБУДОВА СФЕРИЧНИХ 3-КОНСТРУКЦІЙ

We give a construction for spherical 3-designs. This construction is a generalization of Bondarenko's results.

Наведено метод побудови сферичних 3-конструкцій. Цей метод є узагальненням результатів Бондаренка.

1. Introduction. This paper is inspired by [1], which gives an optimal antipodal spherical $(35, 240, 1/7)$ code whose vectors form a spherical 3-design. To explain our results, we review the concept of spherical t-designs and [1].

First, we explain the concept of spherical t-designs.

Definition 1.1 [3]. *For a positive integer* t, *a finite nonempty set* X *in the unit sphere*

$$S^d = \{x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \mid x_1^2 + \ldots + x_{d+1}^2 = 1\}$$

is called a spherical t*-design in* S^d *if the following condition is satisfied*:

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|S^d|}\int\limits_{S^d} f(x)d\sigma(x),$$

for all polynomials $f(x) = f(x_1, \ldots, x_{d+1})$ *of degree not exceeding* t. *Here, the right-hand side involves the surface integral over the sphere and* $|S^d|$, *the volume of sphere* S^d.

The meaning of spherical t-designs is that the average value of the integral of any polynomial of degree up to t on the sphere can be replaced by its average value over a finite set on the sphere.

The following is an equivalent condition of the antipodal spherical designs.

Proposition 1.1 [6]. *An antipodal set* $X = \{x_1, \ldots, x_N\}$ *in* S^d *forms a spherical* 3*-design if and only if*

$$\frac{1}{|X|^2}\sum_{x_i,x_j\in X}(x_i, x_j)^2 = \frac{1}{d+1}.$$

An antipodal set $X = \{x_1, \ldots, x_N\}$ *in* S^d *forms a spherical* 5*-design if and only if*

$$\frac{1}{|X|^2}\sum_{x_i,x_j\in X}(x_i, x_j)^2 = \frac{1}{d+1},$$
$$\frac{1}{|X|^2}\sum_{x_i,x_j\in X}(x_i, x_j)^4 = \frac{3}{(d+3)(d+1)}. \tag{1.1}$$

Next, we review [1]. Let

$$\Delta = \sum_{j=1}^{d+1}\frac{\partial^2}{\partial x_j^2}.$$

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We say that a polynomial P in $\mathbb{R}^{d+1}$ is harmonic if $\Delta P = 0$. For integer $k \geq 1$, the restriction of a homogeneous harmonic polynomial of degree k to S^d is called a spherical harmonic of degree k. We denote by $\mathrm{Harm}_k(S^d)$ the vector space of the spherical harmonics of degree k. Note that (see, for example, [6])

$$\dim \mathrm{Harm}_k(S^d) = \frac{2k+d-1}{k+d-1}\binom{d+k-1}{k}.$$

For $P, Q \in \mathrm{Harm}_k(S^d)$, we denote by $\langle P, Q\rangle$ the usual inner product

$$\langle P, Q\rangle := \int\limits_{S^d} P(x)Q(x)d\sigma(x),$$

where $d\sigma(x)$ is a normalized Lebesgue measure on the unit sphere S^d. For $x \in S^d$, there exists $P_x \in \mathrm{Harm}_k(S^d)$ such that

$$\langle P_x, Q\rangle = Q(x) \quad \text{for all} \quad Q \in \mathrm{Harm}_k(S^d).$$

It is known that

$$P_x(y) = g_{k,d}((x,y)),$$

where $g_{k,d}$ is a Gegenbauer polynomial. Let

$$G_x = \frac{P_x}{g_{k,d}(1)^{1/2}}.$$

We remark that

$$\langle G_x, G_y\rangle = \frac{g_{k,d}((x,y))}{g_{k,d}(1)}.$$

(For a detailed explanation of Gegenbauer polynomials, see [6].) Therefore, if we have a set $X = \{x_1, \ldots, x_N\}$ in S^d, then we obtain the set $G_X = \{G_{x_1}, \ldots, G_{x_N}\}$ in $S^{\dim \mathrm{Harm}_k(S^d)-1}$.

Let $X = \{x_1, \ldots, x_{120}\}$ be an arbitrary subset of 240 normalized minimum vectors of the E_8 lattice such that no pair of antipodal vectors is present in X. Set $P_x(y) = g_{2,7}((x,y))$. A. V. Bondarenko in [1] showed that $G_X \cup -G_X$ is an optimal antipodal spherical $(35, 240, 1/7)$ code whose vectors form a spherical 3-design, where

$$-G_X := \{-G_x \mid G_x \in G_X\}.$$

Furthermore, A. V. Bondarenko in [1] showed that $G_X \cup -G_X$ is a spherical 3-design, using the special properties of the E_8 lattice. However, this fact is an example that extends to a more general setting as follows. The spherical 3-design obtained by A. V. Bondarenko in [1] is a special case of our main result, which is presented as the following theorem.

Theorem 1.1. *Let X be a finite subset of sphere S^d satisfying the condition* (1.1). *We set $P_x(y) = g_{2,d}((x,y))$. Then $G_X \cup -G_X$ is a spherical 3-design in $S^{\dim \mathrm{Harm}_2(S^d)-1}$.*

We denote by $\widetilde{G_X}$ the set $G_X \cup -G_X$ defined in Theorem 1.1.

***Corollary* 1.1.** 1. *Let X be a spherical 4-design in S^d. Then $\widetilde{G_X}$ is a spherical 3-design in $S^{\dim \mathrm{Harm}_2(S^d)-1}$.*

2. *Let X be a spherical 4-design in S^d and an antipodal set. Let X' be an arbitrary subset of X with $|X'| = |X|/2$ such that no pair of antipodal vectors is present in X'. Then $\widetilde{G_{X'}}$ is a spherical 3-design in $S^{\dim \mathrm{Harm}_2(S^d)-1}$.*

In Section 2, we give a proof of Theorem 1.1. In Section 3, we give some examples.

2. Proof of Theorem 1.1. Let $X = \{x_1, \ldots, x_N\}$ be in S^d and $G_X = \{G_{x_1}, \ldots, G_{x_N}\}$ be in $\mathrm{Harm}_2(S^d)$. By Proposition 1.1, we have

$$\frac{1}{|X|^2} \sum_{x_i, x_j \in X} (x_i, x_j)^2 = \frac{1}{d+1},$$

$$\frac{1}{|X|^2} \sum_{x_i, x_j \in X} (x_i, x_j)^4 = \frac{3}{(d+3)(d+1)},$$

since X is a spherical 4-design. We have the following Gegenbauer polynomial of degree 2 on S^d:

$$g_{2,d}(x) = \frac{d+1}{d} x^2 - \frac{1}{d}.$$

It is enough to show that

$$\frac{1}{|X|^2} \sum_{x_i, x_j \in X} \langle G_{x_i}, G_{x_j} \rangle^2 = \frac{2}{d(d+3)},$$

since

$$\dim \mathrm{Harm}_2(S^d) = \frac{d+3}{d+1} \binom{d+1}{2} = \frac{d(d+3)}{2}$$

and $G_X \cup -G_X$ is an antipodal set. We remark that if X is a spherical t-design, then $X \cup -X$ is also a spherical t-design.

In fact,

$$\frac{1}{|X|^2} \sum_{x_i, x_j \in X} \langle G_{x_i}, G_{x_j} \rangle^2 = \frac{1}{|X|^2} \sum_{x_i, x_j \in X} g_{2,d}((x_i, x_j))^2 =$$

$$= \frac{1}{|X|^2} \sum_{x_i, x_j \in X} \left(\frac{(d+1)^2}{d^2} (x_i, x_j)^4 - 2\frac{d+1}{d^2} (x_i, x_j)^2 + \frac{1}{d^2} \right) =$$

$$= \frac{(d+1)^2}{d^2} \frac{3}{(d+3)(d+1)} - 2\frac{d+1}{d^2} \frac{1}{d+1} + \frac{1}{d^2} = \frac{2}{d(d+3)}.$$

Therefore, if $X = \{x_1, \ldots, x_N\}$ is a spherical 4-design, then $G_X \cup -G_X$ is a spherical 3-design.

Theorem 1.1 is proved.

3. Examples. In this section, we give some examples of using Theorem 1.1.

First we recall the concept of a strongly perfect and spherical $(d+1, N, a)$ code.

Definition 3.1 [6]. *A lattice L is called strongly perfect if the minimum vectors of L form a spherical 5-design.*

Definition 3.2 [2]. *An antipodal set $X = \{x_1, \ldots, x_N\}$ in S^d is called an antipodal spherical $(d+1, N, a)$ code if $|(x_i, x_j)| \leq a$ for some $a > 0$ and all $x_i, x_j \in X$, $i \neq j$, are not antipodal.*

Next we give some examples.

***Example* 3.1.** The strongly perfect lattices whose ranks are less than 12 have been classified [4, 5]. Such lattices whose ranks are greater than 1 are as follows:

$$A_2,\ D_4,\ E_6,\ E_6^\sharp,\ E_7,\ E_7^\sharp,\ E_8,\ K_{10},\ K_{10}^\sharp,\ CT_{12}.$$

(For a detailed explanation, see [4, 5].) Let L be one of the above lattices and X be the minimum vectors of L. Then, let X' be an arbitrary subset of X with $|X'| = |X|/2$ such that no pair of antipodal vectors is present in X'.

By Corollary 1.1, $G_X \cup -G_X$ is a spherical 3-design in S^d, where d is as follows:

L	$(d+1, N, a)$ code	$\lvert(x_i, x_j)\rvert$	$\lvert\langle G_{x_i}, G_{x_j}\rangle\rvert$
A_2	$(2, 6, 1/2)$	$\{1/2\}$	$\{1/2\}$
D_4	$(9, 24, 1/3)$	$\{0, 1/2\}$	$\{0, 1/3\}$
E_6	$(20, 72, 1/5)$	$\{0, 1/2\}$	$\{1/10, 1/5\}$
$E_6^\sharp$	$(20, 54, 1/8)$	$\{1/4, 1/2\}$	$\{1/10, 1/8\}$
E_7	$(27, 126, 1/6)$	$\{0, 1/2\}$	$\{1/8, 1/6\}$
$E_7^\sharp$	$(27, 56, 1/27)$	$\{1/3\}$	$\{1/27\}$
E_8	$(35, 240, 1/7)$ [1]	$\{0, 1/2\}$	$\{1/7\}$
K_{10}	$(54, 276, 1/6)$	$\{0, 1/4, 1/2\}$	$\{1/24, 1/9, 1/6\}$
$K_{10}^\sharp$	$(54, 54, 1/6)$	$\{1/8, 1/4, 1/2\}$	$\{1/24, 3/32, 1/6\}$
CT_{12}	$(77, 756, 2/11)$	$\{0, 1/4, 1/2\}$	$\{1/44, 1/11, 2/11\}$

***Example* 3.2.** Let X be the minimum vectors of the Barnes – Wall lattice of rank 16, and let X' be an arbitrary subset of X with $|X'| = |X|/2$ such that no pair of antipodal vectors is present in X'. We remark that X is a spherical 7-design.

By Corollary 1.1, $G_X \cup -G_X$ is a spherical 3-design in S^d, where d is as follows:

L	$(d+1, N, a)$ code	$\lvert(x_i, x_j)\rvert$	$\lvert\langle G_{x_i}, G_{x_j}\rangle\rvert$
BW_{16}	$(135, 4320, 1/5)$	$\{0, 1/4, 1/2\}$	$\{0, 1/15, 1/5\}$

***Example* 3.3.** Let X be the minimum vectors of the Leech lattice, and let X' be an arbitrary subset of X with $|X'| = |X|/2$ such that no pair of antipodal vectors is present in X'. We remark that X is a spherical 11-design.

By Corollary 1.1, $G_X \cup -G_X$ is a spherical 3-design in S^d, where d is as follows:

L	$(d+1, N, a)$ code	$\lvert(x_i, x_j)\rvert$	$\lvert\langle G_{x_i}, G_{x_j}\rangle\rvert$
Leech	$(299, 196560, 5/23)$	$\{0, 1/4, 1/2\}$	$\{1/46, 1/23, 5/23\}$

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